Monotonie und Krümmung

f monoton steigend $\Leftrightarrow f'(x) \geq 0$
f monoton falled $\Leftrightarrow f'(x) \leq 0$
f konvex $\Leftrightarrow f''(x) \geq 0$, $\qquad f$ konkav $\Leftrightarrow f''(x) \leq 0$

Multivariate Analysis

Gradient: $\nabla f(\boldsymbol{x}) = (f_{x_1}(\boldsymbol{x}), \ldots, f_{x_n}(\boldsymbol{x}))$

Richtungsableitung: $\frac{\partial f}{\partial \boldsymbol{h}} = \nabla f(\boldsymbol{x}) \cdot \boldsymbol{h}$, $\quad ||\boldsymbol{h}|| = 1$

Jacobische Matrix:

$$Df(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) & \ldots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{x}) & \ldots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{x}) \end{pmatrix}$$

Kettenregel: $D(g \circ f)(\boldsymbol{x}) = Dg(f(\boldsymbol{x})) \cdot Df(\boldsymbol{x})$

Ableitung der inversen Transformation:
$D(f^{-1})(\boldsymbol{y}) = (Df(\boldsymbol{x}))^{-1}$, $\quad \boldsymbol{y} = f(\boldsymbol{x})$

Ableitung einer impliziten Funktion $\boldsymbol{F}(\boldsymbol{x}) = 0$:

$$\frac{\partial x_i}{\partial x_k} = -\frac{F_{x_k}}{F_{x_i}}$$

Abl. einer impliziten Transformation $F(\boldsymbol{x}, \boldsymbol{y}) = 0$:

$$\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} = -\left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{y}}\right)^{-1} \cdot \left(\frac{\partial \boldsymbol{F}}{\partial \boldsymbol{x}}\right)$$

Hesse-Matrix:

$$\boldsymbol{H}_f(\boldsymbol{x}) = \begin{pmatrix} f_{x_1x_1}(\boldsymbol{x}) & \ldots & f_{x_1x_n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(\boldsymbol{x}) & \ldots & f_{x_nx_n}(\boldsymbol{x}) \end{pmatrix}$$

Optima

$$H_k = \begin{vmatrix} f_{x_1x_1}(\boldsymbol{x}) & \ldots & f_{x_1x_k}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ f_{x_kx_1}(\boldsymbol{x}) & \ldots & f_{x_kx_k}(\boldsymbol{x}) \end{vmatrix}$$

Alle $H_k > 0 \Rightarrow \boldsymbol{x}$ ist lok. Min.

Alle $(-1)^k H_k > 0 \Rightarrow \boldsymbol{x}$ ist lok. Max.
Andernfalls, $|\boldsymbol{H}_f(\boldsymbol{x})| \neq 0 \Rightarrow \boldsymbol{x}$ ist Sattelpunkt.

Lagrange-Funktion:
$\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda\,(c - g(x, y))$

$$\bar{\boldsymbol{H}}(\boldsymbol{x}) = \begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{pmatrix}$$

$|\bar{\boldsymbol{H}}(\boldsymbol{x})| > 0 \Rightarrow \boldsymbol{x}$ ist lok. Max.
$|\bar{\boldsymbol{H}}(\boldsymbol{x})| < 0 \Rightarrow \boldsymbol{x}$ ist lok. Min.

Kuhn-Tucker Bedingung:

$\mathcal{L}_{x_j} \leq 0, \quad x_j \geq 0, \quad x_j\,\mathcal{L}_{x_j} = 0$
$\mathcal{L}_{\lambda_i} \geq 0, \quad \lambda_i \geq 0, \quad \lambda_i\,\mathcal{L}_{\lambda_i} = 0$

Taylorreihen

$$f(x) \approx f(0) + \frac{f'(0)}{1!}\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + R_n(x)$$

Wichtige Taylorreihen:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$f(\boldsymbol{x}) \approx f(\boldsymbol{0}) + \nabla f(\boldsymbol{0})^t \boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^t \boldsymbol{H}_f(\boldsymbol{0})\boldsymbol{x}$$

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^t(\boldsymbol{x} - \boldsymbol{x}_0) + \tfrac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^t \boldsymbol{H}_f(\boldsymbol{0})(\boldsymbol{x} - \boldsymbol{x}_0)$$