Precedence-Type Test Based on Kaplan-Meier Estimator of CDF

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Outline

- Review of the Precedence-type Tests

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  3. Power Comparison under Location-Shift Alternative

Introduction

- Suppose there are two failure time distributions F_X and F_Y. We are interested in testing the hypotheses

$$\begin{aligned} & H_0 : F_X = F_Y \\ \text{against} \quad & H_1 : F_X > F_Y. \end{aligned} \tag{1}$$

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- Nelson (1963, Technometrics) first proposed a Precedence test, which can determine a location difference based on observing only a few failures from the two samples under life testing.
- Nelson (1993, JQT) examined the power of the precedence test when the underlying distributions were normal.

Precedence Test

What is Precedence Test?

- Suppose $X_1, X_2, \ldots, X_{n_1}$ is a random sample from F_X and $Y_1, Y_2, \ldots, Y_{n_2}$ is a random sample from F_Y.

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- All these sample units are placed simultaneously on a life-testing experiment.
- The Precedence test statistic is

$$\begin{aligned} P_{(r)} &= \text{No. of failures from } X\text{-sample that precede} \\ &\quad \text{the } r\text{-th failure from the } Y\text{-sample} \\ &= \sum_{i=1}^{r} M_i \end{aligned}$$

Precedence Test

- We denote the order statistics from the X-sample and the Y-sample by $X_{1:n_1} \leq X_{2:n_1} \leq \ldots \leq X_{n_1:n_1}$ and $Y_{1:n_2} \leq Y_{2:n_2} \leq \ldots \leq Y_{n_2:n_2}$, respectively.

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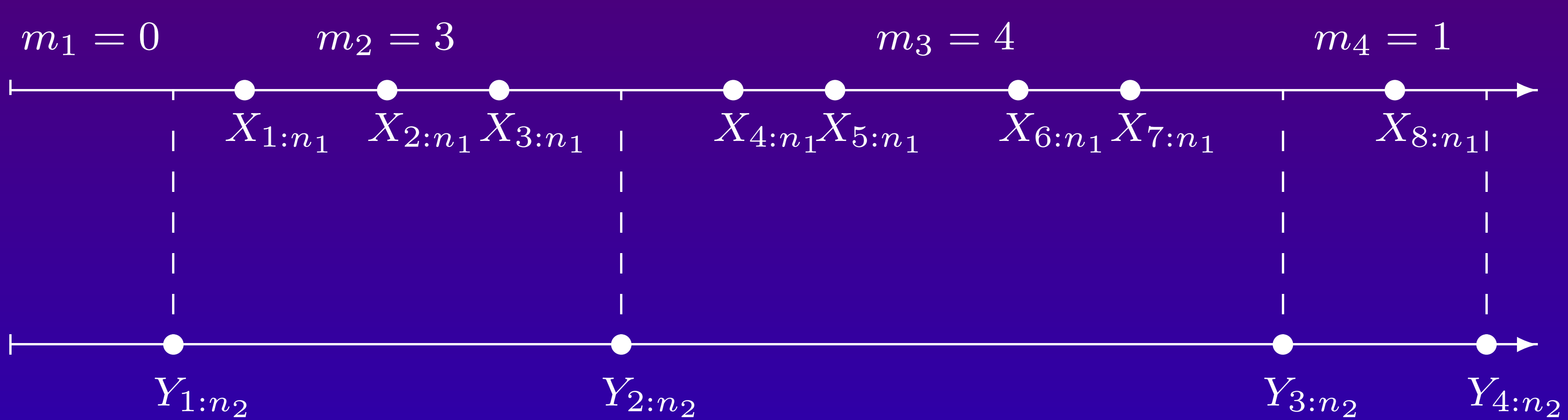
- Let M_1 be the number of X-failures before $Y_{1:n_2}$, and M_i be the number of X-failures between $Y_{i-1:n_2}$ and $Y_{i:n_2}$, $i = 2, 3, \ldots, r$.



Precedence Test Statistics $P_{(4)} = 3 + 4 + 1 = 8$

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  (i) when life-tests involve expensive units

  (ii) to make quick and reliable decisions early on in the life-testing experiment.

  (iii) in medical studies when ethical considerations are taking into account.

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- For specified values of n_1, n_2, s and r, an expression for α is given by

$$\alpha = \sum_{j=s}^{n_1} \frac{\binom{s+r-1}{j}\binom{n_1+n_2-s-r+1}{n_1-j}}{\binom{n_1+n_2}{n_2}}$$

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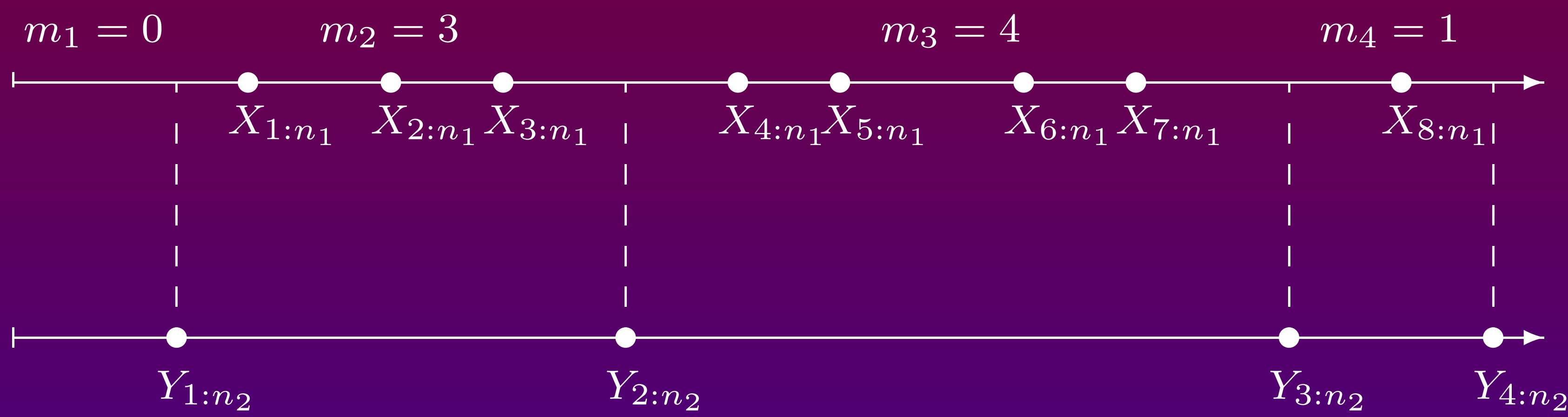
Maximal Precedence Test



Maximal Precedence Test Statistics

$$M_{(4)} = \max\{0, 3, 4, 1\} = 4$$

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Wilcoxon-type Rank-Sum Precedence

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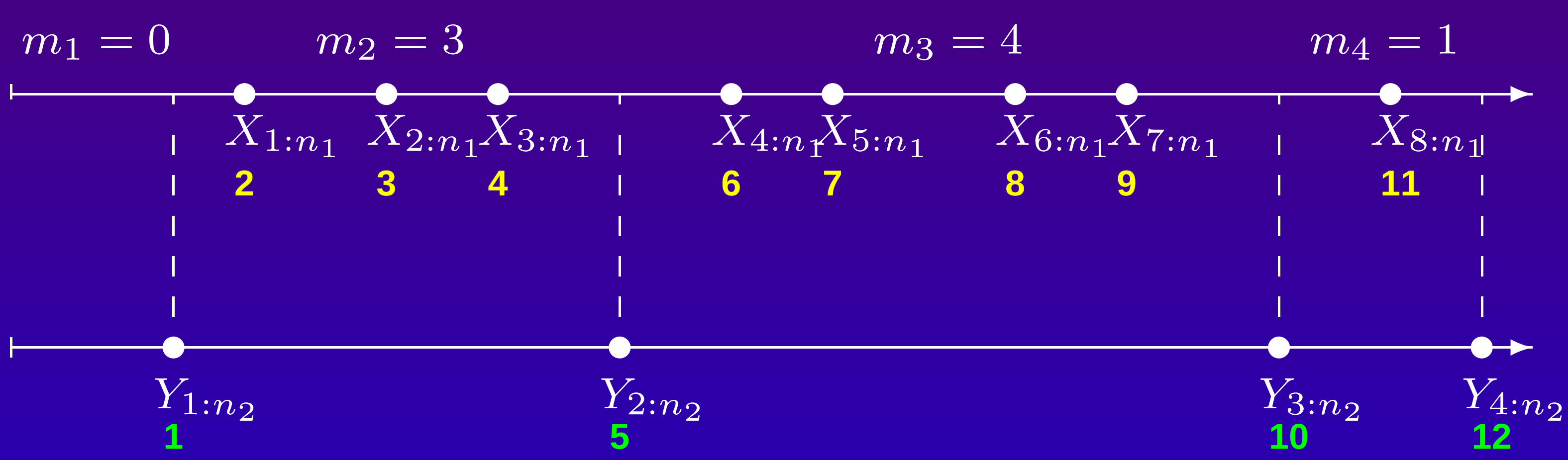
- Let W be the rank-sum of the X-failures that occurred before the r-th Y-failure.
- In our example,

$$W = 2 + 3 + 4 + 6 + 8 + 9 + 11 = 50$$



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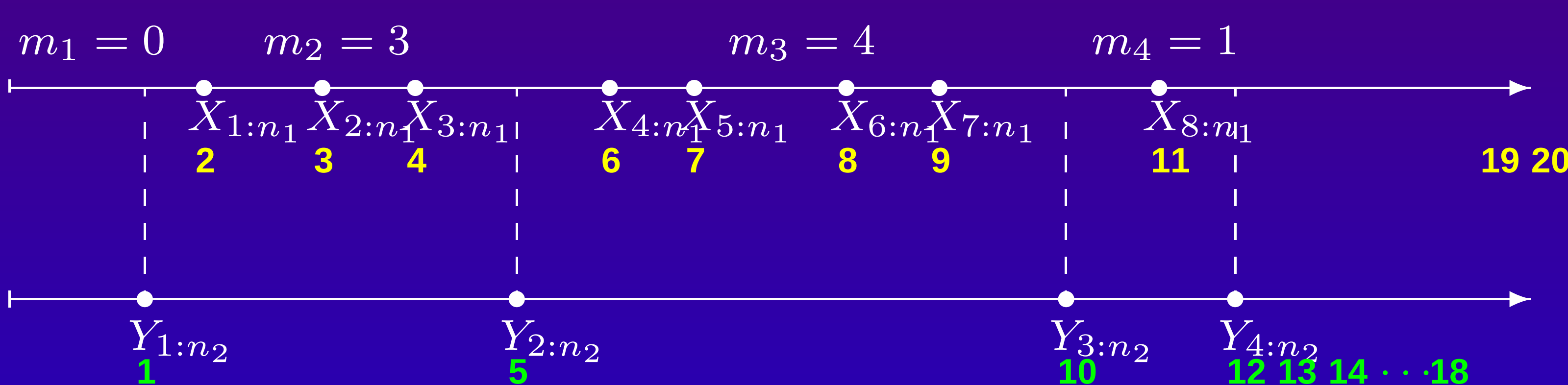
  - Contradiction!!

Maximal Rank-sum Statistic

The Wilcoxon's test statistic will be the largest when all the remaining $\left(n_1 - \sum_{i=1}^{r} m_i\right)$ X-failures occur after the n_2-th Y-failure.

$$W_{\max,r} = W + \left[\left(\sum_{i=1}^{r} m_i + n_2 + 1\right) + \left(\sum_{i=1}^{r} m_i + n_2 + 2\right) + \ldots + (n_1 + n_2)\right]$$

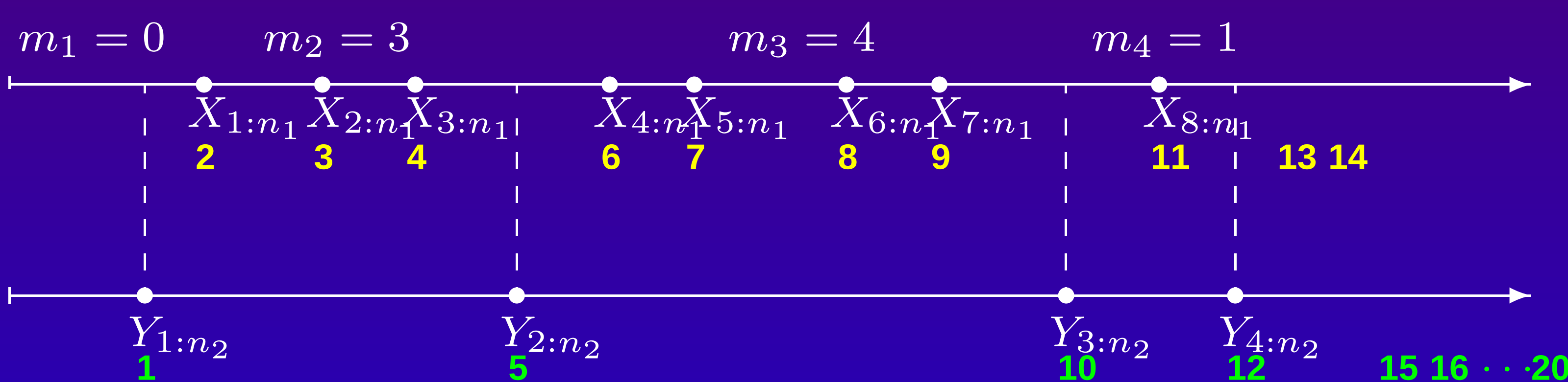
$$= \frac{n_1(n_1 + 2n_2 + 1)}{2} - (n_2 + 1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i.$$



Minimal Rank-sum Statistic

- The Wilcoxon's test statistic will be smallest when all the remaining $\left(n_1 - \sum_{i=1}^{r} m_i\right)$ X-failures occur between the r-th and $(r+1)$-th Y-failures.

$$\begin{aligned} W_{\min,r} &= W + \left[\left(\sum_{i=1}^{r} m_i + r + 1\right) + \left(\sum_{i=1}^{r} m_i + r + 2\right) + \ldots + (n_1 + r)\right] \\ &= \frac{n_1(n_1 + 2r + 1)}{2} - (r+1)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i . \end{aligned}$$



Expected Rank-sum Statistic

- We could similarly propose a rank-sum statistic using the expected rank sums of failures, called the *expected rank-sum statistic*

$$
\begin{aligned}
W_{E,r} &= W + \frac{1}{2}\left[\left(\sum_{i=1}^{r} m_i + r + 1\right) + \cdots + (n_1 + r)\right] \\
&\quad + \frac{1}{2}\left[\left(\sum_{i=1}^{r} m_i + n_2 + 1\right) + \cdots + (n_1 + n_2)\right] \\
&= \frac{n_1(n_1 + n_2 + r + 1)}{2} - \left(\frac{n_2 + r}{2} + 1\right)\sum_{i=1}^{r} m_i + \sum_{i=1}^{r} i m_i .
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- Small values of $W_{\min,r}$, $W_{\max,r}$ and $W_{E,r}$ lead to the rejection of H_0 and in favor of H_1.
- When $r = n_2$, we have

$W_{\min,r} = W_{\max,r} = W_{E,r} =$ classical Wilcoxon's rank-sum statistic

Null Distributions

- For specified values of n_1, n_2, r and the level of significance α, the critical value s (corresponding to a level closest to α) for the maximal rank-sum precedence test can then be found as

$$
\begin{aligned}
\alpha &= \Pr(W_{\max,r} \le s \mid F_X = F_Y) \\
&= \sum_{m_i (i=1,2,\ldots,r)=0}^{n_1} \Pr\left\{M_1 = m_1, \ldots, M_r = m_r \mid F_X = F_Y\right\} I_{\max,s}(m_1, \ldots, m_r),
\end{aligned}
$$

where $I_{\max,s}$ is the indicator function and

$$
\Pr\left\{M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid F_X = F_Y\right\} = \frac{\binom{n_1 + n_2 - \sum_{i=1}^{r} m_i - r}{n_2 - r}}{\binom{n_1 + n_2}{n_2}}.
$$

Weighted Precedence

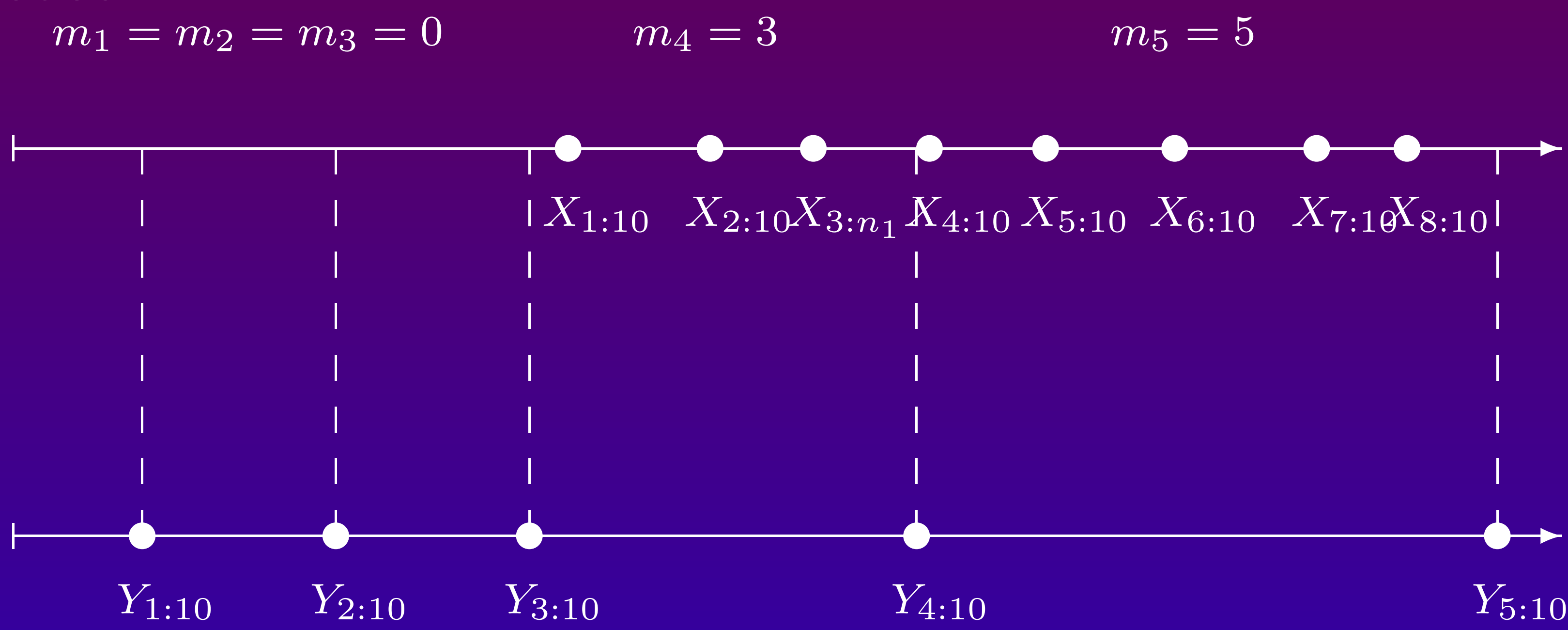
and

Maximal Precedence Tests

Motivation

We are going to introduce another logical extension of the precedence and maximal precedence tests. The motivation is explained by the following two cases with $n_1 = 10$, $n_2 = 10$, $r = 5$:

Case 1:



Precedence test statistic $P_{(5)} = 8$, maximal precedence test statistic $M_{(5)} = 5$.

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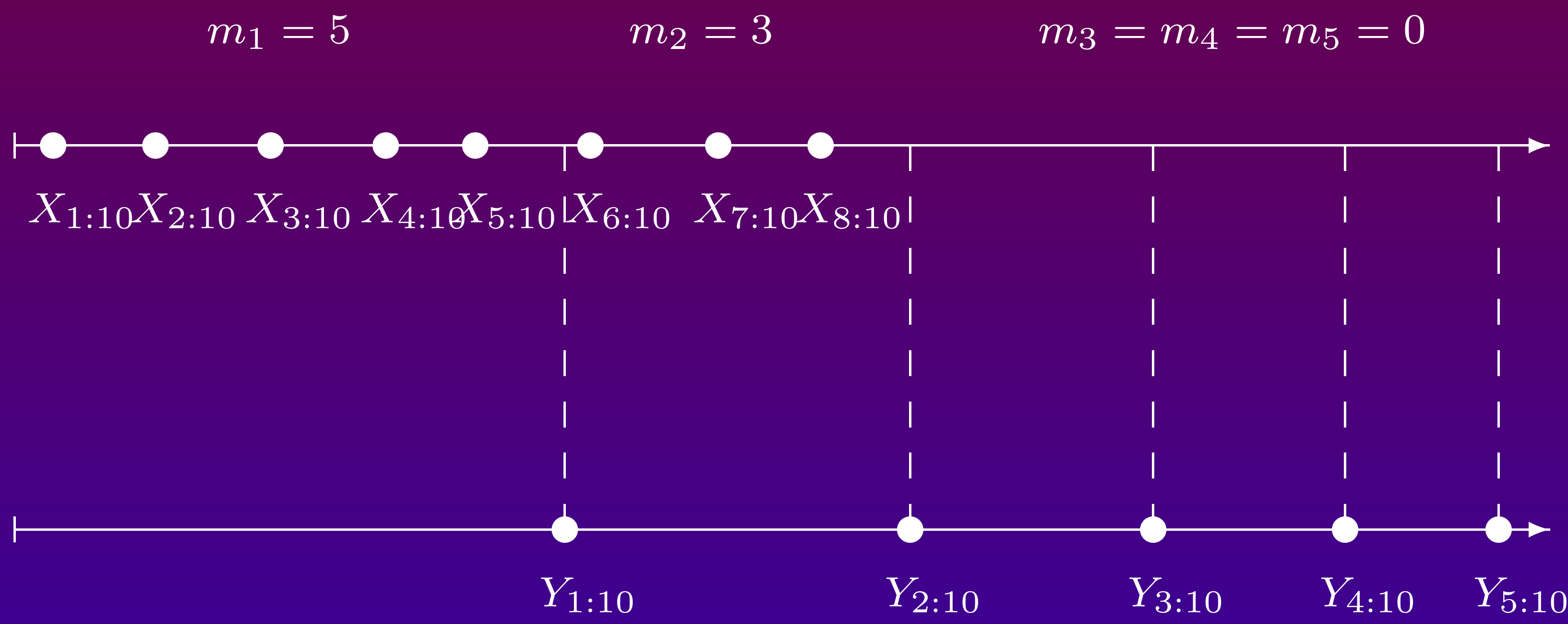
$n_1 = 10$, $n_2 = 10$, $r = 5$:

Case 2



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- The test statistics are $P_{(5)} = 8$ and $M_{(5)} = 5$, we will not reject the null hypothesis that two distributions are equal in both cases at the same level of significance.
- However, we feel that Case 2 provides much more evidence that the Y-sample are better than the X-sample.
- This suggests that we should try to develop a test procedure that distinguishes between Case 1 and Case 2.

Test Statistics

- The weighted precedence and weighted maximal precedence tests give a decreasing weight to m_i as i increases is one such attempt in this direction.

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- The weighted precedence and weighted maximal precedence tests give a decreasing weight to m_i as i increases is one such attempt in this direction.
- The weighted precedence test statistic $P^*_{(r)}$ is thus defined as

$$P^*_{(r)} = \sum_{i=1}^{r} (n_2 - i + 1)\, m_i,$$

and the weighted maximal precedence test statistic $M^*_{(r)}$ as

$$M^*_{(r)} = \max_{1 \le i \le r} \{(n_2 - i + 1)\, m_i\}.$$

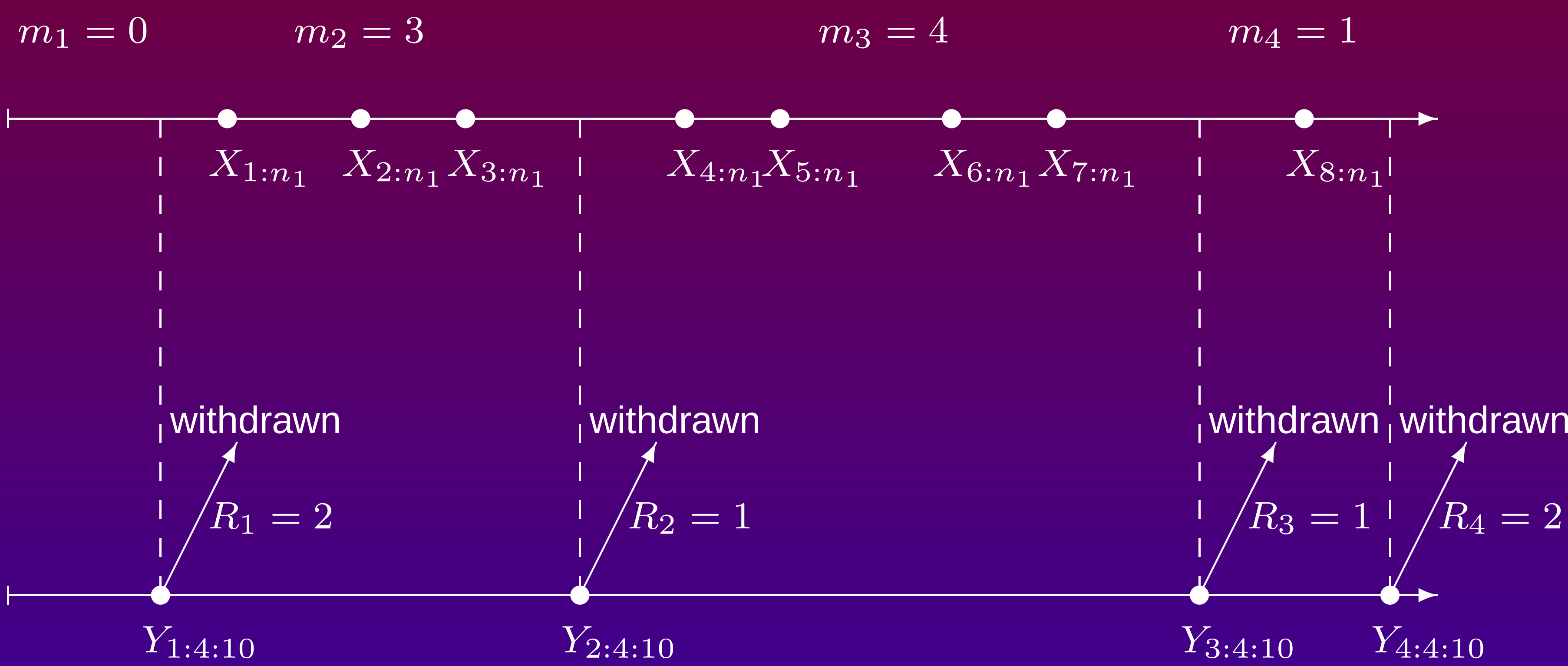
It is clear that large values of $P^*_{(r)}$ or $M^*_{(r)}$ would lead to the rejection of H_0 and in favor of H_1.

An Extension to Type-II Progressive Censored Data

Type-II Progressive Censoring

Experimental schemes of potential use in life-testing in which, not only can the test be terminated at the time of any failure, but in addition, one or more surviving items may be removed from the test (i.e. **censored**) at the time of each failure occurring prior to the termination of the experiment.

Type-II Progressive Censoring



Let us denote such an observed ordered Y-sample by $Y_{1:r:n_2} \leq Y_{2:r:n_2} \leq \cdots \leq Y_{r:r:n_2}$.

Moreover, we denote by M_1 the number of X-failures before $Y_{1:r:n_2}$, and by M_i the number of X-failures between $Y_{i-1:r:n_2}$ and $Y_{i:r:n_2}$, $i = 2, 3, \cdots, r$.

Weighted Precedence & Maximal Precedence

Based on $M_1, M_2, \cdots, M_r$, we propose the weighted precedence test statistic $P^*_{(r)}$, as

$$P^*_{(r)} = \sum_{i=1}^{r} \left[n_2 - \left(\sum_{j=1}^{i-1} R_j \right) - i + 1 \right] m_i,$$

and the weighted maximal precedence test statistic $M^*_{(r)}$, as

$$M^*_{(r)} = \max_{1 \le i \le r} \left\{ \left[n_2 - \left(\sum_{j=1}^{i-1} R_j \right) - i + 1 \right] m_i \right\}.$$

Maximal Rank-sum Statistic for Progressive Censoring

- Similarly, we can apply the idea of maximal Wilcoxon rank-sum precedence statistic for the progressively censored data.

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- Similarly, we can apply the idea of maximal Wilcoxon rank-sum precedence statistic for the progressively censored data.
- The Wilcoxon's rank-sum test statistic will be the largest when all the progressive censored Y-items in an interval fail before the smallest of X-failures in the corresponding interval. The maximal Wilcoxon rank-sum precedence statistic for progressively censored data then becomes

$$W^*_{\max,r} = \sum_{i=1}^{r+1} \left\{ m_i \left[\sum_{j=1}^{i-1} m_j + \sum_{j=1}^{i-1} R_i + (i-1) \right] + \frac{i(m_i+1)}{2} \right\}.$$

where $m_{r+1} = n_1 - \sum_{i=1}^{r} m_i$.

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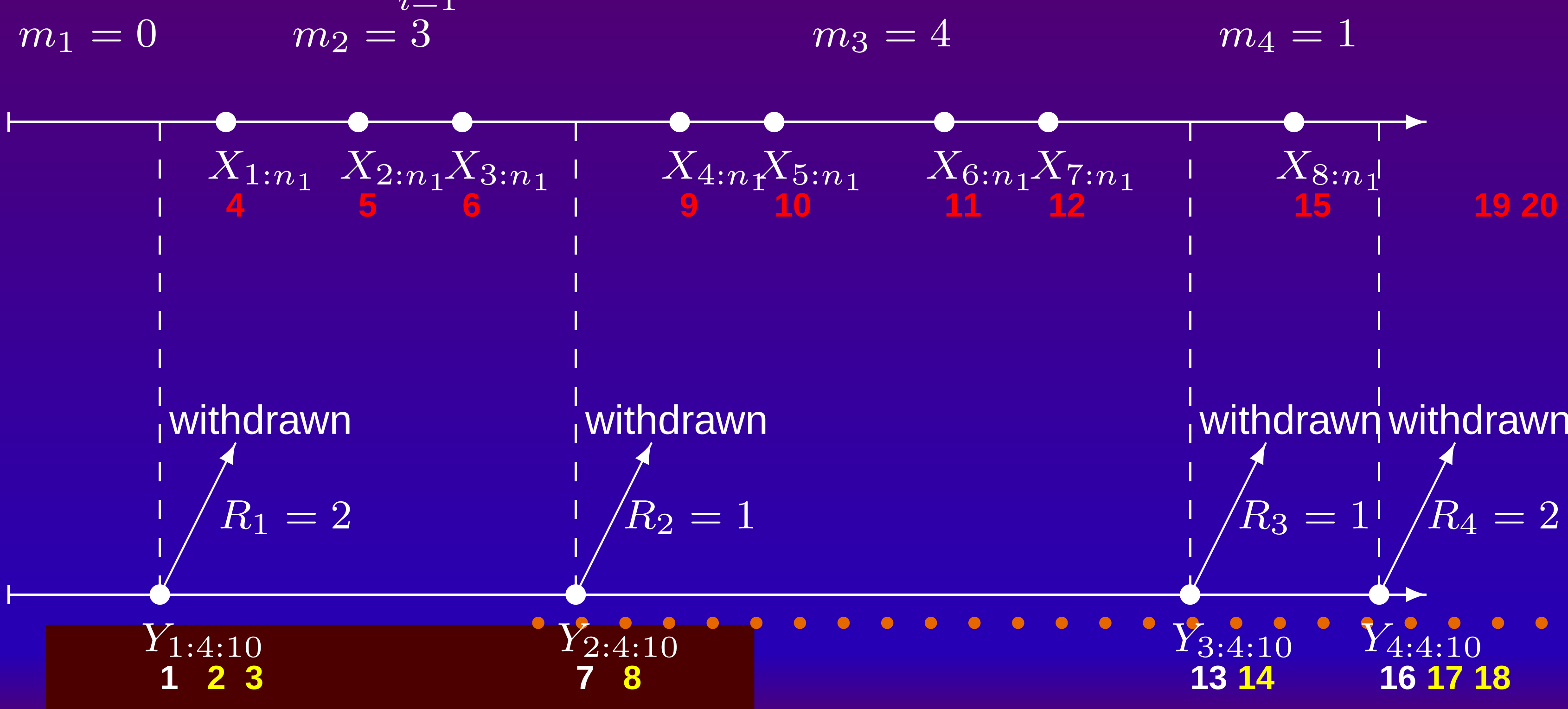


Maximal Rank-sum Statistic for Progressive Censoring

- We can show that the maximal Wilcoxon rank-sum precedence test statistic is equivalent to the weighted precedence test statistic in the case of progressive censoring as follows:

$$
\begin{aligned}
W^*_{\max,r} &= \sum_{i=1}^{r+1} \left\{ m_i \left[\sum_{j=1}^{i-1} m_j + \sum_{j=1}^{i-1} R_j + (i-1) \right] + \frac{m_i(m_i+1)}{2} \right\} \\
&= \frac{n_1(n_1+2n_2+1)}{2} + \sum_{i=1}^{r} \left(m_i \sum_{j=1}^{i-1} R_j \right) - (n_2 - i + 1) \sum_{i=1}^{r} m_i \\
&= \frac{n_1(n_1+2n_2+1)}{2} - P^*_{(r)}.
\end{aligned}
$$

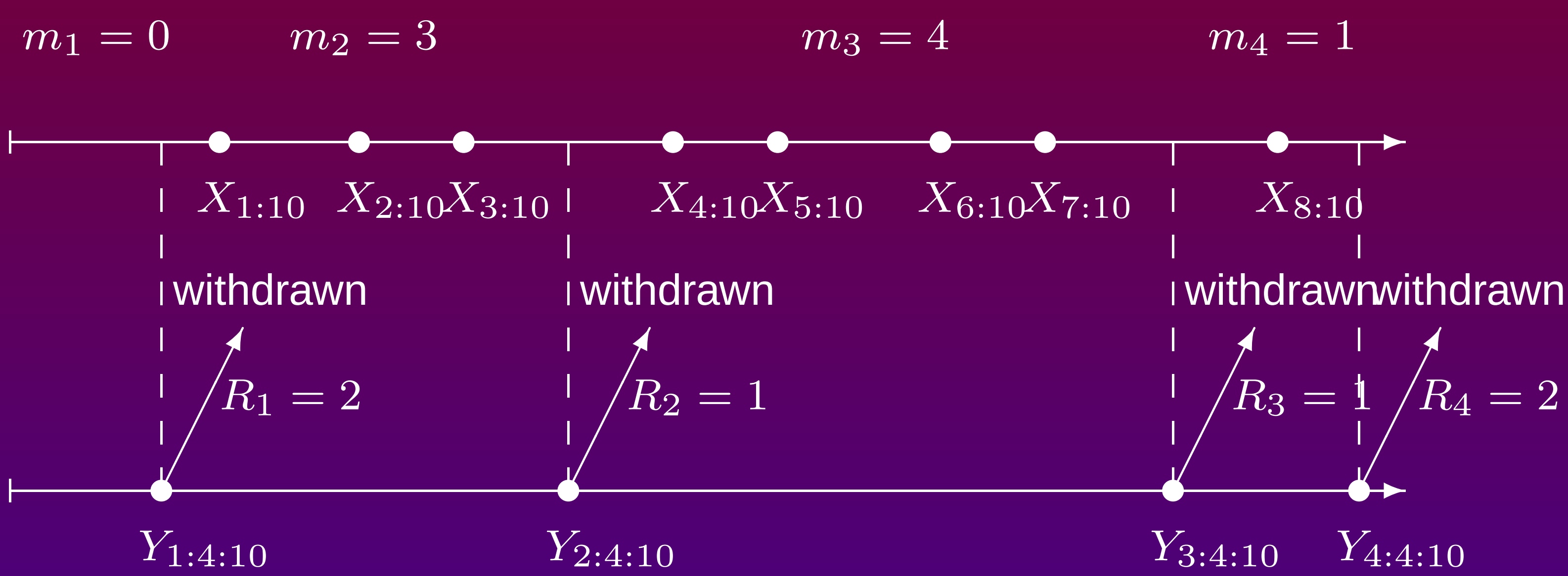
Maximal Rank-sum Statistic for Progressive Censoring

- We can show that the maximal Wilcoxon rank-sum precedence test statistic is equivalent to the weighted precedence test statistic in the case of progressive censoring as follows:

$$
\begin{aligned}
W^*_{\max,r} &= \sum_{i=1}^{r+1} \left\{ m_i \left[\sum_{j=1}^{i-1} m_j + \sum_{j=1}^{i-1} R_j + (i-1) \right] + \frac{m_i(m_i+1)}{2} \right\} \\
&= \frac{n_1(n_1+2n_2+1)}{2} + \sum_{i=1}^{r} \left(m_i \sum_{j=1}^{i-1} R_j \right) - (n_2 - i + 1) \sum_{i=1}^{r} m_i \\
&= \frac{n_1(n_1+2n_2+1)}{2} - P^*_{(r)}.
\end{aligned}
$$

- Since $\{n_1(n_1+2n_2+1)\}/2$ is just a constant, the above relationship readily reveals that the tests based on $P^*_{(r)}$ and $W^*_{\max,r}$ are equivalent, with large values of the former corresponding to small values of the latter.

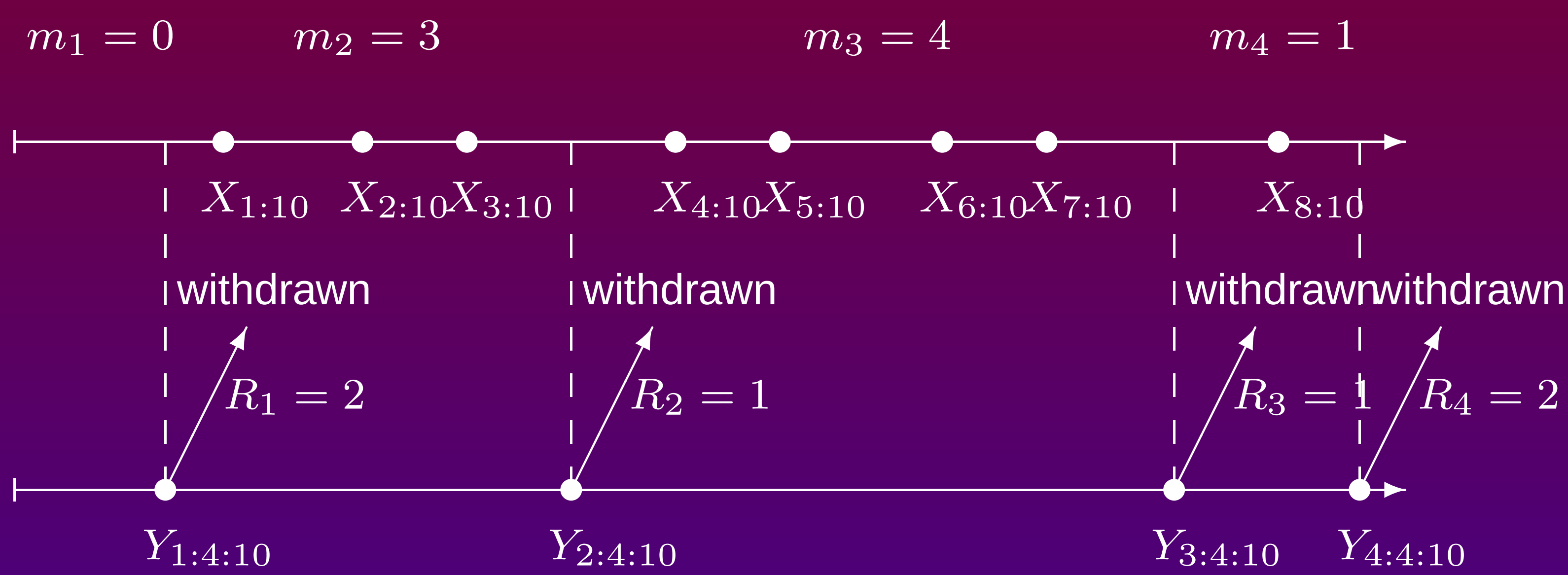
Example



$n_1 = n_2 = 10$, $r = 4$ and the progressive censoring scheme $(2, 1, 1, 2)$

- Weighted precedence test statistic: $P^*_{(4)}$ = $(10 \times 0) + (7 \times 3) + (5 \times 4) + (3 \times 1)$ = 44.

Example



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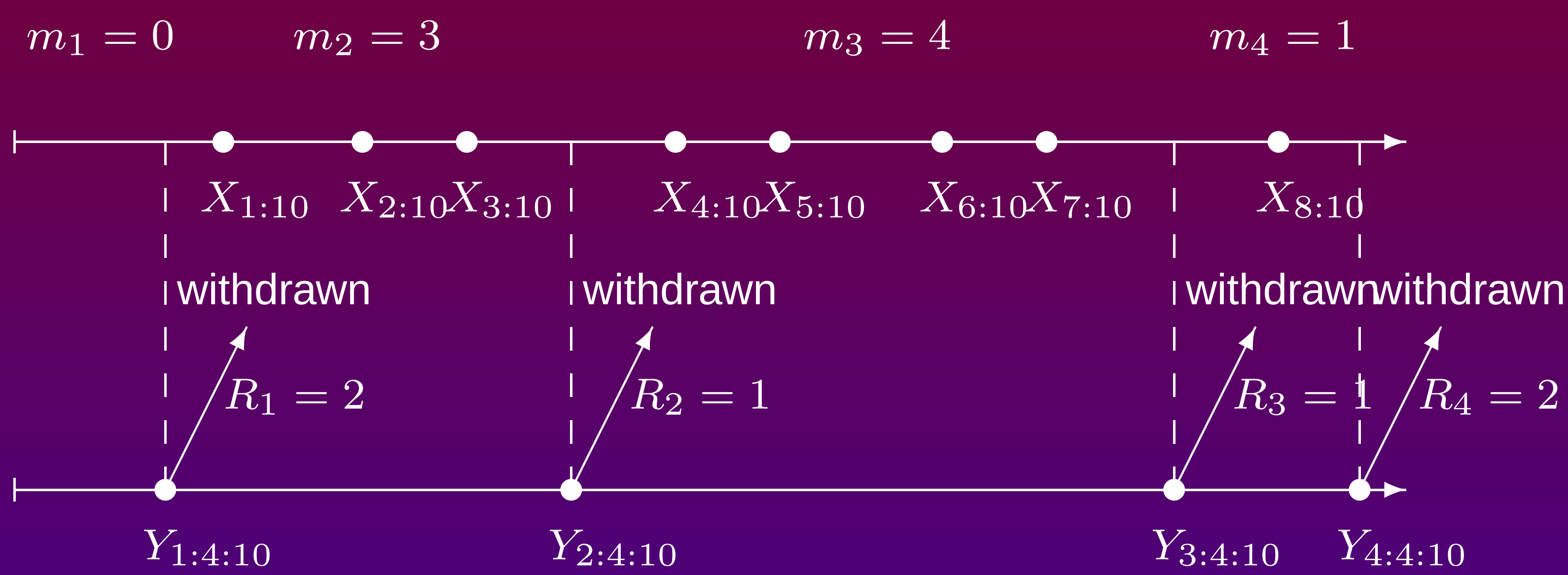

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- It is clear that, in general, large values of $P^*_{(r)}$ or $M^*_{(r)}$ would lead to the rejection of H_0 and in favor of H_1.

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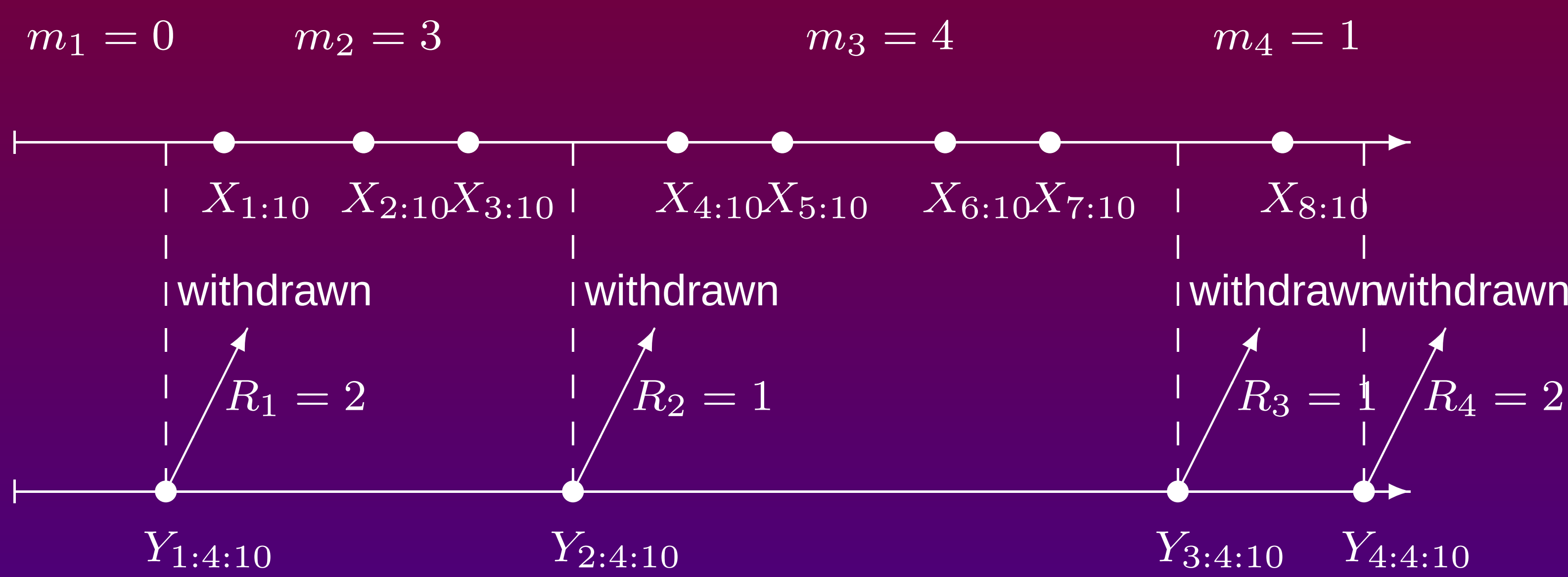

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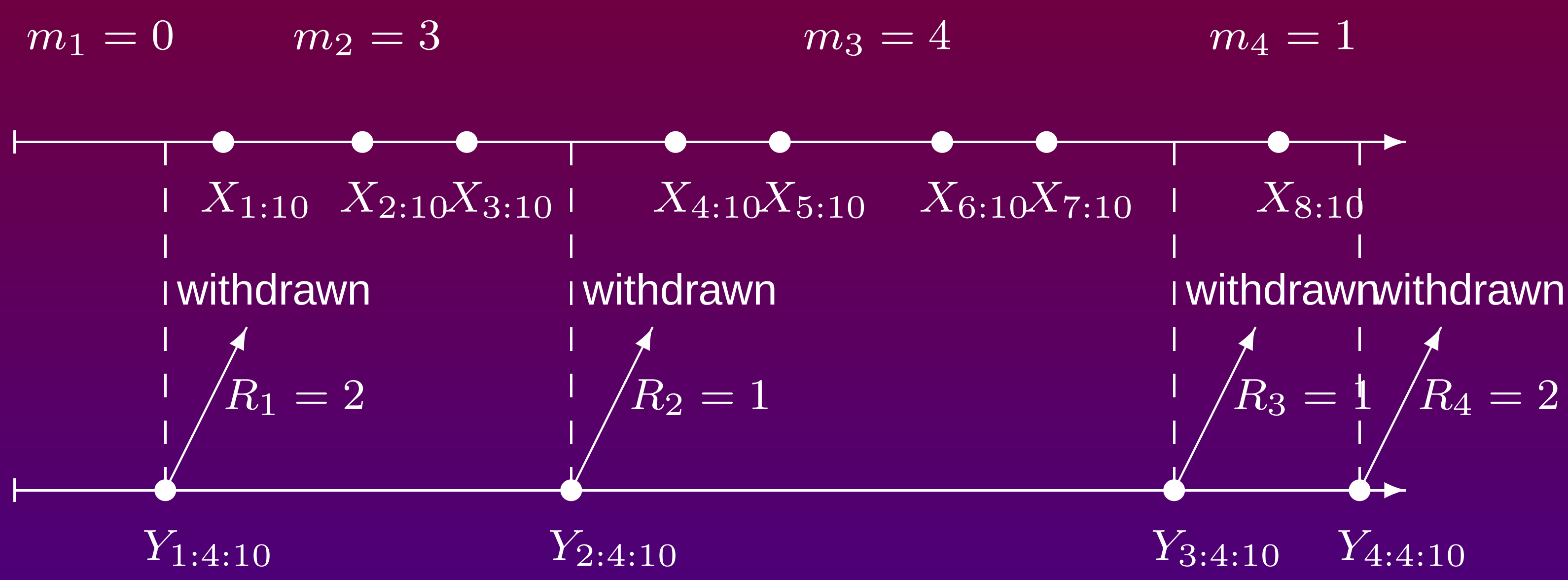

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Weighted Precedence & Maximal Precedence

- With a progressive Type-II censoring on the Y-sample, under the null hypothesis $H_0 : F_X = F_Y$, the joint probability mass function of M_1, M_2, $\cdots$, M_r is given by

$$\Pr(M_1 = m_1, M_2 = m_2, \cdots, M_r = m_r \mid H_0 : F_X = F_Y)$$
$$= C \sum_{j_k=0 \ (k=1,2,\cdots,r-1)}^{\sum_{i=1}^{k} R_i - \sum_{i=1}^{k-1} j_i} \left\{ \prod_{l=1}^{r-1} \binom{\sum_{i=1}^{l} R_i - \sum_{i=1}^{l} j_i}{j_l} \Gamma(m_l + j_{l+1} + 1) \right\}$$
$$\times \frac{\Gamma\left(n_1 + n_2 - r - \sum_{i=1}^{r} m_i - \sum_{i=1}^{r-1} j_i + 1\right)}{\Gamma(n_1 + n_2 + 1)},$$

where

$$C = \frac{n_1! n_2 (n_2 - R_1 - 1) \cdots (n_2 - R_1 - R_2 - \cdots - R_{r-1} - r + 1)}{m_1! m_2! \cdots m_r! \left(n_1 - \sum_{i=1}^{r} m_i\right)!}.$$

PRECEDENCE-TYPE TEST BASED ON KAPLAN-MEIER ESTIMATOR OF CDF

Kaplan-Meier Estimator

- The precedence-type test proposed here is based on the Kaplan–Meier nonparametric estimator [Kaplan and Meier (1958)] of the CDF based on data in which observations reported are exact failure times.

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- The Kaplan–Meier estimator (also called the *product-limit estimator*) of $F_Y(y_{i:r:n_2})$ is given by

$$\hat{F}_Y(y_{i:r:n_2}) \quad = \quad 1 - \prod_{j=1}^{i} \left(1 - \frac{1}{n_j^*}\right), \quad i = 1, \cdots, r,$$

where $n_i^* = n_2 - i + 1 - \sum_{j=0}^{i-1} R_j$ is the risk set at $y_{i:r:n_2}$.

Kaplan-Meier Estimator

- In the case of conventional Type-II right censoring, i.e., $R_1 = \cdots = R_{r-1} = 0$ and $R_r = n_2 - r$, the above estimate of the CDF at $y_{i:n_2}$ simply reduces to

$$\hat{F}_Y(y_{i:n_2}) = \frac{i}{n_2}, \quad i = 1, \cdots, r.$$

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- Similarly, the Kaplan–Meier estimator of the CDF for the X-sample at $x_{i:n_1}$ is given by

$$\hat{F}_X(x_{i:n_1}) = \frac{i}{n_1}, \quad i = 1, \cdots, n_1.$$

Type-II Censoring on Y-sample

Test Statistic

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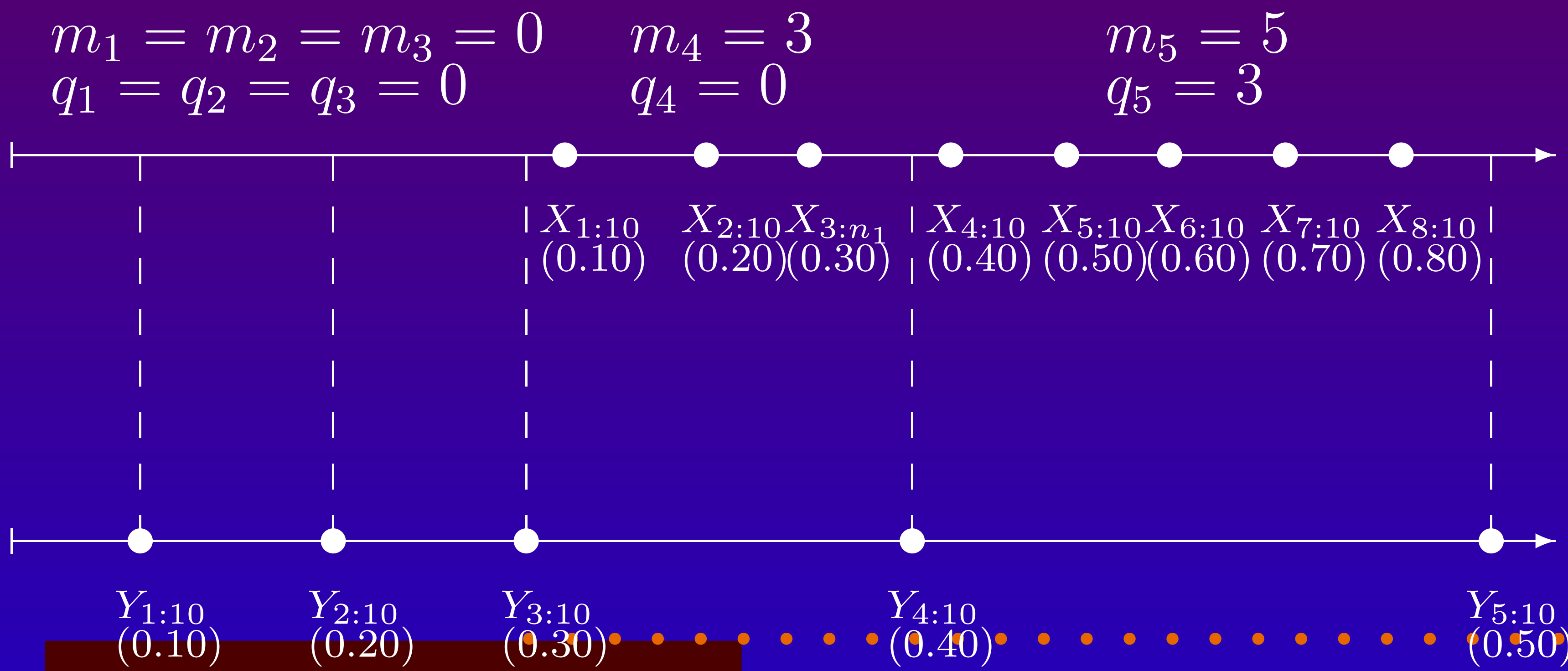

Test Statistics

- If the information after the termination of the experiment at $Y_{r:n_2}$ is not taken into account, we consider the statistic

$$
\begin{aligned}
Q_{(r)}(\mathbf{M}) &= \sum_{i=1}^{r} Q_i \\
&= \max(0, M_1 - 1) + \sum_{i=2}^{r} \sum_{j=S_{i-1}+1}^{S_i} I(j > i),
\end{aligned}
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where $S_i = S_i(\mathbf{M}) = \sum_{k=1}^{i} M_k$.

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- It is evident that large values of $Q_{(r)}$ lead to the rejection of H_0 and in favor of H_1.

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- It is evident that large values of $Q^*_{(r)}$ lead to the rejection of H_0 and in favor of H_1.

Test Statistics

- Then, the test statistic we propose is the average of the two statistics given by

$$\bar{Q}_{(r)}(\mathbf{M}) = \max(0, M_1 - 1) + \sum_{i=2}^{r} \sum_{j=S_{i-1}+1}^{S_i} I(j > i) + \frac{1}{2} \sum_{j=S_r+1}^{n_1} I(j > r + 1),$$

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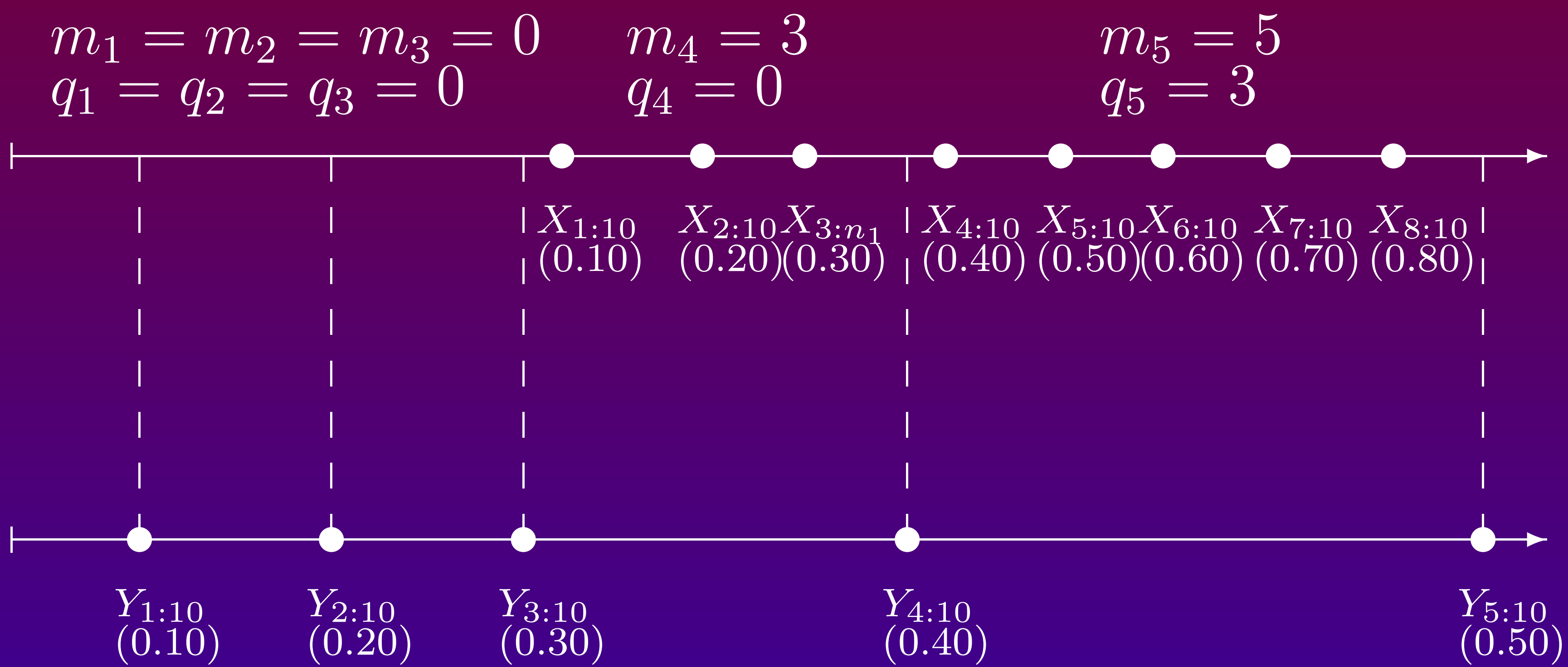
- Large values of $\bar{Q}_{(r)}$ leading to the rejection of H_0 and in favor of H_1.

Example



With $r = 5$, we observe $q_1 = q_2 = q_3 = q_4 = 0$, $q_5 = 3$ with which we obtain $Q_{(5)} = 3$, $Q^*_{(5)} = 5$ and the proposed test statistic $\bar{Q}_{(5)}$ = 4.

Null Distribution

- The null distribution of the test statistic $\bar{Q}_{(r)}$ can be obtained as

$$\Pr(\bar{Q}_{(r)} = q | H_0 : F_X = F_Y) = \sum_{m_i (i=1,2,\ldots,r)} \Pr(\mathbf{M} = \mathbf{m} | F_X = F_Y)\, I(\bar{Q}_{(r)}(\mathbf{m}) = q),$$

where $I(\cdot)$ denotes the indicator function.

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- The critical values s for specified values of n_1, n_2, r and α can be computed from the above formula.

Exact Power under Lehmann Alternative

- Consider the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$ for some $\gamma > 1$ which is a subclass of the alternative $H_1 : F_X > F_Y$ when $\gamma > 1$, and that it is appropriate when some common lifetime distributions are used for modelling in industrial setting.

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- The joint probability mass function of $(M_1, \cdots, M_r)$ under the Lehmann alternative is given by [Balakrishnan and Ng (2001)]

$$
\begin{aligned}
&\Pr(M_1 = m_1, \cdots, M_r = m_r | H_1 : [F_X]^\gamma = F_Y) \\
= \quad & \frac{n_1! n_2! \gamma^r}{m_1!(n_2 - r)!} \left\{ \prod_{j=1}^{r-1} \frac{\Gamma\left(\sum_{i=1}^{j} m_i + j\gamma\right)}{\Gamma\left(\sum_{i=1}^{j+1} m_i + j\gamma + 1\right)} \right\} \\
& \times \sum_{k=0}^{n_2 - r} \binom{n_2 - r}{k} (-1)^k \frac{\Gamma\left(\sum_{i=1}^{r} m_i + (r+k)\gamma\right)}{\Gamma\left(n_1 + (r+k)\gamma + 1\right)}.
\end{aligned}
$$

Exact Power under Lehmann Alternative

- The power function of the proposed precedence-type test under the Lehmann alternative is then given by

$$\begin{aligned}\text{Power} &= \Pr(\bar{Q}_{(r)} \geq s | H_1 : [F_X]^\gamma = F_Y) \\ &= \sum_{m_i (i=1,2,\ldots,r)} \Pr(\mathbf{M} = \mathbf{m} | H_1 : [F_X]^\gamma = F_Y)\, I(\bar{Q}_{(r)}(\mathbf{m}) \geq s),\end{aligned}$$

where s is the chosen critical value.

Power comparison under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3, 4$

Test	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$
$r = 2$						
$P_{(r)}$	0.0286	0.2397	0.4905	0.6730	0.7896	0.8622
$M_{(r)}$	0.0325	0.2210	0.4534	0.6363	0.7606	0.8412
$P^*_{(r)}$	0.0495	0.3568	0.6439	0.8097	0.8962	0.9413
$M^*_{(r)}$	0.0595	0.3573	0.6298	0.7944	0.8845	0.9334
$W_{\min,r}$	0.0595	0.4040	0.6983	0.8532	0.9269	0.9621
$W_{\max,r}$	0.0495	0.3568	0.6439	0.8097	0.8962	0.9413
$W_{E,r}$	0.0495	0.3568	0.6439	0.8097	0.8962	0.9413
$\bar{Q}_{(r)}$	0.0511	0.3673	0.6549	0.8173	0.9008	0.9439
$r = 3$						
$P_{(r)}$	0.0349	0.2458	0.4809	0.6521	0.7646	0.8373
$M_{(r)}$	0.0488	0.2418	0.4680	0.6456	0.7664	0.8449
$P^*_{(r)}$	0.0510	0.3510	0.6314	0.7960	0.8840	0.9314
$M^*_{(r)}$	0.0379	0.2281	0.4582	0.6391	0.7623	0.8422
$W_{\min,r}$	0.0464	0.3643	0.6633	0.8295	0.9119	0.9526
$W_{\max,r}$	0.0510	0.3510	0.6314	0.7960	0.8840	0.9314
$W_{E,r}$	0.0471	0.3462	0.6292	0.7951	0.8837	0.9312
$\bar{Q}_{(r)}$	0.0483	0.3590	0.6438	0.8056	0.8903	0.9352

Exact Power under Lehmann Alternative

Power comparison under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3, 4$

Test	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$
$r = 4$						
$P_{(r)}$	0.0349	0.2168	0.4200	0.5771	0.6883	0.7661
$M_{(r)}$	0.0650	0.2541	0.4746	0.6491	0.7684	0.8460
$P^*_{(r)}$	0.0319	0.2585	0.5159	0.6956	0.8067	0.8742
$M^*_{(r)}$	0.0433	0.2319	0.4600	0.6400	0.7627	0.8424
$W_{\min,r}$	0.0538	0.3949	0.6924	0.8488	0.9234	0.9593
$W_{\max,r}$	0.0507	0.3356	0.6005	0.7615	0.8527	0.9052
$W_{E,r}$	0.0487	0.3482	0.6278	0.7922	0.8808	0.9289
$\bar{Q}_{(r)}$	0.0410	0.3224	0.5958	0.7611	0.8535	0.9061

Progressive Type-II Censoring on Y-sample

Test Statistics

- Let M_1 denote the number of X-failures before $Y_{1:r:n_2}$, M_i the number of X-failures between $Y_{i-1:r:n_2}$ and $Y_{i:r:n_2}$ for $i = 2, \cdots, r$, and Q_i the number of X-failures among the M_i that are between $Y_{i-1:r:n_2}$ and $Y_{i:r:n_2}$ for which $\hat{F}_X(x) > \hat{F}_Y(y_{i:r:n_2})$ for $i = 1, \cdots, r$.

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- If the information after the termination of the experiment at $Y_{r:r:n_2}$ is not taken into account, we consider the statistic

$$\begin{aligned} Q_{(r)}(\mathbf{M}) &= \sum_{i=1}^{r} Q_i \\ &= \max(0, M_1 - 1) + \sum_{i=2}^{r} \sum_{j=S_{i-1}+1}^{S_i} I[j > n_1 \hat{F}_Y(y_{i:r:n_2})], \end{aligned}$$

where $S_i = S_i(\mathbf{M}) = \sum_{k=1}^{i} M_k$.

Test Statistics

- Once again, by assuming that all the remaining X-units will fail before the $(r+1)$-th unobserved Y-failure, we obtain the statistic

$$Q^*_{(r)}(\mathbf{M}) = \max(0, M_1 - 1) + \sum_{i=2}^{r+1} \sum_{j=S_{i-1}+1}^{S_i} I[j > n_1 \hat{F}_Y(y_{i:r:n_2})],$$

where $M_{r+1} = n_1 - S_r$, $S_{(r+1)} = n_1$, and $y_{r+1:r:n_2}$ (with progressive censoring scheme $(R_1, \cdots, R_{r-1}, R_r)$) is taken as the $(r+1)$-th progressively Type-II censored order statistic $y_{r+1:r+1:n_2}$ with progressive censoring scheme $\left(R_1, \cdots, R_{r-1}, 0, n_2 - r - 1 - \sum_{i=1}^{r-1} R_i\right)$.

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$$\begin{aligned}\bar{Q}_{(r)}(\mathbf{M}) &= \max(0, M_1 - 1) + \sum_{i=2}^{r} \sum_{j=S_{i-1}+1}^{S_i} I[j > n_1 \hat{F}_Y(y_{i:r:n_2})] \\ &\quad + \frac{1}{2} \sum_{j=S_r+1}^{n_1} I[j > n_1 \hat{F}_Y(y_{r+1:r:n_2})]\end{aligned}$$

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- Once again, by assuming that all the remaining X-units will fail before the $(r+1)$-th unobserved Y-failure, we obtain the statistic

$$Q^*_{(r)}(\mathbf{M}) = \max(0, M_1 - 1) + \sum_{i=2}^{r+1} \sum_{j=S_{i-1}+1}^{S_i} I[j > n_1 \hat{F}_Y(y_{i:r:n_2})],$$

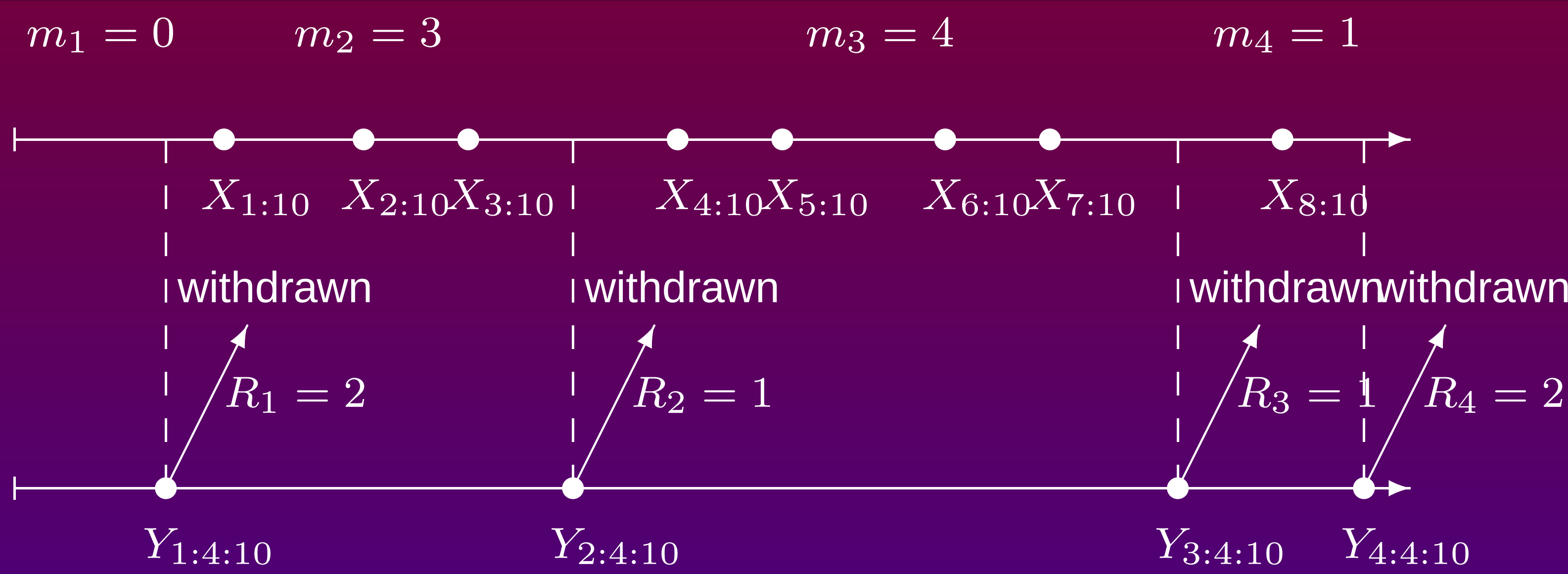
where $M_{r+1} = n_1 - S_r$, $S_{(r+1)} = n_1$, and $y_{r+1:r:n_2}$ (with progressive censoring scheme $(R_1, \cdots, R_{r-1}, R_r)$) is taken as the $(r+1)$-th progressively Type-II censored order statistic $y_{r+1:r+1:n_2}$ with progressive censoring scheme $\left(R_1, \cdots, R_{r-1}, 0, n_2 - r - 1 - \sum_{i=1}^{r-1} R_i\right)$.

- Then, the test statistic we propose is the average of the two statistics given by

$$\begin{aligned}\bar{Q}_{(r)}(\mathbf{M}) &= \max(0, M_1 - 1) + \sum_{i=2}^{r} \sum_{j=S_{i-1}+1}^{S_i} I[j > n_1 \hat{F}_Y(y_{i:r:n_2})] \\ &\quad + \frac{1}{2} \sum_{j=S_r+1}^{n_1} I[j > n_1 \hat{F}_Y(y_{r+1:r:n_2})]\end{aligned}$$

- Large values of $\bar{Q}_{(r)}$ leading to the rejection of H_0 and in favor of H_1.

Example



With $n_1 = n_2 = 10$ and $r = 4$, the Kaplan-Meier estimates of the CDF based on the progressively Type-II censored Y-sample are presented in Table 2, from which we observe $q_1 = 0, q_2 = 1, q_3 = 3, q_4 = 1$ with which we obtain $Q_{(4)} = 5$, $Q^*_{(4)} = 7$ and the proposed test statistic $\bar{Q}_{(4)}$ = 6.

t_i	n_i^*	$1/n_i^*$	$1-(1/n_i^*)$	$\hat{S}_Y(t_i)$	$\hat{F}_Y(t_i)$
$Y_{1:4:10}$	10	0.100	0.900	0.900	0.100
$Y_{2:4:10}$	5	0.200	0.800	0.720	0.280
$Y_{3:4:10}$	4	0.250	0.750	0.540	0.460
$Y_{4:4:10}$	3	0.333	0.667	0.360	0.640
$Y_{5:4:10}$	2	0.500	0.500	0.180	0.820

Exact Power under Lehmann Alternative

- Consider the Lehmann alternative $H_1 : [F_X]^\gamma = F_Y$ for some $\gamma > 1$.

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$$
\begin{aligned}
& \Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y) \\
= \quad & C\gamma^r \sum_{j_i(i=1,2,\ldots,r)=0}^{R_i} \begin{pmatrix} R_1 \\ j_1 \end{pmatrix} \begin{pmatrix} R_2 \\ j_2 \end{pmatrix} \cdots \begin{pmatrix} R_r \\ j_r \end{pmatrix} (-1)^{\left(\sum_{i=1}^{r} j_i\right)} \\
& \times \left\{ \prod_{k=1}^{r-1} B\left(\sum_{i=1}^{k} m_i + \gamma \left(\sum_{i=1}^{k} j_i \right) + k\gamma, m_{k+1} + 1 \right) \right\} \\
& \times B\left(\sum_{i=1}^{r} m_i + \gamma \left(\sum_{i=1}^{r} j_i \right) + r\gamma, n_1 - \sum_{i=1}^{r} m_i + 1 \right).
\end{aligned}
$$

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\begin{aligned}
&\Pr(M_1 = m_1, M_2 = m_2, \ldots, M_r = m_r \mid [F_X]^\gamma = F_Y) \\
&= \quad C\gamma^r \sum_{j_i(i=1,2,\ldots,r)=0}^{R_i} \binom{R_1}{j_1}\binom{R_2}{j_2}\cdots\binom{R_r}{j_r}(-1)^{\left(\sum_{i=1}^{r} j_i\right)} \\
&\qquad \times \left\{ \prod_{k=1}^{r-1} B\left(\sum_{i=1}^{k} m_i + \gamma\left(\sum_{i=1}^{k} j_i\right) + k\gamma, m_{k+1} + 1 \right) \right\} \\
&\qquad \times B\left(\sum_{i=1}^{r} m_i + \gamma\left(\sum_{i=1}^{r} j_i\right) + r\gamma, n_1 - \sum_{i=1}^{r} m_i + 1 \right).
\end{aligned}
$$

- For $n_1 = n_2 = 10, r = 2$ and $\gamma = 2(1)6$, the power values computed from the exact expressions are presented here.

Exact Power under Lehmann Alternative

Power comparison under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3$

PCS	Test	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$
$r = 2$							
(0,8)	$P^*_{(r)}$	0.0495	0.3568	0.6439	0.8097	0.8962	0.9413
	$M^*_{(r)}$	0.0595	0.3573	0.6298	0.7944	0.8845	0.9334
	$\bar{Q}_{(r)}$	0.0511	0.3673	0.6549	0.8173	0.9008	0.9439
(8,0)	$P^*_{(r)}$	0.0511	0.3471	0.6256	0.7931	0.8842	0.9334
	$M^*_{(r)}$	0.0433	0.3147	0.5885	0.7628	0.8622	0.9180
	$\bar{Q}_{(r)}$	0.0433	0.3147	0.5885	0.7628	0.8622	0.9180
(6,2)	$P^*_{(r)}$	0.0478	0.3501	0.6383	0.8073	0.8960	0.9420
	$M^*_{(r)}$	0.0433	0.3147	0.5885	0.7628	0.8622	0.9180
	$\bar{Q}_{(r)}$	0.0447	0.3158	0.5877	0.7594	0.8571	0.9124
(4,4)	$P^*_{(r)}$	0.0504	0.3611	0.6501	0.8157	0.9010	0.9449
	$M^*_{(r)}$	0.0477	0.3229	0.5950	0.7669	0.8647	0.9196
	$\bar{Q}_{(r)}$	0.0472	0.3158	0.5779	0.7451	0.8429	0.9002
(2,6)	$P^*_{(r)}$	0.0521	0.3583	0.6418	0.8069	0.8940	0.9398
	$M^*_{(r)}$	0.0555	0.3437	0.6155	0.7830	0.8763	0.9277
	$\bar{Q}_{(r)}$	0.0651	0.4069	0.6885	0.8391	0.9141	0.9520

Exact Power under Lehmann Alternative

Power comparison under Lehmann alternative for $n_1 = n_2 = 10, r = 2, 3$

PCS	Test	$\gamma = 1$	$\gamma = 2$	$\gamma = 3$	$\gamma = 4$	$\gamma = 5$	$\gamma = 6$
$r = 3$							
(0,0,7)	$P^*_{(r)}$	0.0488	0.2418	0.4680	0.6456	0.7664	0.8449
	$M^*_{(r)}$	0.0510	0.3510	0.6314	0.7960	0.8840	0.9314
	$\bar{Q}_{(r)}$	0.0483	0.3590	0.6438	0.8056	0.8903	0.9352
(7,0,0)	$P^*_{(r)}$	0.0429	0.3228	0.6040	0.7786	0.8752	0.9279
	$M^*_{(r)}$	0.0433	0.3147	0.5885	0.7628	0.8622	0.9180
	$\bar{Q}_{(r)}$	0.0543	0.3149	0.5783	0.7497	0.8498	0.9075
(5,1,1)	$P^*_{(r)}$	0.0491	0.3640	0.6577	0.8239	0.9081	0.9503
	$M^*_{(r)}$	0.0529	0.3288	0.5986	0.7689	0.8658	0.9202
	$\bar{Q}_{(r)}$	0.0383	0.2618	0.4994	0.6638	0.7681	0.8347
(2,3,3)	$P^*_{(r)}$	0.0496	0.3617	0.6507	0.8147	0.8990	0.9426
	$M^*_{(r)}$	0.0557	0.3438	0.6155	0.7830	0.8763	0.9277
	$\bar{Q}_{(r)}$	0.0417	0.3076	0.5702	0.7359	0.8326	0.8898
(0,2,5)	$P^*_{(r)}$	0.0490	0.3487	0.6294	0.7947	0.8834	0.9313
	$M^*_{(r)}$	0.0370	0.2256	0.4560	0.6377	0.7613	0.8416
	$\bar{Q}_{(r)}$	0.0397	0.2747	0.5175	0.6843	0.7899	0.8565

Monte Carlo Power Comparison

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  4. Gamma distribution with shape parameter a and standardized by mean a and standard deviation $\sqrt{a}$;
  5. Standard extreme-value distribution standardized by mean -0.5772156649 (Euler's constant) and standard deviation $\pi/\sqrt{6}$.

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  - Weighted precedence test
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Power comparison under progressive censoring for $n_1 = n_2 = 10$ with θ = 1.0

r	PCS	Test	l.o.s.	Exp(1)	N(0,1)	G(10)	LN(0.1)	EV
3	(0,0,7)	$P^*_{(r)}$	0.051	0.887	0.574	0.832	0.798	0.577
		$M^*_{(r)}$	0.038	0.920	0.395	0.735	0.734	0.401
		$\bar{Q}_{(r)}$	0.048	0.888	0.565	0.840	0.807	0.534
	(7,0,0)	$P^*_{(r)}$	0.048	0.974	0.482	0.880	0.891	0.389
		$M^*_{(r)}$	0.043	0.977	0.423	0.862	0.878	0.322
		$\bar{Q}_{(r)}$	0.054	0.901	0.512	0.804	0.794	0.521
	(2,3,3)	$P^*_{(r)}$	0.050	0.906	0.585	0.849	0.832	0.567
		$M^*_{(r)}$	0.056	0.977	0.493	0.868	0.879	0.442
		$\bar{Q}_{(r)}$	0.042	0.785	0.534	0.750	0.716	0.533
4	(0,0,0,6)	$P^*_{(r)}$	0.051	0.801	0.602	0.774	0.732	0.656
		$M^*_{(r)}$	0.043	0.920	0.402	0.735	0.734	0.422
		$\bar{Q}_{(r)}$	0.041	0.803	0.559	0.778	0.738	0.559
	(6,0,0,0)	$P^*_{(r)}$	0.052	0.972	0.505	0.902	0.900	0.408
		$M^*_{(r)}$	0.043	0.977	0.418	0.873	0.879	0.317
		$\bar{Q}_{(r)}$	0.051	0.815	0.535	0.771	0.736	0.561
	(1,1,2,2)	$P^*_{(r)}$	0.052	0.929	0.607	0.875	0.848	0.579
		$M^*_{(r)}$	0.056	0.977	0.491	0.878	0.881	0.446
		$\bar{Q}_{(r)}$	0.054	0.732	0.604	0.735	0.688	0.640

Power comparison under progressive censoring for $n_1 = n_2 = 10$ with θ = 1.0

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		$\bar{Q}_{(r)}$	0.054	0.901	0.512	0.804	0.794	0.521
	(2,3,3)	$P^*_{(r)}$	0.050	0.906	0.585	0.849	0.832	0.567
		$M^*_{(r)}$	0.056	0.977	0.493	0.868	0.879	0.442
		$\bar{Q}_{(r)}$	0.042	0.785	0.534	0.750	0.716	0.533
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		$M^*_{(r)}$	0.043	0.977	0.418	0.873	0.879	0.317
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	(1,1,2,2)	$P^*_{(r)}$	0.052	0.929	0.607	0.875	0.848	0.579
		$M^*_{(r)}$	0.056	0.977	0.491	0.878	0.881	0.446
		$\bar{Q}_{(r)}$	0.054	0.732	0.604	0.735	0.688	0.640

Power comparison under progressive censoring for $n_1 = n_2 = 10$ with θ = 1.0

r	PCS	Test	l.o.s.	Exp(1)	N(0,1)	G(10)	LN(0.1)	EV
5	(0,0,0,0,5)	$P^*_{(r)}$	0.051	0.823	0.625	0.783	0.741	0.711
		$M^*_{(r)}$	0.045	0.920	0.403	0.735	0.734	0.425
		$\bar{Q}_{(r)}$	0.041	0.863	0.659	0.850	0.821	0.655
	(5,0,0,0,0)	$P^*_{(r)}$	0.052	0.966	0.544	0.902	0.896	0.458
		$M^*_{(r)}$	0.053	0.976	0.463	0.874	0.883	0.417
		$\bar{Q}_{(r)}$	0.065	0.785	0.617	0.789	0.752	0.646
	(1,1,1,1,1)	$P^*_{(r)}$	0.049	0.899	0.629	0.843	0.812	0.6385
		$M^*_{(r)}$	0.070	0.976	0.536	0.885	0.888	0.5066
		$\bar{Q}_{(r)}$	0.039	0.579	0.546	0.609	0.573	0.6142
6	(0,0,0,0,0,4)	$P^*_{(r)}$	0.049	0.792	0.632	0.764	0.728	0.740
		$M^*_{(r)}$	0.045	0.920	0.403	0.735	0.734	0.425
		$\bar{Q}_{(r)}$	0.049	0.757	0.609	0.768	0.735	0.631
	(4,0,0,0,0,0)	$P^*_{(r)}$	0.052	0.959	0.575	0.897	0.894	0.513
		$M^*_{(r)}$	0.049	0.978	0.445	0.872	0.881	0.399
		$\bar{Q}_{(r)}$	0.057	0.639	0.592	0.655	0.626	0.671
	(0,1,1,1,1,0)	$P^*_{(r)}$	0.053	0.869	0.649	0.831	0.800	0.683
		$M^*_{(r)}$	0.041	0.915	0.396	0.738	0.745	0.419
		$\bar{Q}_{(r)}$	0.040	0.506	0.525	0.534	0.500	0.644

Power comparison under progressive censoring for $n_1 = n_2 = 20$ with θ = 1.0

r	PCS	Test	l.o.s.	Exp(1)	N(0,1)	G(10)	LN(0.1)	EV
3	(0,0,17)	$P^*_{(r)}$	0.050	0.999	0.738	0.991	0.354	0.596
		$M^*_{(r)}$	0.044	1.000	0.611	0.988	0.273	0.489
		$\bar{Q}_{(r)}$	0.042	0.999	0.713	0.991	0.332	0.552
	(17,0,0)	$P^*_{(r)}$	0.051	1.000	0.591	0.994	0.998	0.395
		$M^*_{(r)}$	0.053	1.000	0.560	0.993	0.998	0.340
		$\bar{Q}_{(r)}$	0.043	0.991	0.511	0.914	0.915	0.503
	(5,6,6)	$P^*_{(r)}$	0.051	1.000	0.740	0.994	0.994	0.613
		$M^*_{(r)}$	0.052	1.000	0.619	0.987	0.993	0.519
		$\bar{Q}_{(r)}$	0.046	0.994	0.704	0.971	0.964	0.601
4	(0,0,0,16)	$P^*_{(r)}$	0.050	0.998	0.778	0.989	0.361	0.683
		$M^*_{(r)}$	0.054	1.000	0.643	0.988	0.293	0.559
		$\bar{Q}_{(r)}$	0.044	0.998	0.763	0.989	0.356	0.634
	(16,0,0,0)	$P^*_{(r)}$	0.050	1.000	0.605	0.995	0.997	0.410
		$M^*_{(r)}$	0.053	1.000	0.551	0.994	0.997	0.339
		$\bar{Q}_{(r)}$	0.043	0.939	0.577	0.888	0.865	0.565
	(4,4,4,4)	$P^*_{(r)}$	0.046	0.999	0.770	0.993	0.991	0.670
		$M^*_{(r)}$	0.051	1.000	0.628	0.987	0.992	0.516
		$\bar{Q}_{(r)}$	0.053	0.990	0.776	0.976	0.966	0.691

Power comparison under progressive censoring for $n_1 = n_2 = 20$ with θ = 1.0

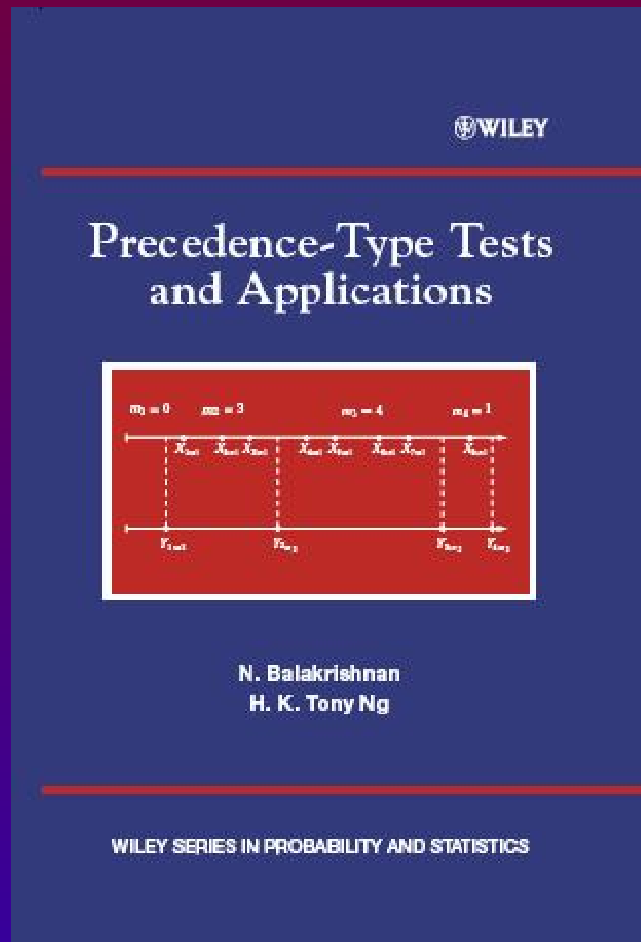
r	PCS	Test	l.o.s.	Exp(1)	N(0,1)	G(10)	LN(0.1)	EV
5	(0,0,0,0,15)	$P^*_{(r)}$	0.050	0.996	0.800	0.984	0.358	0.750
		$M^*_{(r)}$	0.044	1.000	0.573	0.966	0.225	0.541
		$\bar{Q}_{(r)}$	0.045	0.996	0.791	0.986	0.365	0.694
	(15,0,0,0,0)	$P^*_{(r)}$	0.050	1.000	0.641	0.996	0.999	0.455
		$M^*_{(r)}$	0.053	1.000	0.558	0.994	0.998	0.362
		$\bar{Q}_{(r)}$	0.044	0.883	0.629	0.859	0.819	0.630
	(3,3,3,3,3)	$P^*_{(r)}$	0.050	0.999	0.812	0.992	0.990	0.741
		$M^*_{(r)}$	0.050	1.000	0.629	0.986	0.993	0.536
		$\bar{Q}_{(r)}$	0.048	0.966	0.784	0.954	0.929	0.750
6	(0,0,0,0,0,14)	$P^*_{(r)}$	0.050	0.991	0.823	0.978	0.360	0.802
		$M^*_{(r)}$	0.049	1.000	0.580	0.966	0.231	0.564
		$\bar{Q}_{(r)}$	0.043	0.991	0.807	0.981	0.362	0.738
	(14,0,0,0,0,0)	$P^*_{(r)}$	0.051	1.000	0.655	0.996	0.997	0.487
		$M^*_{(r)}$	0.055	1.000	0.571	0.994	0.999	0.417
		$\bar{Q}_{(r)}$	0.049	0.860	0.683	0.856	0.818	0.689
	(2,2,3,3,2,2)	$P^*_{(r)}$	0.049	0.998	0.828	0.991	0.987	0.785
		$M^*_{(r)}$	0.050	1.000	0.629	0.988	0.994	0.541
		$\bar{Q}_{(r)}$	0.049	0.908	0.788	0.899	0.868	0.798

Conclusions

- From the simulation results, the statistic $\bar{Q}_{(r)}$ is in general more powerful than the precedence, maximal precedence, and the weighted maximal precedence test statistics.
- Moreover, it is also more powerful than the weighted precedence statistic in the case of symmetric or near symmetric distributions while it is slightly less powerful than the minimal Wilcoxon-type rank-sum precedence statistic.

Conclusions

- However, the minimal Wilcoxon-type rank-sum precedence statistic handles only conventional Type-II censoring case but not the progressively Type-II censoring case.
- Though the proposed statistic $\bar{Q}_{(r)}$ is slightly less powerful than the minimal Wilcoxon-type rank-sum precedence statistic, it has the advantage that it is applicable for conventional Type-II censoring as well as progressively Type-II censoring cases.



Balakrishnan, N. and Ng, H. K. T. (2006). *Precedence-type Tests and Applications.* Hoboken, NJ: John Wiley & Sons.

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