Step–Stress Tests and Some Exact Inferential Results

N. Balakrishnan

bala@mcmaster.ca

McMaster University

Hamilton, Ontario, Canada

In collaboration with

- **Debasis Kundu**, IIT, Kapur, India
- **H.K. Tony Ng**, Southern Methodist University, Dallas
- **Nandini Kannan**, University of Texas, San Antonio
- **Evans Gouno**, Université de Britagne Sud, France
- **Ananda Sen**, University of Michigan, Ann Arbor
- **Qihao Xie**, McMaster University, Hamilton, Canada
- **Dong-hoon Han**, McMaster University, Hamilton, Canada

Roadmap

1. Step-stress Problem
2. Models
3. Cumulative Exposure Model
4. Likelihood under Type-II Censoring and MLEs
5. Exact Conditional Distributions of MLEs
6. Confidence Intervals & Bootstrap Intervals
7. Simulation Results & Comments
8. Illustrative Example
9. Time-Constrained Step-Stress Test
10. Progressive Type-II Censoring and Results
11. Other Extensions and Generalizations
12. Further Problems of Interest
13. Bibliography

1. Step-stress Problem

- In industrial experiments, products that are tested are often extremely reliable with large mean times to failure under normal operating conditions. So, an experimenter may resort to *accelerated life-testing* (ALT) wherein the units are subjected to higher stress levels than normal. Examples include assessing the effects of temperature, voltage, load, vibration, etc. on the lifetime of a product.

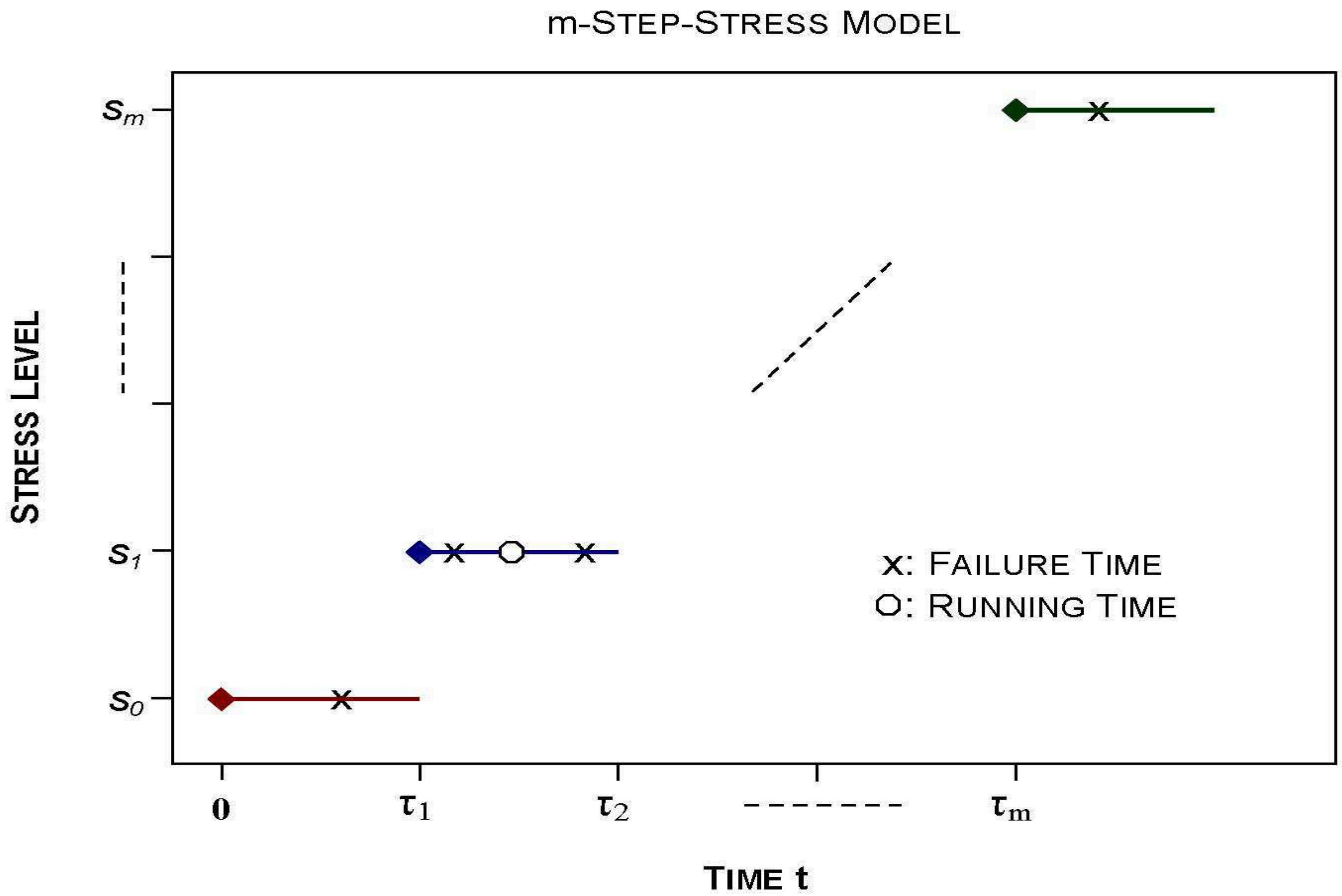
1. Step-stress Problem

- In industrial experiments, products that are tested are often extremely reliable with large mean times to failure under normal operating conditions. So, an experimenter may resort to *accelerated life-testing* (ALT) wherein the units are subjected to higher stress levels than normal. Examples include assessing the effects of temperature, voltage, load, vibration, etc. on the lifetime of a product.
- Step-stress testing is one such ALT.

1. Step-stress Problem

- In industrial experiments, products that are tested are often extremely reliable with large mean times to failure under normal operating conditions. So, an experimenter may resort to *accelerated life-testing* (ALT) wherein the units are subjected to higher stress levels than normal. Examples include assessing the effects of temperature, voltage, load, vibration, etc. on the lifetime of a product.
- Step-stress testing is one such ALT.
- Some key references on ALT are:
  - Nelson (1990)
  - Meeker and Escobar (1998)
  - Bagdonavicius and Nikulin (2002)

1. Step-stress Problem



2. Models

- There are three important models discussed in the literature:
  - *Tampered Random Variable Model*

    [DeGroot and Goel (1979)]
  - *Tampered Hazard Model*

    [Bhattacharyya and Zanzawi (1989)]
  - *Cumulative Exposure Model*

    [Nelson (1980)]

3. Cumulative Exposure Model

- The cumulative exposure model [Nelson (1980, 1990)] relates the life distribution of an unit at one stress level to the life distribution of that unit at the next stress level by assuming that the residual life of the unit depends only on the cumulative exposure that unit had experienced, with no memory of how this exposure was accumulated.

3. Cumulative Exposure Model

- In the case of a simple step-stress model, with the life distributions as $Exp(\theta_1)$ and $Exp(\theta_2)$ at stress levels s_0 and s_1, the cumulative exposure distribution (CED) of T (time-to-failure of unit) becomes

3. Cumulative Exposure Model

- In the case of a simple step-stress model, with the life distributions as $Exp(\theta_1)$ and $Exp(\theta_2)$ at stress levels s_0 and s_1, the cumulative exposure distribution (CED) of T (time-to-failure of unit) becomes

$$G(t) = \begin{cases} G_1(t) = F_1(t;\theta_1) & \text{if } 0 < t < \tau \\ G_2(t) = F_2\left(t - \left(1 - \frac{\theta_2}{\theta_1}\right)\tau;\theta_2\right) & \text{if } \tau \leq t < \infty, \end{cases}$$

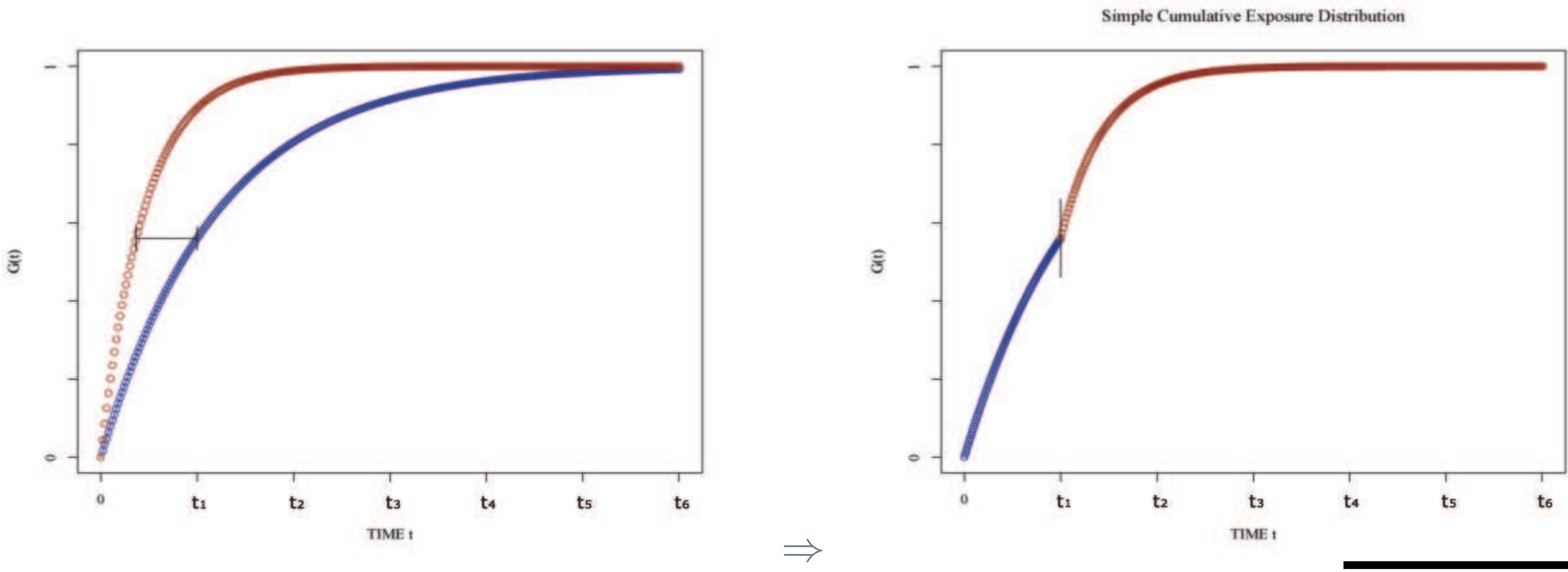
where

$$F_k(t;\theta_k) = 1 - e^{-t/\theta_k}, \quad t \geq 0, \ \ \theta_k > 0, \ \ k = 1, 2.$$

3. Cumulative Exposure Model

The corresponding PDF is

$$g(t) = \begin{cases} g_1(t) = \frac{1}{\theta_1} e^{-\frac{1}{\theta_1} t} & \text{if } 0 < t < \tau \\ g_2(t) = \frac{1}{\theta_2} e^{-\frac{1}{\theta_2}(t-\tau) - \frac{1}{\theta_1}\tau} & \text{if } \tau \le t < \infty. \end{cases}$$



4. Likelihood under Type-II Censoring & MLEs

- Suppose n units are placed under a step-stress test at an initial stress level of s_0, and at a pre-fixed time τ the stress level will be changed to s_1.

4. Likelihood under Type-II Censoring & MLEs

- Suppose n units are placed under a step-stress test at an initial stress level of s_0, and at a pre-fixed time τ the stress level will be changed to s_1.
- Under Type-II censoring, the experiment will terminate when a required number (say, r) of the n units fail. If r is taken as n, then a complete sample would be observed from the step-stress test.

4. Likelihood under Type-II Censoring & MLEs

- Suppose n units are placed under a step-stress test at an initial stress level of s_0, and at a pre-fixed time τ the stress level will be changed to s_1.
- Under Type-II censoring, the experiment will terminate when a required number (say, r) of the n units fail. If r is taken as n, then a complete sample would be observed from the step-stress test.
- Let n_1 denote the (random) number of failures that occur before τ.

4. Likelihood under Type-II Censoring & MLEs

- The likelihood of the observed Type-II censored data $t_{1:n} < \cdots < t_{r:n}$ is

4. Likelihood under Type-II Censoring & MLEs

- The likelihood of the observed Type-II censored data $t_{1:n} < \cdots < t_{r:n}$ is

$$L(\theta_1,\theta_2) = \begin{cases} \frac{c_1}{\theta_1^r} e^{-\frac{1}{\theta_1}\left[\sum_{i=1}^r t_{i:n}+(n-r)t_{r:n}\right]} \\ \qquad \text{if} \quad n_1 = r \\ \frac{c_1}{\theta_2^r} e^{-\frac{1}{\theta_2}\left[\sum_{i=1}^r \left(\frac{\theta_2}{\theta_1}\tau+t_{i:n}-\tau\right)+(n-r)\left(\frac{\theta_2}{\theta_1}\tau+t_{r:n}-\tau\right)\right]} \\ \qquad \text{if} \quad n_1 = 0 \\ \frac{c_2}{\theta_1^{n_1}\theta_2^{r-n_1}} e^{-\frac{1}{\theta_1}\left\{\sum_{i=1}^{n_1} t_{i:n}+(n-n_1)\tau\right\}} \\ \qquad \times\, e^{-\frac{1}{\theta_2}\left\{\sum_{i=n_1+1}^{r}(t_{i:n}-\tau)+(n-r)(t_{r:n}-\tau)\right\}} \\ \qquad \text{if} \quad 1 \le n_1 \le r-1. \end{cases}$$

4. Likelihood under Type-II Censoring & MLEs

- It is evident that the MLEs of θ_1 and θ_2 exist only when $1 \leq n_1 \leq r - 1$, and they are:

4. Likelihood under Type-II Censoring & MLEs

- It is evident that the MLEs of θ_1 and θ_2 exist only when $1 \leq n_1 \leq r-1$, and they are:

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n_1} t_{i:n} + (n-n_1)\tau}{n_1},$$

$$\hat{\theta}_2 = \frac{\sum_{i=n_1+1}^{r} (t_{i:n} - \tau) + (n-r)(t_{r:n} - \tau)}{r - n_1}.$$

4. Likelihood under Type-II Censoring & MLEs

- It is evident that the MLEs of θ_1 and θ_2 exist only when $1 \le n_1 \le r-1$, and they are:

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n_1} t_{i:n} + (n-n_1)\tau}{n_1},$$
$$\hat{\theta}_2 = \frac{\sum_{i=n_1+1}^{r}(t_{i:n}-\tau) + (n-r)(t_{r:n}-\tau)}{r-n_1}.$$

- These are the conditional MLEs of θ_1 and θ_2, conditional on $1 \le n_1 \le r-1$.

4. Likelihood under Type-II Censoring & MLEs

- It is evident that the MLEs of θ_1 and θ_2 exist only when $1 \le n_1 \le r-1$, and they are:

$$\hat{\theta}_1 = \frac{\sum_{i=1}^{n_1} t_{i:n} + (n-n_1)\tau}{n_1},$$

$$\hat{\theta}_2 = \frac{\sum_{i=n_1+1}^{r} (t_{i:n} - \tau) + (n-r)(t_{r:n} - \tau)}{r-n_1}.$$

- These are the conditional MLEs of θ_1 and θ_2, conditional on $1 \le n_1 \le r-1$.
- The inference we develop here will be exact and conditional.

5. Exact Conditional Distributions of MLEs

- Denote the CMGFs of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \leq n_1 \leq r-1$, by $M_1(t)$ and $M_2(t)$, respectively.

5. Exact Conditional Distributions of MLEs

- Denote the CMGFs of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \leq n_1 \leq r-1$, by $M_1(t)$ and $M_2(t)$, respectively.

$$M_1(t) = E\left[e^{t\hat{\theta}_1} | 1 \leq n_1 \leq r-1\right],$$

$$M_2(t) = E\left[e^{t\hat{\theta}_2} | 1 \leq n_1 \leq r-1\right].$$

5. Exact Conditional Distributions of MLEs

- Denote the CMGFs of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \leq n_1 \leq r-1$, by $M_1(t)$ and $M_2(t)$, respectively.

$$M_1(t) = E\left[e^{t\hat{\theta}_1} | 1 \leq n_1 \leq r-1\right],$$

$$M_2(t) = E\left[e^{t\hat{\theta}_2} | 1 \leq n_1 \leq r-1\right].$$

- For deriving $M_k(t)$, we may write

$$M_k(t) = \sum_{j=1}^{r-1} E\left[e^{t\hat{\theta}_k} | n_1 = j\right] \times P\left[n_1 = j | 1 \leq n_1 \leq r-1\right].$$

5. Exact Conditional Distributions of MLEs

- **Lemma 1**: The number of failures occurring before τ, viz. n_1, is a binomial random variable with PMF (for $j = 0, 1, \cdots, n$)

$$P\left[n_1 = j\right] = \binom{n}{j}\left(1 - e^{-\frac{\tau}{\theta_1}}\right)^j e^{-\frac{\tau}{\theta_1}(n-j)} = p_j \ (\text{say}).$$

5. Exact Conditional Distributions of MLEs

- **Lemma 1**: The number of failures occurring before τ, viz. n_1, is a binomial random variable with PMF (for $j = 0, 1, \cdots, n$)

$$P\left[n_1 = j\right] = \binom{n}{j}\left(1 - e^{-\frac{\tau}{\theta_1}}\right)^j e^{-\frac{\tau}{\theta_1}(n-j)} = p_j \text{ (say)}.$$

So,

$$P\left[n_1 = j | 1 \le n_1 \le r-1\right] = \frac{p_j}{\sum_{i=1}^{r-1} p_i}.$$

5. Exact Conditional Distributions of MLEs

- **Lemma 2**: Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a sample of size n from PDF $f(x)$ and CDF $F(x)$. Let D be the number of order statistics $\leq \tau$ (fixed time).

5. Exact Conditional Distributions of MLEs

- **Lemma 2**: Let $X_{1:n} < \cdots < X_{n:n}$ be the order statistics of a sample of size n from PDF $f(x)$ and CDF $F(x)$. Let D be the number of order statistics $\leq \tau$ (fixed time).

  The conditional joint PDF of $X_{1:n}, \cdots, X_{D:n}$, given that $D = j$, is same as the joint PDF of all order statistics from a sample of size j from the right truncated density

$$f_\tau(t) = \begin{cases} \frac{f(t)}{F(\tau)} & \text{for} \quad 0 < t < \tau \\ 0 & \text{otherwise.} \end{cases}$$

5. Exact Conditional Distributions of MLEs

- **Lemma 3**: Let Z be a right-truncated exponential random variable with PDF

$$f_Z(z) = \frac{\frac{1}{\theta_1} e^{-\frac{z}{\theta_1}}}{1 - e^{-\frac{\tau}{\theta_1}}} \quad \text{for} \quad 0 < z < \tau.$$

5. Exact Conditional Distributions of MLEs

- **Lemma 3**: Let Z be a right-truncated exponential random variable with PDF

$$f_Z(z) = \frac{\frac{1}{\theta_1} e^{-\frac{z}{\theta_1}}}{1 - e^{-\frac{\tau}{\theta_1}}} \quad \text{for} \quad 0 < z < \tau.$$

  Then, the MGF of Z is

$$M_Z(t) = E(e^{tZ}) = \frac{1 - e^{-\frac{\tau}{\theta_1}(1-\theta_1 t)}}{(1 - e^{-\frac{\tau}{\theta_1}})(1 - \theta_1 t)}.$$

5. Exact Conditional Distributions of MLEs

- **Lemma 4**: Let Y be a $Gamma(\alpha, \lambda)$, i.e., a gamma random variable with shape parameter α and scale parameter λ. The PDF of Y is given by

$$f_G(y; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} \quad \text{for} \quad y > 0.$$

5. Exact Conditional Distributions of MLEs

- **Lemma 4**: Let Y be a $Gamma(\alpha, \lambda)$, i.e., a gamma random variable with shape parameter α and scale parameter λ. The PDF of Y is given by

$$f_G(y; \alpha, \lambda) = \frac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda y} \quad \text{for} \quad y > 0.$$

For any constant A, the MGF of $Y + A$ is

$$M_{Y+A}(t) = e^{tA} \left(1 - \frac{t}{\lambda}\right)^{-\alpha}.$$

5. Exact Conditional Distributions of MLEs

- By using the Lemmas, it can be shown that

$$E\left(e^{t\hat{\theta}_1}|n_1 = j\right) = \frac{e^{\frac{t}{j}(n-j)\tau}\left(1 - e^{-\frac{\tau}{\theta_1}\left(1-\frac{\theta_1 t}{j}\right)}\right)^j}{\left(1 - e^{-\frac{\tau}{\theta_1}}\right)^j\left(1 - \frac{\theta_1 t}{j}\right)^j}.$$

5. Exact Conditional Distributions of MLEs

- By using the Lemmas, it can be shown that

$$E\left(e^{t\hat{\theta}_1} | n_1 = j\right) = \frac{e^{\frac{t}{j}(n-j)\tau}\left(1 - e^{-\frac{\tau}{\theta_1}\left(1-\frac{\theta_1 t}{j}\right)}\right)^j}{\left(1 - e^{-\frac{\tau}{\theta_1}}\right)^j \left(1 - \frac{\theta_1 t}{j}\right)^j}.$$

So,

$$E\left(e^{t\hat{\theta}_1} | 1 \leq n_1 \leq r-1\right)$$

$$= \sum_{j=1}^{r-1} \sum_{k=0}^{j} c_{j,k} \left(1 - \frac{\theta_1 t}{j}\right)^{-j} e^{\frac{t\tau}{j}(n-j+k)}.$$

5. Exact Conditional Distributions of MLEs

- **Theorem 1**: The PDF of $\hat{\theta}_1$, conditional on $1 \le n_1 \le r-1$, is given by

$$f_{\hat{\theta}_1}(t) = \sum_{j=1}^{r-1} \sum_{k=0}^{j} c_{j,k}\, f_G\left(t - \tau_{j,k}; j, \frac{j}{\theta_1}\right);$$

5. Exact Conditional Distributions of MLEs

- **Theorem 1**: The PDF of $\hat{\theta}_1$, conditional on $1 \leq n_1 \leq r-1$, is given by

$$f_{\hat{\theta}_1}(t) = \sum_{j=1}^{r-1} \sum_{k=0}^{j} c_{j,k} \, f_G\left(t - \tau_{j,k}; j, \frac{j}{\theta_1}\right);$$

here,

$$c_{j,k} = \frac{(-1)^k}{\sum_{i=1}^{r-1} p_i} \binom{n}{j} \binom{j}{k} e^{-\frac{\tau}{\theta_1}(n-j+k)},$$

$$\tau_{j,k} = \frac{\tau}{j}(n-j+k).$$

5. Exact Conditional Distributions of MLEs

- **Lemma 5**: Let $T_{1:n} < \cdots < T_{r:n}$ be the first r order statistics from the cumulative exposure density. The conditional joint PDF of $T_{n_1+1:n}, \cdots, T_{r:n}$, given that $n_1 = j$, where $1 \leq j \leq r-1$, is given by

5. Exact Conditional Distributions of MLEs

- **Lemma 5**: Let $T_{1:n} < \cdots < T_{r:n}$ be the first r order statistics from the cumulative exposure density. The conditional joint PDF of $T_{n_1+1:n}, \cdots, T_{r:n}$, given that $n_1 = j$, where $1 \leq j \leq r-1$, is given by

$$f_{T_{n_1+1:n},\cdots,T_{r:n}|(n_1=j)}(t_{n_1+1:n},\cdots,t_{r:n}) = \frac{c_3}{\theta_2^{r-j}} e^{-\left\{\left(\frac{t_{j+1:n}-\tau}{\theta_2}\right)+\cdots+\left(\frac{t_{r:n}-\tau}{\theta_2}\right)+(n-r)\left(\frac{t_{r:n}-\tau}{\theta_2}\right)\right\}}$$

for $\tau < t_{j+1:n} < \cdots < t_{r:n} < \infty$, and c_3 is the normalizing constant.

5. Exact Conditional Distributions of MLEs

- By using the preceding Lemma, it can be shown that the CMGF of $\hat{\theta}_2$ is

$$M_2(t) = \sum_{j=1}^{r-1} \frac{p_{r-j}}{\sum_{i=1}^{r-1} p_i} \left(1 - \frac{t\theta_2}{j}\right)^{-j}.$$

5. Exact Conditional Distributions of MLEs

- By using the preceding Lemma, it can be shown that the CMGF of $\hat{\theta}_2$ is

$$M_2(t) = \sum_{j=1}^{r-1} \frac{p_{r-j}}{\sum_{i=1}^{r-1} p_i} \left(1 - \frac{t\theta_2}{j}\right)^{-j}.$$

- **Theorem 2**: The PDF of $\hat{\theta}_2$, conditional on $1 \leq n_1 \leq r-1$, is given by

$$f_{\hat{\theta}_2}(t) = \sum_{j=1}^{r-1} \frac{p_{r-j}}{\sum_{i=1}^{r-1} p_i} f_G\left(t; j, \frac{j}{\theta_2}\right).$$

5. Exact Conditional Distributions of MLEs

- **Theorem 3**: The first two moments of $\hat{\theta}_1$ and $\hat{\theta}_2$ are as follows:

5. Exact Conditional Distributions of MLEs

- **Theorem 3**: The first two moments of $\hat{\theta}_1$ and $\hat{\theta}_2$ are as follows:

$$E\left(\hat{\theta}_1\right) = \sum_{j=1}^{r-1}\sum_{k=0}^{j} c_{j,k}\left(\tau_{j,k}+\theta_1\right),$$

$$E\left(\hat{\theta}_1^2\right) = \sum_{j=1}^{r-1}\sum_{k=0}^{j} c_{j,k}\left(\frac{(j+1)}{j}\theta_1^2+\tau_{j,k}^2+2\theta_1\tau_{j,k}\right),$$

$$E(\hat{\theta}_2) = \theta_2\sum_{j=1}^{r-1}\frac{p_{r-j}}{\sum_{i=1}^{r-1}p_i} = \theta_2,$$

$$E(\hat{\theta}_2^2) = \theta_2^2\sum_{j=1}^{r-1}\frac{p_{r-j}}{\sum_{i=1}^{r-1}p_i}\,\frac{(j+1)}{j}.$$

5. Exact Conditional Distributions of MLEs

- **Remark 1**: The conditional PDF of $\hat{\theta}_2$ in Theorem 2 is a true mixture of gamma densities.

5. Exact Conditional Distributions of MLEs

- **Remark 1**: The conditional PDF of $\hat{\theta}_2$ in Theorem 2 is a true mixture of gamma densities.
- **Remark 2**: The expressions of the expected values in Theorem 3 clearly reveal that $\hat{\theta}_1$ is a biased estimator of θ_1, while $\hat{\theta}_2$ is an unbiased estimator of θ_2.

5. Exact Conditional Distributions of MLEs

- **Remark 1**: The conditional PDF of $\hat{\theta}_2$ in Theorem 2 is a true mixture of gamma densities.
- **Remark 2**: The expressions of the expected values in Theorem 3 clearly reveal that $\hat{\theta}_1$ is a biased estimator of θ_1, while $\hat{\theta}_2$ is an unbiased estimator of θ_2.
- **Remark 3**: The expressions of the second moments in Theorem 3 can be used for finding standard errors of the estimates.

- **Theorem 4**: The tail probability of $\hat{\theta}_1$ is

$$P_{\theta_1}\left(\hat{\theta}_1 \geq b\right) = \sum_{j=1}^{r-1}\sum_{k=0}^{j} c_{j,k}\,\Gamma\left(j, \frac{j}{\theta_1} < b - \tau_{j,k} >\right),$$

where $\Gamma(a,z) = \frac{1}{\Gamma(a)}\int_z^\infty t^{a-1}e^{-t}dt$ is incomplete gamma, and $< x >= \max\{x, 0\}$.

- **Theorem 4**: The tail probability of $\hat{\theta}_1$ is

$$P_{\theta_1}\left(\hat{\theta}_1 \geq b\right) = \sum_{j=1}^{r-1}\sum_{k=0}^{j} c_{j,k}\,\Gamma\left(j, \frac{j}{\theta_1} < b - \tau_{j,k} >\right),$$

where $\Gamma(a,z) = \frac{1}{\Gamma(a)}\int_z^{\infty} t^{a-1}e^{-t}dt$ is incomplete gamma, and $< x >= \max\{x, 0\}$.

Similarly, the tail probability of $\hat{\theta}_2$ is

$$P_{\theta_2}\left(\hat{\theta}_2 \geq b\right) = \sum_{j=1}^{r-1}\frac{p_{r-j}}{\sum_{i=1}^{r-1} p_i}\,\Gamma\left(j, \frac{bj}{\theta_2}\right).$$

6. Confidence Intervals & Bootstrap Intervals

- **Exact CIs**: Using the results in Theorem 4, exact conditional CIs can be constructed for θ_1 and θ_2.

6. Confidence Intervals & Bootstrap Intervals

- **Exact CIs**: Using the results in Theorem 4, exact conditional CIs can be constructed for θ_1 and θ_2.
- **Approximate CIs**: Using the observed Fisher information matrix, approximate CIs can be constructed for θ_1 and θ_2 using the asymptotic normality of the MLEs.

6. Confidence Intervals & Bootstrap Intervals

- **Exact CIs**: Using the results in Theorem 4, exact conditional CIs can be constructed for θ_1 and θ_2.
- **Approximate CIs**: Using the observed Fisher information matrix, approximate CIs can be constructed for θ_1 and θ_2 using the asymptotic normality of the MLEs.
- **BCA Bootstrap CIs**: Using the bias-corrected and accelerated percentile bootstrap method [Efron and Tibshirani (1982)], bootstrap CIs can be constructed for θ_1 and θ_2.

7. Simulation Results & Comments

- We simulated the coverage probabilities (CP) of all three CIs for different values of n, r and τ; see Tables 1 and 2.

7. Simulation Results & Comments

- We simulated the coverage probabilities (CP) of all three CIs for different values of n, r and τ; see Tables 1 and 2.
  - Exact conditional CI has its CP to be quite close to the nominal level.
  - Approximate CI has its CP to be always smaller than the nominal level, and so it will often be unduly narrower.
  - Bootstrap CI has its CP to be close to the nominal level for θ_2, but is not satisfactory for θ_1, and especially worse for small n.

7. Simulation Results & Comments

Table 1: Estimated coverage probabilities (in %) based on 1000 simulations with $\theta_1 = 12.0$ and $\theta_2 = 4.5$, $n = 20$, $r = 16$, $B = 1000$

C.I. of θ_1	90% C.I.			95% C.I.		
τ	Boot	Approx	Exact	Boot	Approx	Exact
1	97.7	74.0	90.6	99.0	74.7	95.8
2	98.5	83.6	88.0	99.6	84.3	94.6
3	85.4	83.5	89.0	85.7	86.6	94.0
4	84.2	81.8	88.4	92.1	87.0	94.2
5	93.8	85.4	91.4	96.8	89.0	95.8
6	95.3	87.1	90.7	97.3	89.8	95.8

7. Simulation Results & Comments

Table 2: Estimated coverage probabilities (in %) based on 1000 simulations with $\theta_1 = 12.0$ and $\theta_2 = 4.5$, $n = 20$, $r = 16$, $B = 1000$

C.I. of θ_2	90% C.I.			95% C.I.		
τ	Boot	Approx	Exact	Boot	Approx	Exact
1	90.7	88.7	90.9	94.8	92.8	95.8
2	90.1	86.1	90.5	94.3	91.1	95.8
3	89.8	87.1	91.9	94.2	91.0	96.1
4	89.4	86.2	90.5	94.5	90.2	96.1
5	89.7	86.6	91.0	93.8	89.9	96.0
6	88.3	84.7	91.0	93.4	87.4	96.2

8. Illustrative Example

- Let us consider the data presented by Xiong (1998). The data $n = 20$, $r = 16$ and $\tau = 5$ are as follows:

8. Illustrative Example

- Let us consider the data presented by Xiong (1998). The data $n = 20$, $r = 16$ and $\tau = 5$ are as follows:

Stress level	Failure Time					
$\theta_1 = e^{2.5}$	2.01	3.60	4.12	4.34		
$\theta_2 = e^{1.5}$	5.04	5.94	6.68	7.09	7.17	7.49
	7.60	8.23	8.24	8.25	8.69	12.05

8. Illustrative Example

- In this example, we have

$$n_1 = 4 \quad \text{and} \quad n_2 = r - n_1 = 12.$$

8. Illustrative Example

- In this example, we have

$$n_1 = 4 \quad \text{and} \quad n_2 = r - n_1 = 12.$$

- We then find

$$\hat{\theta}_1 = 23.5175 \quad \text{and} \quad \hat{\theta}_2 = 5.0558$$

8. Illustrative Example

- In this example, we have

$$n_1 = 4 \quad \text{and} \quad n_2 = r - n_1 = 12.$$

- We then find

$$\hat{\theta}_1 = 23.5175 \quad \text{and} \quad \hat{\theta}_2 = 5.0558$$

- The confidence intervals for θ_1 and θ_2 are presented in Table 3.

8. Illustrative Example

Table 3: Confidence intervals for θ_1 and θ_2

CI for θ_1	90%	95%
Bootstrap CI	(7.78, 34.05)	(6.03, 34.05)
Approx CI	(0.00, 35.66)	(0.00, 39.36)
Exact CI	(11.70, 72.95)	(10.35, 94.78)
CI for θ_2	90%	95%
Bootstrap CI	(5.76, 11.43)	(5.51, 12.80)
Approx CI	(2.66, 7.46)	(2.20, 7.92)
Exact CI	(3.33, 8.80)	(3.07, 9.86)

8. Illustrative Example

From Table 3, we observe the following:

8. Illustrative Example

From Table 3, we observe the following:

- Exact CIs are wider in general than the other two intervals;

8. Illustrative Example

From Table 3, we observe the following:

- Exact CIs are wider in general than the other two intervals;
- Approximate CIs are always narrower while Bootstrap CIs are sometimes narrower and sometimes wider;

8. Illustrative Example

From Table 3, we observe the following:

- Exact CIs are wider in general than the other two intervals;
- Approximate CIs are always narrower while Bootstrap CIs are sometimes narrower and sometimes wider;
- This is so because the CPs for the approximate method are lower than the nominal level while those of the bootstrap method are sometimes lower and sometimes higher;

8. Illustrative Example

From Table 3, we observe the following:

- Exact CIs are wider in general than the other two intervals;
- Approximate CIs are always narrower while Bootstrap CIs are sometimes narrower and sometimes wider;
- This is so because the CPs for the approximate method are lower than the nominal level while those of the bootstrap method are sometimes lower and sometimes higher;
- CIs for θ_2 are considerably narrower than those for θ_1. This is so since when τ is small relative to θ_1, relatively small (large) numbers of failures would occur before (after) τ.

9. Time-Constrained Step-Stress Test

- Consider the step-stress testing when there is a time constraint on the experiment.

9. Time-Constrained Step-Stress Test

- Consider the step-stress testing when there is a time constraint on the experiment.
- n identical units are tested at an initial stress level s_0. The stress level is changed to s_1 at time τ_1, and the testing is terminated at time τ_2, where $0 < \tau_1 < \tau_2 < \infty$ are pre-fixed.

9. Time-Constrained Step-Stress Test

- Consider the step-stress testing when there is a time constraint on the experiment.
- n identical units are tested at an initial stress level s_0. The stress level is changed to s_1 at time τ_1, and the testing is terminated at time τ_2, where $0 < \tau_1 < \tau_2 < \infty$ are pre-fixed.
- Let

$$\begin{aligned} N_1 &= \text{no. of units that fail before time } \tau_1; \\ N_2 &= \text{no. of units that fail before time } \tau_2 \\ &\qquad \text{at stress level } s_1. \end{aligned}$$

9. Time-Constrained Step-Stress Test

- With these notation, we will observe the following data:

9. Time-Constrained Step-Stress Test

- With these notation, we will observe the following data:

$$\mathbf{t} = \Big\{ t_{1:n} < \cdots < t_{N_1:n} \le \tau_1 < t_{N_1+1:n} < \cdots < t_{N_1+N_2:n} \le \tau_2 \Big\}.$$

9. Time-Constrained Step-Stress Test

- With these notation, we will observe the following data:

$$\mathbf{t} = \Big\{ t_{1:n} < \cdots < t_{N_1:n} \leq \tau_1 < t_{N_1+1:n} < \cdots < t_{N_1+N_2:n} \leq \tau_2 \Big\}.$$

- We obtain the likelihood function of θ_1 and θ_2 based on the above Type-I censored sample as follows:

9. Time-Constrained Step-Stress Test

- If $N_1 = n$ and $N_2 = 0$, the likelihood is

$$L(\theta_1, \theta_2 | \mathbf{t}) = n! \prod_{k=1}^{n} g_1(t_{k:n}) = \frac{n!}{\theta_1^n} \exp\left\{ -\frac{1}{\theta_1} \sum_{k=1}^{n} t_{k:n} \right\},$$
$$0 < t_{1:n} < \cdots < t_{n:n} < \tau_1;$$

9. Time-Constrained Step-Stress Test

- If $N_1 = n$ and $N_2 = 0$, the likelihood is

$$\begin{aligned} L(\theta_1, \theta_2|\mathbf{t}) &= n! \prod_{k=1}^{n} g_1\left(t_{k:n}\right) = \frac{n!}{\theta_1^n} \exp\left\{ -\frac{1}{\theta_1} \sum_{k=1}^{n} t_{k:n} \right\}, \\ & 0 < t_{1:n} < \cdots < t_{n:n} < \tau_1; \end{aligned}$$

- In all other cases, the likelihood function is

$$L(\theta_1, \theta_2|\mathbf{t}) = \frac{n!}{(n-r)!\, \theta_1^{N_1} \theta_2^{N_2}} \exp\left\{ -\frac{1}{\theta_1} D_1 - \frac{1}{\theta_2} D_2 \right\},$$

$$0 < t_{1:n} < \cdots < t_{N_1:n} < \tau_1 \leq t_{N_1+1:n} < \cdots < t_{r:n} < \tau_2,$$

9. Time-Constrained Step-Stress Test

where

$$\begin{aligned} r &= N_1 + N_2 \; (2 \le r \le n), \\ D_1 &= \sum_{k=1}^{N_1} t_{k:n} + (n - N_1)\tau_1 , \\ D_2 &= \sum_{k=N_1+1}^{r} (t_{k:n} - \tau_1) + (n - r)(\tau_2 - \tau_1). \end{aligned}$$

9. Time-Constrained Step-Stress Test

where

$$\begin{aligned} r &= N_1 + N_2 \; (2 \le r \le n), \\ D_1 &= \sum_{k=1}^{N_1} t_{k:n} + (n - N_1)\tau_1, \\ D_2 &= \sum_{k=N_1+1}^{r} (t_{k:n} - \tau_1) + (n-r)(\tau_2 - \tau_1). \end{aligned}$$

- We observe that if at least one failure occurs before τ_1 and between τ_1 and τ_2, the MLE of (θ_1, θ_2) exists, and (D_1, D_2) is joint complete sufficient for (θ_1, θ_2).

9. Time-Constrained Step-Stress Test

- In this situation, the log-likelihood function of θ_1 and θ_2 is

$$l(\theta_1, \theta_2|\mathbf{t}) = \log \frac{n!}{(n-r)!} - N_1 \log \theta_1 - N_2 \log \theta_2 - \frac{D_1}{\theta_1} - \frac{D_2}{\theta_2}.$$

9. Time-Constrained Step-Stress Test

- In this situation, the log-likelihood function of θ_1 and θ_2 is

$$l(\theta_1, \theta_2 | \mathbf{t}) = \log \frac{n!}{(n-r)!} - N_1 \log \theta_1 - N_2 \log \theta_2 - \frac{D_1}{\theta_1} - \frac{D_2}{\theta_2}.$$

- The MLEs of θ_1 and θ_2 are obtained as

$$\hat{\theta}_1 = \frac{D_1}{N_1} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{D_2}{N_2}.$$

9. Time-Constrained Step-Stress Test

- In this situation, the log-likelihood function of θ_1 and θ_2 is

$$l(\theta_1, \theta_2|\mathbf{t}) = \log \frac{n!}{(n-r)!} - N_1 \log \theta_1 - N_2 \log \theta_2 - \frac{D_1}{\theta_1} - \frac{D_2}{\theta_2}.$$

- The MLEs of θ_1 and θ_2 are obtained as

$$\hat{\theta}_1 = \frac{D_1}{N_1} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{D_2}{N_2}.$$

- We can similarly develop here conditional inference (conditioned on $N_1 \geq 1$ and $N_2 \geq 1$), basing it on truncated trinomial distribution.

9. Time-Constrained Step-Stress Test

- **Theorem 5**: The conditional PDF of $\hat{\theta}_1$, given $\left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$, is

$$f_{\hat{\theta}_1}(x) = C_n \sum_{i=1}^{n-1} \sum_{k=0}^{i} C_{ik}\, f_G\left(x - \tau_{ik}; i, \frac{i}{\theta_1}\right),$$

where $f_G(\cdot)$ is the gamma density as before,

$$\begin{aligned}
\tau_{ik} &= \frac{1}{i}(n-i+k)\tau_1, \qquad p_1 = G_1(\tau_1) = 1 - e^{-\tau_1/\theta_1}, \\
p_2 &= G_2(\tau_2) - G_1(\tau_1) = (1-p_1)\left\{1 - e^{-(\tau_2-\tau_1)/\theta_2}\right\}, \\
p_3 &= 1 - p_1 - p_2, \qquad C_n = \frac{1}{1-(1-p_1)^n - (1-p_2)^n + p_3^n}, \\
C_{ik} &= (-1)^k \binom{n}{i}\binom{i}{k}\left\{(1-p_1)^{n-i} - p_3^{n-i}\right\}(1-p_1)^k.
\end{aligned}$$

9. Time-Constrained Step-Stress Test

- **Theorem 6**: The conditional PDF of $\hat{\theta}_2$, given $\left\{1 \leq N_1 \leq n-1 \text{ and } 1 \leq N_2 \leq n - N_1\right\}$, is

$$f_{\hat{\theta}_2}(x) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk}\, f_G\left(x - \tau_{ijk}; j, \frac{j}{\theta_2}\right),$$

where

$$\tau_{ijk} = \frac{1}{j}(n-i-j+k)(\tau_2 - \tau_1),$$

$$C_{ijk} = (-1)^k \binom{n}{i, j, n-i-j} \binom{j}{k} p_1^i p_3^{n-i-j+k} (1-p_1)^{j-k}.$$

9. Time-Constrained Step-Stress Test

- **Theorem 6**: The conditional PDF of $\hat{\theta}_2$, given $\left\{1 \le N_1 \le n-1 \text{ and } 1 \le N_2 \le n - N_1\right\}$, is

$$f_{\hat{\theta}_2}(x) = C_n \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} \sum_{k=0}^{j} C_{ijk}\, f_G\left(x - \tau_{ijk}; j, \frac{j}{\theta_2}\right),$$

where

$$\begin{aligned} \tau_{ijk} &= \frac{1}{j}(n-i-j+k)(\tau_2 - \tau_1), \\ C_{ijk} &= (-1)^k \binom{n}{i, j, n-i-j} \binom{j}{k} p_1^i p_3^{n-i-j+k} (1-p_1)^{j-k}. \end{aligned}$$

- We obtained the following CIs for θ_1 and θ_2 based on earlier data for different choices of the time constraint, viz., τ_2.

9. Time-Constrained Step-Stress Test

Table 4: Interval estimation for θ_1 with $\tau_1 = 5$ and different τ_2

τ_2	Method	90%	95%
6.00	Bootstrap (BCa)	(11.8452, 97.0986)	(10.4813, 97.9780)
	Approximation	(0.0000, 35.1448)	(0.0000, 38.8501)
	Exact	(11.4823, 71.8781)	(10.1474, 93.3925)
8.00	Bootstrap (BCa)	(11.5395, 95.6438)	(10.2748, 97.3843)
	Approximation	(0.0000, 35.6525)	(0.0000, 39.3578)
	Exact	(11.6965, 72.9479)	(10.3429, 94.7722)
12.05	Bootstrap (BCa)	(11.4043, 96.3999)	(10.4989, 98.1858)
	Approximation	(0.0000, 35.6561)	(0.0000, 39.3614)
	Exact	(11.7003, 72.9524)	(10.3472, 94.7775)

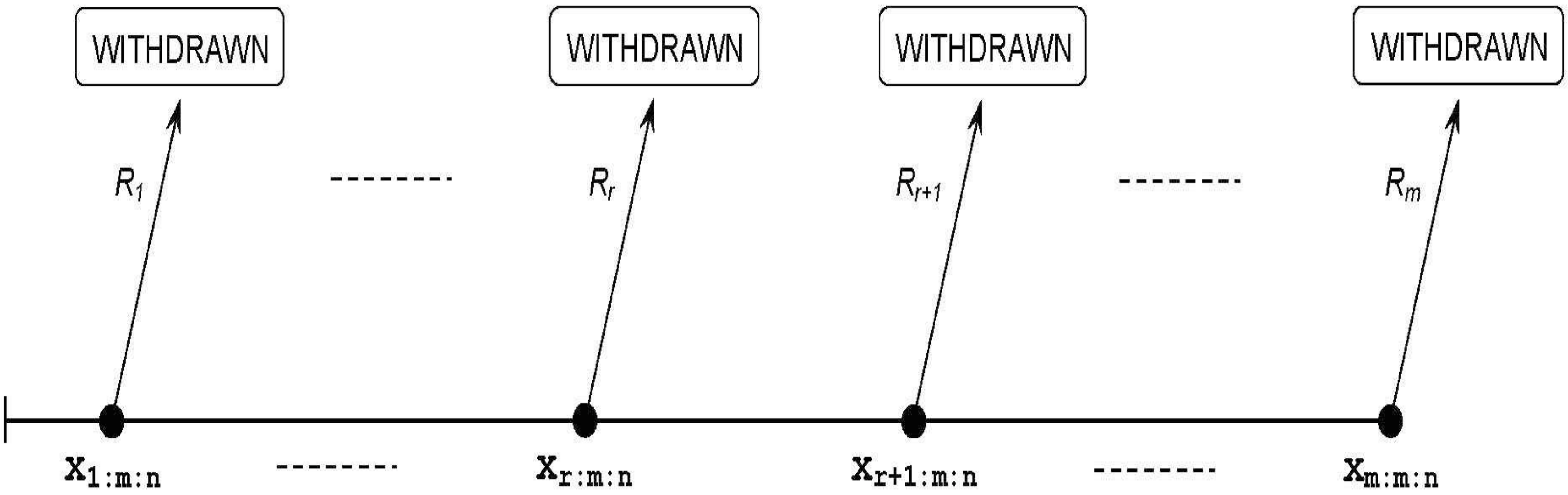
9. Time-Constrained Step-Stress Test

Table 5: Interval estimation for θ_2 with $\tau_1 = 5$ and different τ_2

τ_2	Method	90%	95%
6.00	Bootstrap (BCa)	(2.8799, 16.6801)	(2.4932, 17.3774)
	Approximation	(0.0000, 14.9771)	(0.0000, 16.6460)
	Exact	(2.7403, 61.6015)	(2.3523,117.4822)
8.00	Bootstrap (BCa)	(2.5731, 8.2473)	(2.2987, 9.8267)
	Approximation	(1.2354, 8.1647)	(0.5717, 8.8284)
	Exact	(3.1190, 11.2912)	(2.8251, 13.2468)
12.05	Bootstrap (BCa)	(3.3126, 6.3473)	(2.86336, 7.1828)
	Approximation	(2.5111, 7.9818)	(1.98708, 8.5058)
	Exact	(3.5491, 9.4128)	(3.27812, 10.5409)

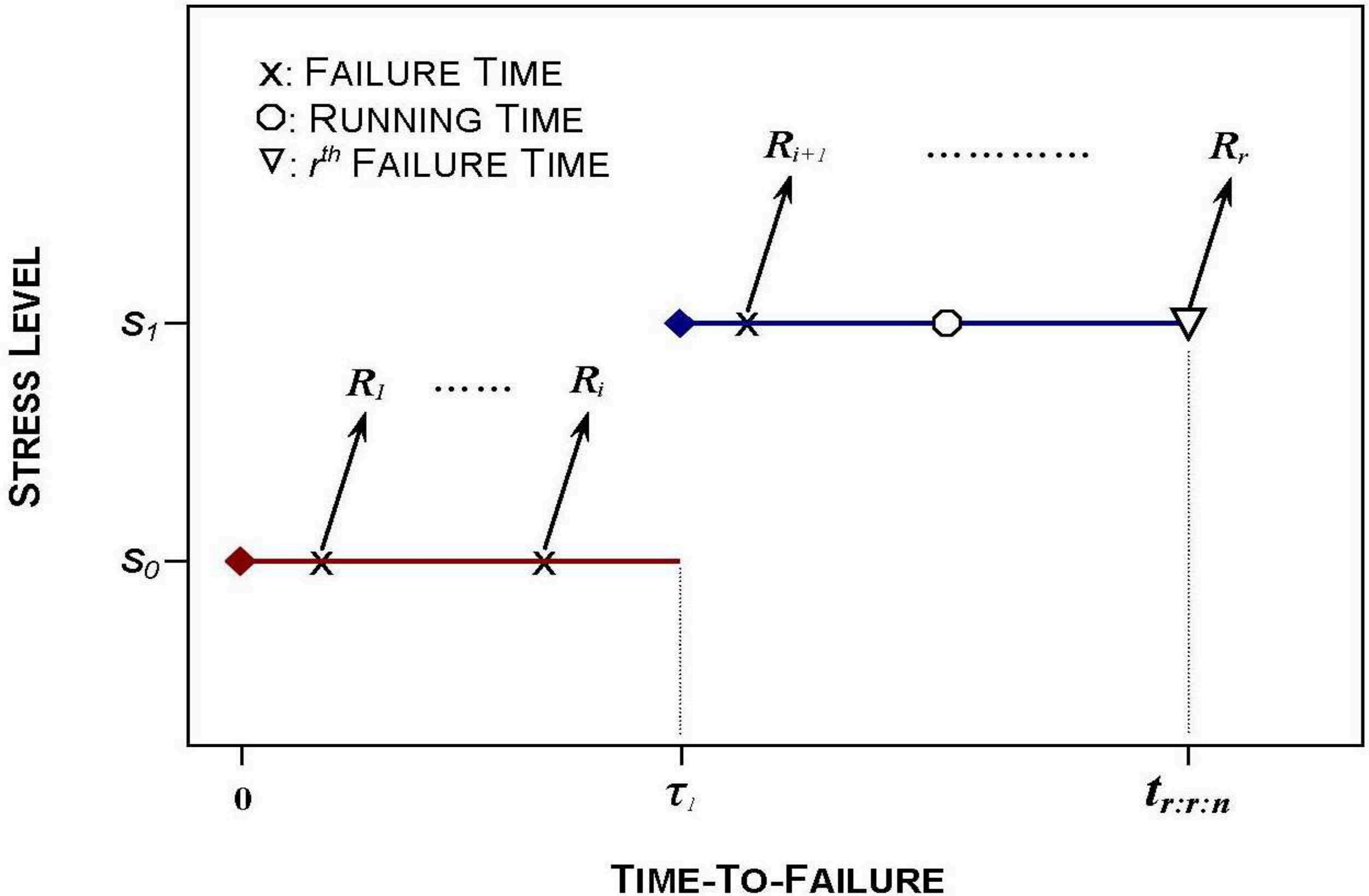
10. Progressive Type-II Censoring and Results

- Progressive Type-II Censoring



10. Progressive Type-II Censoring and Results

Model Description



10. Progressive Type-II Censoring and Results

- Set-up

10. Progressive Type-II Censoring and Results

- **Set-up**
  - Let us consider a simple step-stress model when the observed failure data are progressively Type-II censored.
  - $Exp(\theta_1)$ and $Exp(\theta_2)$ are the distributions.
  - Let

$$\begin{aligned} N_1 &= \text{no. of units that fail before time } \tau \text{ at stress level } s_0 \\ N_2 &= \text{no. of units that fail after time } \tau \text{ at stress level } s_1 . \end{aligned}$$

  - Observed data are

$$\boldsymbol{t} = \left\{ t_{1:r:n} < \cdots < t_{N_1:r:n} \leq \tau < t_{N_1+1:r:n} < \cdots < t_{r:r:n} \right\}.$$

10. Progressive Type-II Censoring and Results

- Maximum Likelihood Estimation

Maximum Likelihood Estimation

The likelihood function is

$$L(\theta_1, \theta_2|\mathbf{t}) = C_p \cdot \left\{ \prod_{k=1}^{r} g(t_{k:r:n}) \left[1 - G(t_{k:r:n})\right]^{R_k} \right\}$$

$$= \begin{cases} \frac{C_p}{\theta_1^r} \exp\left\{ -\frac{1}{\theta_1} \sum_{k=1}^{r} (R_k+1) t_{k:r:n} \right\} & \text{if } N_1 = r \text{ and } N_2 = 0 \\ \frac{C_p}{\theta_2^r} \exp\left\{ -\frac{1}{\theta_2} \sum_{k=1}^{r} (R_k+1)(t_{k:r:n} - \tau) - \frac{1}{\theta_1} \sum_{k=1}^{r} (R_k+1)\tau \right\} & \\ & \text{if } N_1 = 0 \text{ and } N_2 = r \\ \frac{C_p}{\theta_1^{N_1} \theta_2^{N_2}} e^{-\frac{1}{\theta_1} D_1 - \frac{1}{\theta_2} D_2} & \text{otherwise,} \end{cases}$$

where $r = N_1 + N_2$ $(2 \le r \le n)$, $C_p = \prod_{j=1}^{r} \sum_{k=j}^{r} (R_k + 1)$ and

$$D_1 = \sum_{k=1}^{N_1} (R_k+1) t_{k:r:n} + \tau \sum_{k=N_1+1}^{r} (R_k+1), \quad D_2 = \sum_{k=N_1+1}^{r} (R_k+1)(t_{k:r:n} - \tau).$$

10. Progressive Type-II Censoring and Results

- The MLEs of θ_1 and θ_2 exist only if at least one failure occurs before τ and after τ.

10. Progressive Type-II Censoring and Results

- The MLEs of θ_1 and θ_2 exist only if at least one failure occurs before τ and after τ.
- They are

$$\hat{\theta}_1 = \frac{D_1}{N_1} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{D_2}{N_2}.$$

10. Progressive Type-II Censoring and Results

- The MLEs of θ_1 and θ_2 exist only if at least one failure occurs before τ and after τ.
- They are

$$\hat{\theta}_1 = \frac{D_1}{N_1} \qquad \text{and} \qquad \hat{\theta}_2 = \frac{D_2}{N_2}.$$

- In this case, we can develop exact conditional inference based on the conditional PMF

$$\mathrm{P}_{\theta_1,\theta_2,c}\{N_1 = i\} = \mathrm{P}\Big\{N_1 = i \,\Big|\, 1 \le N_1 \le r-1\Big\}, \; i = 1, \cdots, r.$$

10. Progressive Type-II Censoring and Results

- Notation

10. Progressive Type-II Censoring and Results

- **Notation**

  Let us denote $M_{12}(\nu, \omega | N_1)$ for the joint CMGF of $\hat{\theta}_1$ and $\hat{\theta}_2$, and $M_k(\omega | N_1)$ for the CMGF of $\hat{\theta}_k,\ k = 1, 2$.

10. Progressive Type-II Censoring and Results

Notation

Let us denote $M_{12}(\nu, \omega | N_1)$ for the joint CMGF of $\hat{\theta}_1$ and $\hat{\theta}_2$, and $M_k(\omega|N_1)$ for the CMGF of $\hat{\theta}_k,\ k = 1, 2$.

Then,

$$
\begin{aligned}
M_{12}(\nu, \omega | N_1) &= \mathrm{E}\Big\{ e^{\nu\hat{\theta}_1 + \omega\hat{\theta}_2} \Big| 1 \le N_1 \le r-1 \Big\} \\
&= \sum_{i=1}^{r-1} \mathrm{E}_{\theta_1,\theta_2}\Big\{ e^{\nu\hat{\theta}_1 + \omega\hat{\theta}_2} \Big| N_1 = i \Big\} \cdot \mathrm{P}_{\theta_1,\theta_2,c}\{N_1 = i\}, \\
M_k(\omega|N_1) &= \mathrm{E}\Big\{ e^{\omega\hat{\theta}_k} \Big| 1 \le N_1 \le r-1 \Big\} \\
&= \sum_{i=1}^{r-1} \mathrm{E}_{\theta_1,\theta_2}\Big\{ e^{\omega\hat{\theta}_k} \Big| N_1 = i \Big\} \cdot \mathrm{P}_{\theta_1,\theta_2,c}\{N_1 = i\}.
\end{aligned}
$$

10. Progressive Type-II Censoring and Results

Lemma 6: Let $T_{1:r:n} < \cdots < T_{r:r:n}$ denote the progressively Type-II censored sample from the cumulative exposure PDF $g(t)$.

10. Progressive Type-II Censoring and Results

Lemma 6: Let $T_{1:r:n} < \cdots < T_{r:r:n}$ denote the progressively Type-II censored sample from the cumulative exposure PDF $g(t)$.

Then, the joint density function of $T_{1:r:n}, \cdots, T_{r:r:n}$ is [see Balakrishnan and Aggarwala (2000)]

$$
\begin{aligned}
f(t_1, \ldots, t_r) \quad &= \quad C_p \cdot \left\{ \prod_{k=1}^{N_1} g_1(t_k) \left[1 - G_1(t_k)\right]^{R_k} \right\} \\
&\qquad \times \left\{ \prod_{k=N_1+1}^{r} g_2(t_k) \left[1 - G_2(t_k)\right]^{R_k} \right\}, \\
&\quad\; 0 < t_1 < \cdots < t_{N_1} \leq \tau < t_{N_1+1} < \cdots < t_r \leq \infty;
\end{aligned}
$$

10. Progressive Type-II Censoring and Results

Lemma 6: Let $T_{1:r:n} < \cdots < T_{r:r:n}$ denote the progressively Type-II censored sample from the cumulative exposure PDF $g(t)$.

Then, the joint density function of $T_{1:r:n}, \cdots, T_{r:r:n}$ is [see Balakrishnan and Aggarwala (2000)]

$$
\begin{aligned}
f(t_1, \ldots, t_r) &= C_p \cdot \left\{ \prod_{k=1}^{N_1} g_1(t_k) \left[1 - G_1(t_k)\right]^{R_k} \right\} \\
&\quad \times \left\{ \prod_{k=N_1+1}^{r} g_2(t_k) \left[1 - G_2(t_k)\right]^{R_k} \right\}, \\
&\quad 0 < t_1 < \cdots < t_{N_1} \le \tau < t_{N_1+1} < \cdots < t_r \le \infty;
\end{aligned}
$$

further, the probability of the event $\{N_1 = i,\ i = 1, \ldots, r-1\}$ is

$$
\mathrm{P}\{N_1 = i\} = C_p \cdot \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} \frac{C_{k,i}(\mathbf{S}_i) C_{l,r-i-1}(\mathbf{S}_{i+l})}{B_{l.r-i}(\mathbf{S}_{i+l})} \cdot \exp\left\{ -\frac{\tau}{\theta_1} \sum_{j=i-k+1}^{r} S_j \right\},
$$

10. Progressive Type-II Censoring and Results

where

$$S_j = R_j + 1, \quad \boldsymbol{S_i} = (S_1, \ldots, S_i), \quad \boldsymbol{S}_{i+\boldsymbol{l}} = (S_{i+1}, \ldots, S_{i+l}),$$

$$B_{l.r-i}(\boldsymbol{S}_{i+\boldsymbol{l}}) = \sum_{j=r-i-l}^{r-i} S_{i+j} \quad \text{with} \quad \sum_{j=i}^{0} A_j \equiv 0,$$

$$C_{k,i}(\boldsymbol{S_i}) = \frac{(-1)^k}{\left\{\prod_{j=1}^{k} \sum_{m=i-k+1}^{i-k+j} S_m\right\}\left\{\prod_{j=1}^{i-k} \sum_{m=j}^{i-k} S_m\right\}},$$

$$C_{l,r-i-1}(\boldsymbol{S}_{i+\boldsymbol{l}}) = \frac{(-1)^l}{\left\{\prod_{j=1}^{l} \sum_{m=r-i-l}^{r-i-l+j-1} S_{i+m}\right\}\left\{\prod_{j=1}^{r-i-l-1} \sum_{m=j}^{r-i-l-1} S_{i+m}\right\}},$$

with $\prod_{j=1}^{0} A_j \equiv 1$ and C_p is as given earlier.

10. Progressive Type-II Censoring and Results

Lemma 7: The joint conditional density of $T_{1:r:n}, \ldots, T_{r:r:n}$, given $N_1 = i$, is given by [see Balakrishnan and Aggarwala (2000)]

$$\begin{aligned} & f\left(t_1, \ldots, t_r | N_1 = i\right) \\ = \; & \frac{C_p}{\mathrm{P}\{N_1 = i\}} \cdot \left\{ \prod_{k=1}^{i} g_1\left(t_k\right) \left[1 - G_1\left(t_k\right)\right]^{R_k} \right\} \\ & \times \left\{ \prod_{k=i+1}^{r} g_2\left(t_k\right) \left[1 - G_2\left(t_k\right)\right]^{R_k} \right\}, \\ & 0 < t_1 < \cdots < t_i \le \tau < t_{i+1} < \cdots < t_r \le \infty. \end{aligned}$$

10. Progressive Type-II Censoring and Results

- Conditional MGFs

Conditional MGFs

Using Lemmas 6 and Lemma 7, it can be shown that

$$M_{12}(\nu,\omega|N_1) = D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\frac{e^{\frac{\tau}{i}\sum_{j=i-k+1}^{r}S_j\nu}}{\left(1-\frac{\theta_1}{i}\nu\right)^i\left(1-\frac{\theta_2}{r-i}\omega\right)^{r-i}},$$

$$M_{1}(\omega|N_1) = D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\frac{e^{\frac{\tau}{i}\sum_{j=i-k+1}^{r}S_j\omega}}{\left(1-\frac{\theta_1}{i}\omega\right)^i},$$

$$M_{2}(\omega|N_1) = D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\frac{1}{\left(1-\frac{\theta_2}{r-i}\omega\right)^{r-i}},$$

where

$$D=\frac{C_p}{\sum_{j=1}^{r-1}\mathrm{P}\{N_1=j\}},\quad D_{ikl}=\frac{C_{k,i}(\boldsymbol{S_i})C_{l,r-i-1}(\boldsymbol{S_{i+l}})}{B_{l.r-i}(\boldsymbol{S_{i+l}})}\exp\left\{-\frac{\tau}{\theta_1}\sum_{j=i-k+1}^{r}S_j\right\}$$

and C_p is as defined earlier.

Theorem 7: The joint conditional PDF of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \leq N_1 \leq r-1$, is

$$f_{\hat{\theta}_1,\hat{\theta}_2}(x,y) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot f_G\left(x - \tau_{ik}; i, \frac{\theta_1}{i}\right) \cdot f_G\left(y; r-i, \frac{\theta_2}{r-i}\right),$$

where $\tau_{ik} = \frac{\tau}{i} \sum_{j=i-k+1}^{r} (R_j + 1)$.

10. Progressive Type-II Censoring and Results

Theorem 8: The conditional PDF of $\hat{\theta}_1$, given $1 \le N_1 \le r-1$, is

$$f_{\hat{\theta}_1}(x) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot f_G\left(x - \tau_{ik}; i, \frac{\theta_1}{i}\right).$$

10. Progressive Type-II Censoring and Results

Theorem 8: The conditional PDF of $\hat{\theta}_1$, given $1 \le N_1 \le r-1$, is

$$f_{\hat{\theta}_1}(x) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot f_G\left(x - \tau_{ik}; i, \frac{\theta_1}{i}\right).$$

Theorem 9: The conditional PDF of $\hat{\theta}_2$, given $1 \le N_1 \le r-1$, is

$$f_{\hat{\theta}_2}(x) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot f_G\left(x; r-i, \frac{\theta_2}{r-i}\right).$$

10. Progressive Type-II Censoring and Results

Theorem 10: The mean and variance of $\hat{\theta}_1$ and $\hat{\theta}_2$ are

10. Progressive Type-II Censoring and Results

Theorem 10: The mean and variance of $\hat{\theta}_1$ and $\hat{\theta}_2$ are

$$\begin{aligned}
\mathrm{E}(\hat{\theta}_1) &= \theta_1 + D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\tau_{ik},\\
\mathrm{Var}(\hat{\theta}_1) &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\left(\tau_{ik}^2+\frac{\theta_1^2}{i}\right)\\
&\quad -\left(D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\tau_{ik}\right)^2,\\
\mathrm{E}(\hat{\theta}_2) &= \theta_2,\\
\mathrm{Var}(\hat{\theta}_2) &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\frac{\theta_2^2}{r-i}.
\end{aligned}$$

10. Progressive Type-II Censoring and Results

Remark 4: We observe that $\hat{\theta}_1$ is a biased estimator of θ_1 while $\hat{\theta}_2$ is an unbiased estimator of θ_2.

10. Progressive Type-II Censoring and Results

Remark 4: We observe that $\hat{\theta}_1$ is a biased estimator of θ_1 while $\hat{\theta}_2$ is an unbiased estimator of θ_2.

Furthermore, from the joint density of $\hat{\theta}_1$ and $\hat{\theta}_2$ in Theorem 7, we obtain

$$\begin{aligned} \mathrm{E}\big(\hat{\theta}_1\hat{\theta}_2\big) &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\left(\tau_{ik} + i\frac{\theta_1}{i}\right)\left[(r-i)\frac{\theta_2}{r-i}\right] \\ &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\big(\tau_{ik}+\theta_1\big)\theta_2 \\ &= \mathrm{E}\big(\hat{\theta}_1\big)\mathrm{E}\big(\hat{\theta}_2\big) \end{aligned}$$

so that $\mathrm{Cov}\big(\hat{\theta}_1, \hat{\theta}_2\big) = 0$.

10. Progressive Type-II Censoring and Results

Theorem 11: The tail probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \leq N_1 \leq r-1$, are

10. Progressive Type-II Censoring and Results

Theorem 11: The tail probabilities of $\hat{\theta}_1$ and $\hat{\theta}_2$, given $1 \le N_1 \le r-1$, are

$$\begin{aligned} P_{\theta_1}\left\{\hat{\theta}_1 > \xi\right\} &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\Gamma\left(\frac{i}{\theta_1}\langle \xi - \tau_{ik}\rangle; i\right), \\ P_{\theta_2}\left\{\hat{\theta}_2 > \xi\right\} &= D\sum_{i=1}^{r-1}\sum_{k=0}^{i}\sum_{l=0}^{r-i-1} D_{ikl}\cdot\Gamma\left(\frac{r-i}{\theta_2}\langle \xi\rangle; r-i\right), \end{aligned}$$

where $\langle w\rangle = \max\{0, w\}$, and

$$\Gamma(w;\alpha) = \int_w^\infty f_G(x;\alpha,1)\,dx = \int_w^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x}\,dx.$$

10. Progressive Type-II Censoring and Results

- Simulation Results and Comments

10. Progressive Type-II Censoring and Results

Simulation Results and Comments

The coverage probabilities (in $\%$) of CIs for θ_1 and θ_2 based on 1000 simulations and $M = 1000$ replications with $n = 20,\ r = 8,\ \theta_1 = e^{2.5}$ and $\theta_2 = e^{1.5}$ are presented in the following Tables 6 and 7.

Table 6: Coverage Probabilities of CIs for θ_1

		90% C.I.					95% C.I.				
PCS	τ	Bootstrap			App.	Exact	Bootstrap			App.	Exact
		P	St	BCa			P	St	BCa		
(7★0, 12)	1	96.9	51.0	82.0	75.4	89.9	97.5	52.4	85.9	73.5	96.7
	2	96.7	61.6	79.3	83.6	88.9	100.0	62.8	85.3	81.7	94.4
	3	89.7	63.3	75.8	78.8	90.9	95.6	65.1	81.4	83.6	94.8
	4	94.2	66.2	76.5	83.1	88.5	93.2	65.2	80.0	90.5	94.6
	5	83.2	65.8	73.0	79.1	92.1	91.5	67.0	80.2	87.0	96.0
	6	88.9	71.8	70.7	82.3	90.8	93.9	74.8	85.6	89.4	96.0
	7	92.3	69.0	74.9	79.9	90.1	96.0	71.3	89.0	85.7	94.3
	8	89.1	70.1	78.3	79.1	91.2	94.7	72.7	87.5	86.9	95.5
	9	87.9	65.0	76.7	84.0	89.8	92.6	78.7	86.3	88.4	94.2
	10	88.9	72.2	70.0	82.5	90.6	94.2	73.6	89.9	87.7	95.9
(12, 7★0)	1	88.8	69.0	80.7	75.7	89.2	96.4	76.3	87.6	79.4	94.8
	2	96.1	69.9	83.5	79.6	89.0	97.6	76.1	89.6	84.8	95.4
	3	94.2	74.7	86.8	79.5	90.0	97.4	77.7	92.8	86.1	96.0
	4	90.5	75.3	87.9	82.4	90.6	96.3	80.7	94.1	83.7	93.8
	5	91.5	80.2	91.8	80.5	89.0	94.3	81.2	94.4	86.6	95.0
	6	88.6	78.6	86.9	84.3	91.0	94.1	84.2	93.3	84.5	95.8
	7	91.5	80.1	88.0	79.3	88.8	93.5	81.1	90.8	88.0	95.2
	8	89.3	77.9	86.7	81.5	90.2	95.0	82.5	91.3	87.5	95.5
	9	88.4	75.8	84.1	82.0	91.7	94.1	80.1	89.4	86.8	95.9
	10	89.8	75.7	85.6	82.5	90.0	94.5	81.4	90.4	86.0	95.0
(6★0, 6, 6)	1	97.0	50.3	79.8	75.9	89.8	99.4	53.6	86.5	74.5	94.6
	2	96.3	70.7	83.0	81.4	89.8	99.5	72.1	87.8	83.3	95.3
	3	87.8	67.3	77.4	76.4	88.5	95.0	74.4	84.1	86.3	94.8
	4	92.4	63.6	76.0	82.7	90.3	94.5	67.2	83.8	88.9	95.6
	5	90.5	65.4	72.7	80.1	88.1	92.6	71.1	83.3	87.3	95.2
	6	88.9	69.2	69.3	82.0	89.4	93.0	75.6	79.5	88.8	94.1
	7	92.8	69.8	74.5	81.9	89.2	91.1	76.1	86.9	87.4	94.5
	8	87.8	67.9	79.2	83.9	89.9	93.9	73.5	86.6	88.7	95.1
	9	90.8	63.2	73.1	80.5	90.0	95.9	75.2	86.9	88.8	95.3
	10	92.3	63.1	73.5	81.8	89.7	96.0	74.6	85.8	88.9	95.9

Table 7: Coverage Probabilities of CIs for θ_2

PCS	τ	90% C.I.					95% C.I.				
		Bootstrap			App.	Exact	Bootstrap			App.	Exact
		P	St	BCa			P	St	BCa		
(7⋆0, 12)	1	83.6	88.8	89.7	84.1	91.3	91.1	96.1	96.7	88.4	96.1
	2	84.2	89.6	90.6	83.9	91.3	88.1	94.9	95.0	87.7	95.5
	3	81.4	90.5	88.9	80.7	90.7	86.1	95.5	93.5	81.2	95.4
	4	80.1	90.3	86.7	77.0	88.8	83.1	95.8	91.8	81.1	95.6
	5	79.3	90.1	83.5	75.6	87.5	79.6	95.3	88.5	79.7	95.4
	6	78.3	90.9	83.4	74.7	90.5	79.2	94.9	88.4	79.7	93.7
	7	78.4	91.5	83.1	73.1	89.6	79.1	94.6	87.1	78.4	94.9
	8	77.6	89.7	80.7	73.6	89.8	78.6	95.3	86.7	78.7	95.3
	9	76.3	88.8	79.8	73.6	90.0	78.3	94.3	86.1	76.2	94.9
	10	76.0	89.9	79.5	72.7	91.6	77.2	95.4	85.7	75.8	94.1
(12, 7⋆0)	1	86.8	90.6	90.9	86.2	89.4	92.5	95.5	95.2	88.7	95.0
	2	87.4	89.4	89.7	83.6	89.2	92.3	95.5	95.9	85.9	94.3
	3	86.0	88.7	89.6	83.1	89.7	89.8	95.8	95.6	86.0	94.4
	4	87.3	90.4	91.3	84.9	90.0	92.1	95.4	95.7	87.6	94.3
	5	84.5	90.7	90.7	83.5	91.1	90.8	96.2	96.4	85.6	96.5
	6	83.8	89.3	90.2	82.6	89.5	90.6	95.9	96.8	86.4	95.3
	7	83.1	89.4	89.7	82.2	90.0	89.1	94.7	94.9	85.4	96.0
	8	81.9	89.8	89.2	80.2	91.1	89.4	95.4	95.2	81.7	95.8
	9	83.7	91.9	91.9	78.4	88.9	88.6	94.8	94.2	84.5	94.6
	10	82.3	91.2	90.7	77.0	90.4	86.9	95.4	93.2	84.2	95.0
(6⋆0, 6, 6)	1	86.8	89.2	90.1	84.9	90.8	91.1	95.0	96.1	87.4	94.8
	2	85.7	89.7	91.4	80.3	88.6	91.2	95.4	95.2	86.4	95.1
	3	81.2	88.7	88.6	79.4	91.1	88.9	96.2	95.4	83.7	95.0
	4	79.4	90.1	87.3	79.4	91.4	85.0	96.7	93.1	80.7	95.7
	5	77.5	90.3	83.9	73.3	89.1	81.0	96.2	89.4	78.9	94.2
	6	73.9	91.6	82.7	74.0	89.6	80.1	96.5	88.5	76.3	94.8
	7	72.2	91.0	79.3	72.2	89.9	80.5	95.7	86.8	75.2	94.5
	8	73.3	91.8	80.1	74.4	91.0	77.7	97.2	86.0	74.3	95.3
	9	71.3	90.2	78.3	71.4	89.3	76.7	95.8	83.2	73.8	94.5
	10	67.1	88.7	75.5	70.3	89.1	75.4	95.7	82.7	76.3	93.2

10. Progressive Type-II Censoring and Results

- **Comments**:

10. Progressive Type-II Censoring and Results

- **Comments**:
  - The approximate CIs and the Studentized-*t* bootstrap CIs are both unsatisfactory in terms of coverage probabilities.
  - The percentile bootstrap method seems to be sensitive for small values of τ_1 and τ_2, the method does improve for larger sample size.
  - Among all three bootstrap methods, the adjusted percentile method seems to be the one with somewhat satisfactory coverage probabilities (not so for θ_1 when τ_1 is small).
  - Use the exact method whenever possible, and use the adjusted percentile method in case of large sample size when the computation of the exact CIs becomes difficult.

10. Progressive Type-II Censoring and Results

- **Optimal Sampling Scheme**:

10. Progressive Type-II Censoring and Results

- **Optimal Sampling Scheme**:

  With $(R_1, \ldots, R_r)$ as the progressive censoring scheme, we may consider the optimal choice of $\boldsymbol{R} = (R_1, \ldots, R_r)$, denoted by $\boldsymbol{R}^* = (R_1^*, \ldots, R_r^*)$.

- **Optimal Sampling Scheme**:

With $(R_1, \ldots, R_r)$ as the progressive censoring scheme, we may consider the optimal choice of $\boldsymbol{R} = (R_1, \ldots, R_r)$, denoted by $\boldsymbol{R}^* = (R_1^*, \ldots, R_r^*)$.

Variance Optimality

$$\psi(\boldsymbol{R}) = \mathrm{Var}(\hat{\theta}_1 + \hat{\theta}_2) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot \left(\tau_{ik}^2 + \frac{\theta_1^2}{i} + \frac{\theta_2^2}{r-i} \right) - \left(D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot \tau_{ik} \right)^2 .$$

10. Progressive Type-II Censoring and Results

- **Optimal Sampling Scheme**:

  With $(R_1, \ldots, R_r)$ as the progressive censoring scheme, we may consider the optimal choice of $\boldsymbol{R} = (R_1, \ldots, R_r)$, denoted by $\boldsymbol{R}^* = (R_1^*, \ldots, R_r^*)$.

  Variance Optimality

$$\psi(\boldsymbol{R}) = \mathrm{Var}(\hat{\theta}_1 + \hat{\theta}_2) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot \left(\tau_{ik}^2 + \frac{\theta_1^2}{i} + \frac{\theta_2^2}{r-i} \right) - \left(D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot \tau_{ik} \right)^2 .$$

  MSE Optimality

$$\varphi(\boldsymbol{R}) = \mathrm{MSE}(\hat{\theta}_1) + \mathrm{Var}(\hat{\theta}_2) = D \sum_{i=1}^{r-1} \sum_{k=0}^{i} \sum_{l=0}^{r-i-1} D_{ikl} \cdot \left(\tau_{ik}^2 + \frac{\theta_1^2}{i} + \frac{\theta_2^2}{r-i} \right).$$

10. Progressive Type-II Censoring and Results

- Determination of Optimal Censoring Schemes:

10. Progressive Type-II Censoring and Results

- **Determination of Optimal Censoring Schemes**:

  For $\theta_1 = e^{1.5}$ and $\theta_2 = e^{0.5}$, we present in Tables 8 and 9 the best and worst censoring schemes determined under variance optimality and mean square error optimality, respectively, for different choices of $n,\ r$ and τ.

10. Progressive Type-II Censoring and Results

- **Determination of Optimal Censoring Schemes**:

  For $\theta_1 = e^{1.5}$ and $\theta_2 = e^{0.5}$, we present in Tables 8 and 9 the best and worst censoring schemes determined under variance optimality and mean square error optimality, respectively, for different choices of $n,\ r$ and τ.

  The relative efficiency values of worst to best censoring schemes presented in these two tables reveal the distinct advantage of adopting an optimal censoring scheme in the simple step-stress life-test.

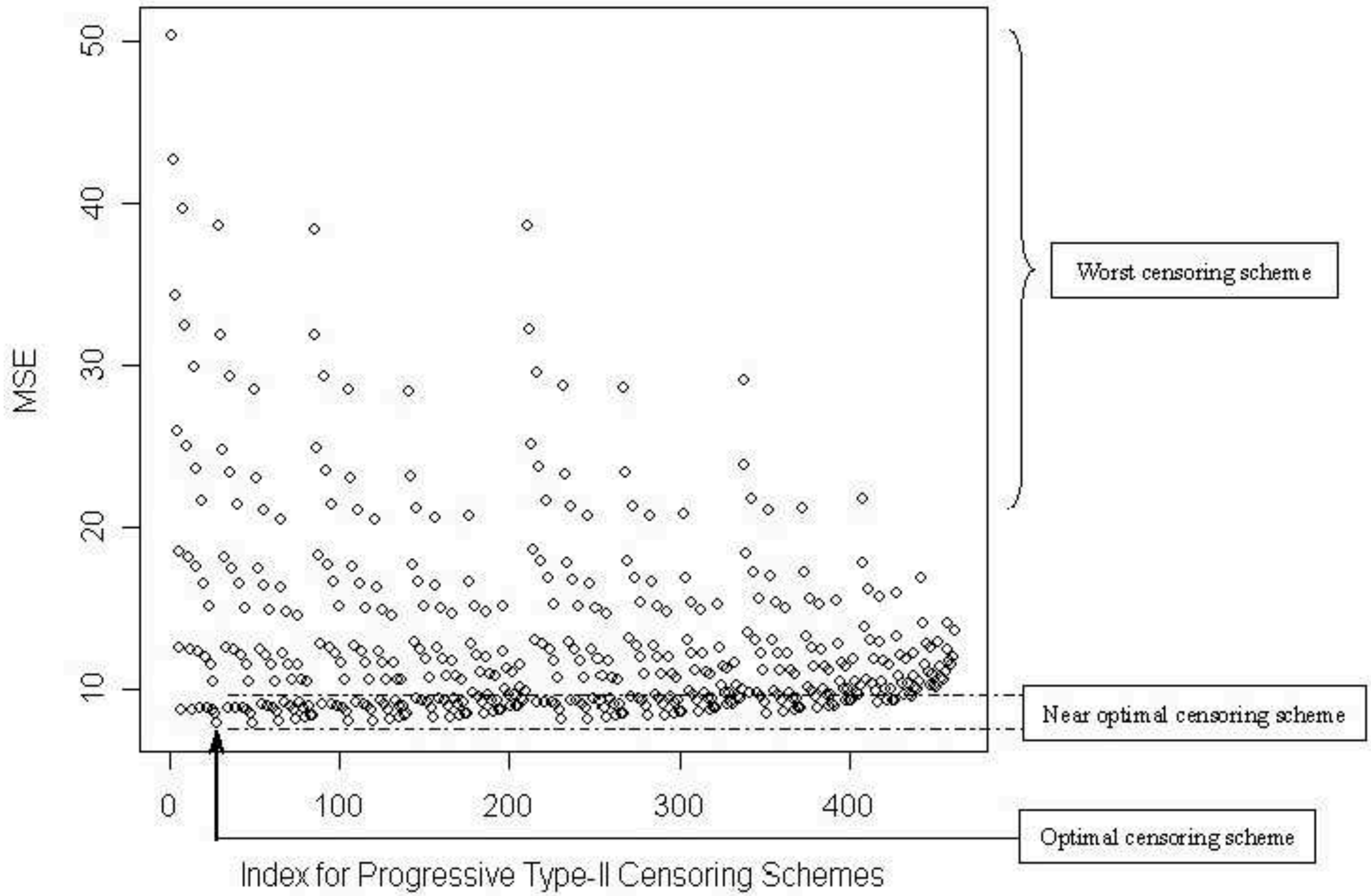
Table 8: Optimal Censoring Schemes under Variance Optimality

n	r	τ	Best PCS	Var	Worst PCS	Var	RE
10	4	1	(6, 3⋆0)	6.67	(0, 6, 2⋆0)	11.48	58%
		3	(0, 6, 2⋆0)	16.20	(3⋆0, 6)	27.33	59%
		5	(2⋆0, 6, 0)	11.10	(3⋆0, 6)	28.76	39%
		7	(2⋆0, 6, 0)	9.58	(3⋆0, 6)	26.00	37%
		9	(2⋆0, 6, 0)	9.43	(3⋆0, 6)	22.82	41%
	6	1	(4, 5⋆0)	6.97	(0, 4, 4⋆0)	10.13	69%
		3	(0, 4, 4⋆0)	14.90	(4, 5⋆0)	19.74	75%
		5	(3⋆0, 4, 2⋆0)	8.50	(4, 5⋆0)	13.99	61%
		7	(3⋆0, 4, 2⋆0)	6.99	(4, 5⋆0)	10.11	69%
		9	(3⋆0, 3, 0, 1)	6.74	(4, 5⋆0)	8.41	80%
	8	1	(2, 7⋆0)	7.90	(0, 2, 6⋆0)	9.40	84%
		3	(2⋆0, 2, 5⋆0)	14.18	(2, 7⋆0)	16.82	84%
		5	(3⋆0, 2, 4⋆0)	7.40	(2, 7⋆0)	9.45	78%
		7	(4⋆0, 1, 2⋆0, 1)	5.71	(2, 7⋆0)	6.73	85%
		9	(7⋆0, 2)	5.28	(1, 5⋆0, 1, 0)	5.91	89%
12	6	1	(6, 5⋆0)	8.87	(0, 6, 4⋆0)	13.21	67%
		3	(2⋆0, 6, 3⋆0)	11.75	(6, 5⋆0)	19.40	61%
		5	(3⋆0, 6, 2⋆0)	7.22	(6, 5⋆0)	13.38	54%
		7	(3⋆0, 5, 0, 1)	6.70	(6, 5⋆0)	9.86	68%
		9	(3⋆0, 4, 0, 2)	6.66	(6, 5⋆0)	8.30	80%
	8	1	(4, 7⋆0)	9.80	(0, 4, 6⋆0)	12.46	79%
		3	(2⋆0, 4, 5⋆0)	11.01	(4, 7⋆0)	16.11	68%
		5	(4⋆0, 3, 2⋆0, 1)	6.05	(4, 7⋆0)	8.96	68%
		7	(4⋆0, 2, 2⋆0, 2)	5.23	(4, 7⋆0)	6.59	79%
		9	(4⋆0, 1, 2⋆0, 3)	5.08	(3, 5⋆0, 1, 0)	5.88	86%
	10	1	(2, 9⋆0)	10.82	(2⋆0, 2, 7⋆0)	12.01	90%
		3	(3⋆0, 2, 6⋆0)	10.61	(2, 9⋆0)	12.91	82%
		5	(9⋆0, 2)	5.40	(2, 9⋆0)	6.47	83%
		7	(9⋆0, 2)	4.46	(8⋆0, 2, 0)	5.08	88%
		9	(9⋆0, 2)	4.29	(8⋆0, 2, 0)	4.98	86%

Table 9: Optimal Censoring Schemes under MSE Optimality

n	r	τ	Best PCS	MSE	Worst PCS	MSE	RE (%)
10	4	1	(6, 3⋆0)	6.69	(0, 6, 2⋆0)	11.62	57%
		3	(0, 6, 2⋆0)	17.47	(3⋆0, 6)	68.45	26%
		5	(2⋆0, 6, 0)	14.02	(3⋆0, 6)	150.75	9%
		7	(2⋆0, 6, 0)	15.04	(3⋆0, 6)	274.81	5%
		9	(2⋆0, 6, 0)	18.43	(3⋆0, 6)	444.74	4%
	6	1	(4, 5⋆0)	7.06	(0, 4, 4⋆0)	10.40	68%
		3	(2⋆0, 4, 3⋆0)	15.67	(5⋆0, 4)	21.87	72%
		5	(3⋆0, 4, 2⋆0)	9.26	(5⋆0, 4)	26.67	35%
		7	(3⋆0, 4, 2⋆0)	8.54	(5⋆0, 4)	45.33	19%
		9	(3⋆0, 4, 2⋆0)	9.62	(5⋆0, 4)	75.64	13%
	8	1	(2, 7⋆0)	8.24	(0, 2, 6⋆0)	9.82	84%
		3	(2⋆0, 2, 5⋆0)	14.94	(2, 7⋆0)	18.06	83%
		5	(3⋆0, 2, 4⋆0)	7.74	(2, 7⋆0)	9.96	78%
		7	(4⋆0, 2, 3⋆0)	6.31	(7⋆0, 2)	10.46	60%
		9	(4⋆0, 2, 3⋆0)	6.53	(7⋆0, 2)	15.21	43%
12	6	1	(6, 5⋆0)	9.21	(2⋆0, 6, 3⋆0)	14.00	66%
		3	(2⋆0, 6, 3⋆0)	12.19	(5⋆0, 6)	24.64	49%
		5	(3⋆0, 6, 2⋆0)	7.94	(5⋆0, 6)	43.25	18%
		7	(3⋆0, 6, 2⋆0)	8.37	(5⋆0, 6)	84.83	10%
		9	(2, 2⋆0, 4, 2⋆0)	9.63	(5⋆0, 6)	146.24	7%
	8	1	(4, 7⋆0)	10.51	(2⋆0, 4, 5⋆0)	13.45	78%
		3	(2⋆0, 4, 5⋆0)	11.46	(4, 7⋆0)	17.22	67%
		5	(4⋆0, 4, 3⋆0)	6.31	(7⋆0, 4)	11.96	53%
		7	(4⋆0, 4, 3⋆0)	5.97	(7⋆0, 4)	20.72	29%
		9	(1, 2⋆0, 1, 2, 3⋆0)	6.51	(7⋆0, 4)	36.08	18%
	10	1	(2, 9⋆0)	11.83	(2⋆0, 2, 7⋆0)	13.12	90%
		3	(3⋆0, 2, 6⋆0)	11.09	(2, 9⋆0)	13.70	81%
		5	(4⋆0, 2, 5⋆0)	5.57	(2, 9⋆0)	6.72	83%
		7	(5⋆0, 2, 4⋆0)	4.83	(9⋆0, 2)	6.58	73%
		9	(5⋆0, 2, 4⋆0)	5.03	(9⋆0, 2)	9.35	54%

10. Progressive Type-II Censoring and Results



11. Other Extensions and Generalizations

- Some other extensions and generalizations have been carried out:

11. Other Extensions and Generalizations

- Some other extensions and generalizations have been carried out:
  - Other forms of censoring, such as hybrid censoring, have been considered;
  - Extensions of these results to the multiple-step stress model have been done.

12. Further Problems of Interest

- Generalization to k-step stress model and discuss exact conditional inference for the full-parameter model;

12. Further Problems of Interest

- Generalization to k-step stress model and discuss exact conditional inference for the full-parameter model;
- With a link function (connecting the mean lifetimes to the stress levels), discuss inference for the parameters in this reduced-parameter model and compare efficiency;

12. Further Problems of Interest

- Generalization to k-step stress model and discuss exact conditional inference for the full-parameter model;
- With a link function (connecting the mean lifetimes to the stress levels), discuss inference for the parameters in this reduced-parameter model and compare efficiency;
- Test for the suitability of this reduced-parameter model;

12. Further Problems of Interest

- Generalization to k-step stress model and discuss exact conditional inference for the full-parameter model;
- With a link function (connecting the mean lifetimes to the stress levels), discuss inference for the parameters in this reduced-parameter model and compare efficiency;
- Test for the suitability of this reduced-parameter model;
- Generalizations to other life-time models such as Weibull and lognormal.

This talk is based on

[1] Balakrishnan, N., Kundu, D., Ng, H. K. T. and Kannan, N. (2006). Point and interval estimation for a simple step-stress model with Type-II censoring,
Journal of Quality Technology (to appear).

[2] Balakrishnan, N., Xie, Q. and Kundu, D. (2006). Exact inference for a simple step-stress model from the exponential distribution under time constraint,
Ann. Inst. Statist. Math. (to appear).

[3] Balakrishnan, N. and Xie, Q. (2006a). Exact inference for a simple step-stress model with Type-I hybrid censored data from the exponential distribution,
J. Statist. Plann. Inf. (to appear).

[4] Balakrishnan, N. and Xie, Q. (2006b). Exact inference for a simple step-stress model with Type-II hybrid censored data from the exponential distribution,
J. Statist. Plann. Inf. (to appear).

[5] Balakrishnan, N., Xie, Q. and Han, D. (2006). Exact inference and optimal censoring scheme for a simple step-stress model under progressive Type-II censoring,
(Submitted for publication).

[6] Gouno, E., Sen, A. and Balakrishnan, N. (2004). Optimal step-stress test under progressive Type-I censoring,
IEEE Trans. on Reliab., **53**, 383-393.

13. Bibliography

Arnold, B. C., Balakrishnan, N. and Nagaraja, H.N.(1992). *A First Course in Order Statistics*, John Wiley & Sons, New York.

Bagdonavicius, V. and Nikulin, M. (2002). *Accelerated Life Models: Modeling and Statistical Analysis*, Chapman and Hall/CRC Press, Boca Raton, Florida.

Bai, D. S., Kim, M. S. and Lee, S. H. (1989). Optimum simple step-stress accelerated life test with censoring, *IEEE Transactions on Reliability*, **38**, 528-532.

Balakrishnan, N. and Aggarwala, R. (2000). *Progressive Censoring: Theory, Methods, and Applications*. Birkhäuser, Boston.

Bhattacharyya, G. K. and Zanzawi, S. (1989). A tampered failure rate model for step-stress accelerated life test, *Communications in Statistics – Theory and Methods*, **18**, 1627-1643.

DeGroot, M. H. and Goel, P. K. (1979). Bayesian estimation and optimal design in partially accelerated life testing, *Naval Research Logistics Quarterly*, **26**, 223-235.

Efron, B. and Tibshirani, R. (1993). *An Introduction to the Booststrap*, Chapman and Hall, New York.

13. Bibliography

Khamis, I. H. and Higgins, J. J. (1998). A new model for step-stress testing, *IEEE Transactions on Reliability*, **47**, 131-134.

Madi, M. T. (1993). Multiple step-stress accelerated life test: the tampered failure rate model, *Communications in Statistics – Theory and Methods*, **22**, 2631-2639.

Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*, John Wiley & Sons, New York.

Nelson, W. (1980). Accelerated life testing: step-stress models and data analysis, *IEEE Transactions on Reliability*, **29**, 103-108.

Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans, and Data Analyses*, John Wiley & Sons, New York.

Xiong, C. (1998). Inference on a simple step-stress model with Type-II censored exponential data, *IEEE Transactions on Reliability*, **47**, 142-146.

Xiong, C. and Milliken, G.A. (1999). Step-stress life-testing with random stress change times for exponential data, *IEEE Transactions on Reliability*, **48**, 141-148.