Graphical Modelling

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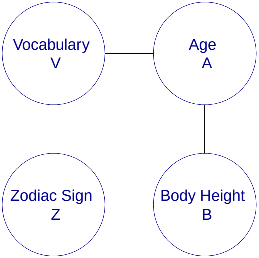
Introduction

- What is a graphical model ...
- ... and when is a model a “graphical” model?
- What are graphical models so famous for?
  - intuitive properties of graphical illustrations
  - graphical properties allow for keen reduction of model complexity
- The magic that allows for all that is ...
  - conditional independence

Independence I

- Conditional Independence
  - V and B are conditionally independent given A, i.e. keeping A constant, the relationship between B and V vanishes
- Independence
  - V, A and B are independent of Z given the "empty set", i.e. A is independent of Z, as are V and B

- Example:
  $A \perp Z \mid ()$,
  $V \perp Z \mid ()$,
  $B \perp Z \mid ()$,
  $B \perp V \mid (rest) i.e. (A, Z)$



Independence II

- Independence:
  A and B are independent if
  $P(A \cap B) = P(A)P(B)$ or $P(A|B) = P(A)$
  $f_{X,Y}(x, y) = f_X(x)f_Y(y)$ or $f_{Y|X}(y|x) = f_Y(y)$
- Conditional Independence:
  X and Y are conditionally independent given Z if
  $f_{X|Y,Z}(x|y, z) = f_{X,Z}(x, z)$
  written as $X \perp Y | Z$

Types of Graphical Models

- Undirected graphical models
  - i.e. Markov graphical models
    - discrete models
    - continuous models
    - mixed models
- Directed graphical models
  - i.e. Bayesian graphical models
    - discrete models
    - continuous models
    - mixed models

Notations

- Independence notation
  - Conditional independence
    - $V \perp B | A$
  - Independence
    - $V \perp Z$ and $A \perp Z$ and $B \perp Z$
- Model notation
  - [VA] [AB] [Z]

Graphical & Non-Graphical Models

- Graphical models
  - *Def.: A model is graphical if, whenever the model contains all two-factor terms generated by a higher-order interaction, the model also contains the higher order interaction.*
  - i.e. if [12] [23] [13] must include [123]
  - therefore [123] is the graphical model
- Non-graphical model
  - [12] [23] [34] [41] [24] is not graphical ...
  - ... [124] [243] is the graphical model
  - [12] [23] [34] [41] [24] [13] is not graphical ...
  - ... [1234] is the graphical model

Graphical or Not?

- [123] [124]
- [12] [23] [34] [41]
- [12] [23] [34] [41] [42]
- [123] [34] [42] [456]

Maximal Cliques

- Characteristics
- A clique is complete set (subgraph).
- A maximal clique is a clique where no other node can be added such that the subset is still a clique.
- The maximal cliques of a graph determine the graphical model.
- Every graph has a graphical model but not every model is a graphical model. (to be discussed ...)

Chains and Circles

- Chain
  - ... is a sequence of edges from one vertex to another
  - ... must include all intermediate vertexes
  - ... a chain from 1 to 2 and back to 1 is call a "degenerated chain"
  - Chains account for marginal independence
- Circle
  - ... is a closed chain
  - ... a chain with the same start and end point.
  - ... not allowed to use a vertex more then once.
  - ... a circle has at least length 3

Pairwise Markov Property

- In a Markov graph the absence of an edge implies that the corresponding random variables are conditionally independent given the variables at the other vertexes.
- altern. Any non-adjacent pair of variables are conditionally independent given the rest
- No edge joining X and Y ... it follows ... $X \perp Y | (rest)$

Local Markov Property

- Every variable is conditionally independent of the remaining, given it's neighbors

Global Markov Property

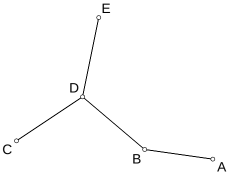
- Theorem
  - *Let the sets A,B, and C denote disjoint subsets of the factors in a graphical model. The factors in A are independent of the factors in B given C iff every chain between a factor in A and a factor in B involves at least one factor in C. (proven)*
  - ... also implies marginal independence relations.
  - ... breaks the graph into conditionally independent pieces.
  - ... i.e. increases efficiency, decreases complexity!

Collapsing Three-Factor Tables

- a) If the model [13] [23] holds, then the relationship between factors 2 and 3 can be examined in the marginal table $n_{.jk}$ and the relationship between factors 1 and 3 can be examined in the marginal table $n_{i.k}$.
- b) If either model [13] [23] or [12] [23] holds, then the relationship between factors 2 and 3 can be examined in the marginal table $n_{.jk}$.
- c) If the model [1] [23] holds, then the relationship between factors 2 and 3 can be examined in the marginal table $n_{.jk}$.

Applications of Graphical Models to Relational Data (Networks)

- Nodes represent actors
- Edges represent relations between actors




```
> network

  A B C D E
A 0 1 0 0 0
B 1 0 0 1 0
C 0 0 0 1 0
D 0 0 1 0 1
E 0 0 0 1 0
```

Network Models

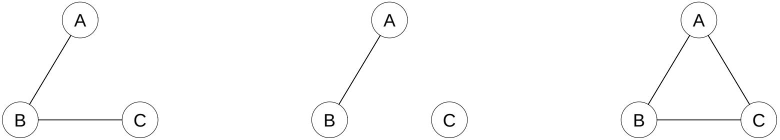
- Bernoulli Graphs: Edges on a fixed node set exist with a certain constant probability p.
  - $P(X = x) = \frac{1}{\kappa(\theta)} \exp(\theta L(x))$
    where
    X is a random process generating a network of fixed size,
    x is the observed network,
    $\kappa(\theta) = \sum_X \exp(\theta L(x))$ is a normalizing contant,
    θ is a parameter, and
    $L(x)$ is the number of edges in x.
  - Example in R:

```
> library(ergm)
> data(florentine)
> summary(ergm(flomarriage ~ edges))
```

    Interpretation: Significant negative parameter indicates that the number of observed edges is smaller than expected.

Edge Independence

- Assumption of independence between edges is unrealistic.
- Structural balance.



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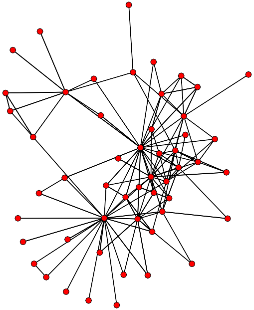
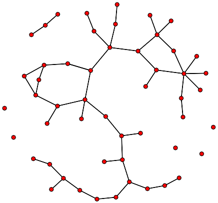
Real Graph vs. Bernoulli Graph (Equal Size)



A real graph: Short average path length and strong tendency for clustering.

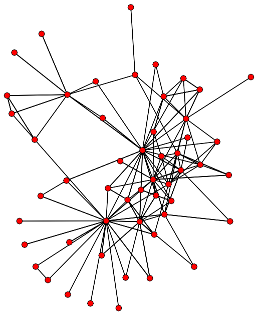


A Bernoulli random graph: Short average path length but low tendency for clustering.

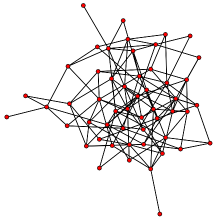
Markov Graphs

- Using the concept of grafical models to model social networks
  - Nodes are no longer variables
  - but the concept of subgraph decomposition is kept.
- Markov Graphs: Dependence between edges.
  - $P(X = x) = \frac{1}{\kappa} \exp\left(\sum_A \lambda_A z_A(x)\right)$
    where
    A is a subgraph of X (dyad, triad...),
    $z_A(x) = \prod_{ij \in A} x_{ij}$ is the "network statistic" of A,
    κ is a normalizing constant, and
    λ_A is a parameter vector.

Real Graph vs. Markov Random Graph (Equal Size)



A real graph.



A Markov random graph.

Markov Graphs in R: The `ergm` Package.

- Example in R:

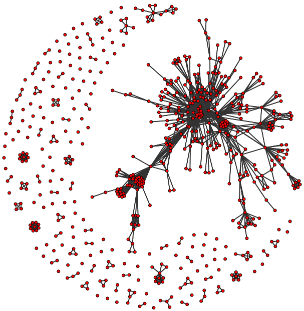
  > *summary(ergm(flomarriage ~ triangle))*

  Interpretation: Significant negative parameter indicates that there are less triangles (smaller propensity to clustering) than expected.

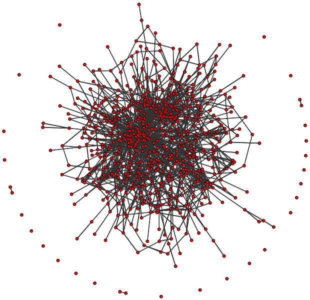
Extensions

- Other subgraph configurations:
  - k-stars: Graphs with one node having a degree of k and k nodes having a degree of one.
  - k-cycles: Paths of length k that start and end at the same node.
- Nodal attributes: E.g. Wealth.
- Edge covariages: Edge weights.
- Directed networks.
  - Receiver, sender ,mutuality
  - Triad census: In directed triads there are 16 possible ways how the three nodes can be connected. Which of these occure more/less often than expected?
- R: See `ergm-terms`.

R-Forge Collaboration Network



Real



Simulated with edge parameter -4.775016.

References

- David Edwards. Introduction to Graphical Modelling. 2000. Springer.
- Trevor Hastie. The Elements of Statistical Learning. 2008. Springer.
- Peter J. Carrington et al. Models and Methods in Social Network Analysis. 2009. Cambridge.