Bridging Course Mathematics

(MSc Economics)

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Introduction

Prior Knowledge

Prior knowledge in mathematics (and statistics) of students of the master programme in *economics* differ heavily:

- Students with courses in mathematics with a total of 25 ECTS points (or more) in their bachelor programme.
- Students who did not attend any mathematics course at all.

Prior knowledge differ in

- Basic *skills* (like computations with “symbols”)
- *Tools* (like methods for optimization)
- Mathematical *reasoning* (proving your claim)

Knowledge Gap

The following problems cause issues for quite a few students:

- Drawing (or sketching) of graphs of functions.
- Transform equations into equivalent ones.
- Handling inequalities.
- Correct handling of fractions.
- Calculations with exponents and logarithms.
- Obstructive multiplying of factors.
- Usage of mathematical notation.

Presented *"solutions"* of such calculation subtasks are surprisingly often *wrong*.

Learning Objectives

This **bridging course** is intended to help participants to

- *close* possible knowledge gaps, and
- *raise* prior knowledge in **basic** mathematical **skills** to the *same higher* level.

Further courses:

- *Foundations of Mathematics* (Msc Economics):

  Essential mathematical *tools*.
  (matrix algebra, Taylor series, implicit functions, static optimization, Hessian, Lagrange multiplier, difference equations, . . .)

- *Mathematics 1 and 2* (science track only):

  Advanced (new) tools and mathematical reasoning.

Learning Methods

- *Revision* of mathematical notions and concepts by the instructor.
- Solve problems *collectively* during the course.
- Solve *homework problems*.
  Solutions are discussed during the next course.
- The subject matter may not be presented in a linear way.
- There will be *no exams*.
- For a positive grade (*"erfolgreich teilgenommen"*) you have to be **present** in at least **8 units**.

Solutions of Problems

- A problem is **solved** when the problem question is *answered*.
- It is *not sufficient* when you just present the computations that are *necessary* to answer the question.
- In particular, fragments of computations that start and end at some point are not considered as correct solution of a (homework) problem.
- You have to show that you can draw the right conclusions from your computations.

Maxima – Computer Algebra System (CAS)

Maxima is a so called **Computer Algebra System** (CAS),
i.e., one can

- handle algebraic expressions,
- solve (in-) equalities with parameters,
- differentiate and integrate analytically,
- handle abstract matrices,
- plot univariate and bivariate functions,
- …

Program *wxMaxima* provides a GUI:

`http://wxmaxima.sourceforge.net/`

The manuscript *Introduction to Maxima for Economics* can be downloaded from the webpage of this course.

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May you do well!

Chapter 1

Sets and Maps

Set

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

A **set** is a collection of *distinct* objects.

An object a of a set A is called an **element** of the set. We write:

$$\boxed{a \in A}$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\ldots\}$.

$A = \{1, 2, 3, 4, 5, 6\} \qquad B = \{x \mid x \text{ is an integer divisible by 2}\}$

Important Sets

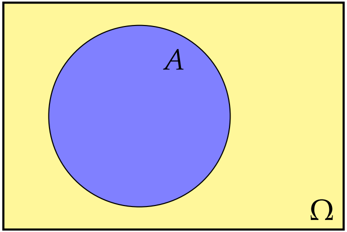
Symbol	Description
$\emptyset$	empty set sometimes: $\{\}$
$\mathbb{N}$	natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	rational numbers $\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$
$\mathbb{R}$	real numbers
$[a, b]$	closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a, b)	open interval[a] $\{x \in \mathbb{R} \mid a < x < b\}$
$[a, b)$	half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$
$\mathbb{C}$	complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$

[a] also: $]a, b[$

Venn Diagram

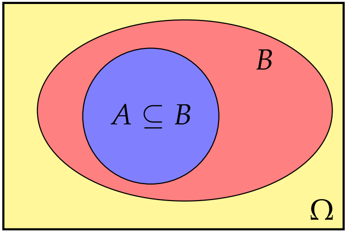
We assume that all *sets* are subsets of some universal **superset** Ω.

Sets can be represented by **Venn diagrams** where Ω is a rectangle and sets are depicted as circles or ovals.



Subset and Superset

Set A is a **subset** of B, $\boxed{A \subseteq B}$, if each element of A is also an element of B, i.e., $x \in A \Rightarrow x \in B$.



Vice versa, B is then called a **superset** of A, $\boxed{B \supseteq A}$.

Set A is a **proper subset** of B, $\boxed{A \subset B}$ (or: $A \subsetneq B$),
if $A \subseteq B$ and $A \neq B$.

Problem 1.1

Which of the the following sets is a subset of

$$A = \{x \mid x \in \mathbb{R} \text{ and } 10 < x < 200\}$$

(a) $\{x \mid x \in \mathbb{R} \text{ and } 10 < x \leq 200\}$

(b) $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 121\}$

(c) $\{x \mid x \in \mathbb{R} \text{ and } 4\pi < x < \sqrt{181}\}$

(d) $\{x \mid x \in \mathbb{R} \text{ and } 20 < |x| < 100\}$

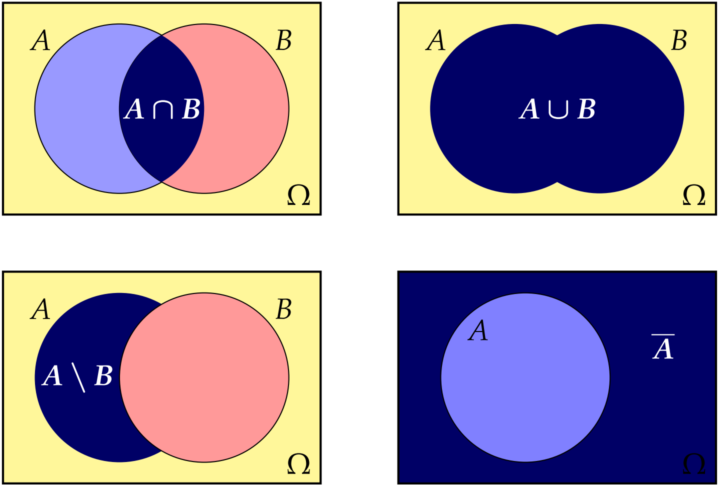
Basic Set Operations

Symbol	Definition	Name
$A \cap B$	$\{x \mid x \in A$ and $x \in B\}$	**intersection**
$A \cup B$	$\{x \mid x \in A$ or $x \in B\}$	**union**
$A \setminus B$	$\{x \mid x \in A$ and $x \notin B\}$	**set-theoretic difference**[a]
$\overline{A}$	$\Omega \setminus A$	**complement**

[a] also: $A - B$

Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Basic Set Operations



Problem 1.2

The set $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ has subsets $A = \{1,3,6,9\}$, $B = \{2,4,6,10\}$ and $C = \{3,6,7,9,10\}$.

Draw the Venn diagram and give the following sets:

(a) $A \cup C$

(b) $A \cap B$

(c) $A \setminus C$

(d) $\overline{A}$

(e) $(A \cup C) \cap B$

(f) $(\overline{A} \cup B) \setminus C$

(g) $\overline{(A \cup C)} \cap B$

(h) $(\overline{A} \setminus B) \cap (\overline{A} \setminus C)$

(i) $(A \cap B) \cup (A \cap C)$

Problem 1.3

Mark the following set in the corresponding Venn diagram:

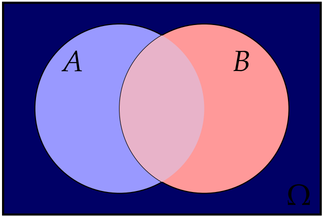
$$(A \cap \overline{B}) \cup (A \cap B)$$

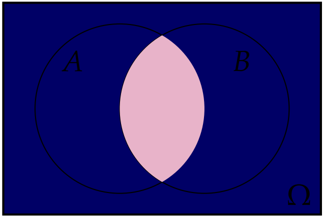
Rules for Basic Operations

Rule	Name
$A \cup A = A \cap A = A$	Idempotence
$A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$	Identity
$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$	Associativity
$A \cup B = B \cup A$ and $A \cap B = B \cap A$	Commutativity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributivity
$\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$	

De Morgan's Law

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$





Problem 1.4

Simplify the following set-theoretic expression:

$$(A \cap \overline{B}) \cup (A \cap B)$$

Problem 1.5

Simplify the following set-theoretic expressions:

(a) $\overline{(A \cup B)} \cap \overline{B}$

(b) $(A \cup \overline{B}) \cap (A \cup B)$

(c) $((\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{B})}) \cap A$

(d) $(C \cup B) \cap \overline{(\overline{C} \cap \overline{B})} \cap (C \cup \overline{B})$

Cartesian Product

The set

$$A \times B = \{(x, y) | x \in A, y \in B\}$$

is called the **Cartesian product** of *sets* A and B.

Given two sets A and B the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set A and the second element is from set B.

In general we have $\quad A \times B \neq B \times A$.

Cartesian Product

The Cartesian product of $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ is

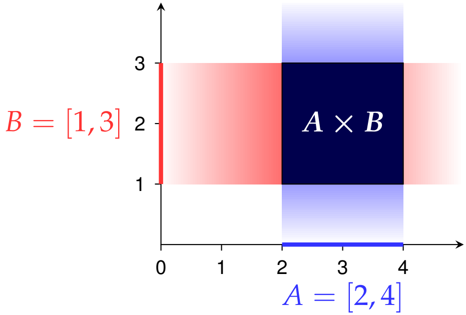
$$A \times B = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}.$$

$A \times B$	2	3	4
0	$(0,2)$	$(0,3)$	$(0,4)$
1	$(1,2)$	$(1,3)$	$(1,4)$

Cartesian Product

The Cartesian product of $A = [2, 4]$ and $B = [1, 3]$ is

$$A \times B = \{(x, y) \mid x \in [2, 4] \text{ and } y \in [1, 3]\}.$$



Problem 1.6

Describe the Cartesian products of

(a) $A = [0, 1]$ and $P = \{2\}$.

(b) $A = [0, 1]$ and $Q = \{(x, y) \colon 0 \leq x, y \leq 1\}$.

(c) $A = [0, 1]$ and $O = \{(x, y) \colon 0 < x, y < 1\}$.

(d) $A = [0, 1]$ and $C = \{(x, y) \colon x^2 + y^2 \leq 1\}$.

(e) $A = [0, 1]$ and $\mathbb{R}$.

(f) $Q_1 = \{(x, y) \colon 0 \leq x, y \leq 1\}$ and $Q_2 = \{(x, y) \colon 0 \leq x, y \leq 1\}$.

Map

A **map** (or **mapping**) f is defined by

(i) a **domain** D_f,

(ii) a **codomain** (**target set**) W_f and

(iii) a **rule**, that maps each element of D to *exactly one* element of W.

$$f\colon D \to W, \quad x \mapsto y = f(x)$$

- x is called the **independent** variable, y the **dependent** variable.
- y is the **image** of x, x is the **preimage** of y.
- $f(x)$ is the **function term**, x is called the **argument** of f.
- $f(D) = \{y \in W\colon y = f(x) \text{ for some } x \in D\}$
  is the **image** (or **range**) of f.

Other names: *function*, *transformation*

Problem 1.7

We are given map

$$\varphi\colon [0,\infty) \to \mathbb{R},\ x \mapsto y = x^\alpha \qquad \text{for some } \alpha > 0$$

What are

- function name,
- domain,
- codomain,
- image (range),
- function term,
- argument,
- independent and dependent variable?

Injective · Surjective · Bijective

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

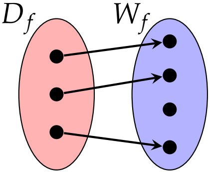
- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Injections have the important property

$$f(x) \neq f(y) \quad \Leftrightarrow \quad x \neq y$$

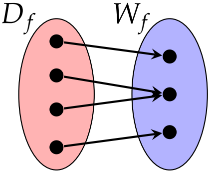
Injective · Surjective · Bijective

Maps can be visualized by means of arrows.



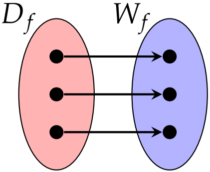
one-to-one

(not onto)



onto

(not one-to-one)

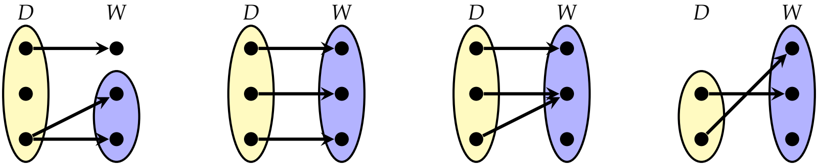


one-to-one and onto

(bijective)

Problem 1.8

Which of these diagrams represent maps?
Which of these maps are one-to-one, onto, both or neither?



(a) (b) (c) (d)

Problem 1.9

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

(a) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^2$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^{-2}$

(c) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto x^2$

(d) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$

(e) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$

(f) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \{y \in [0,\infty)\colon x = y^2\}$

Problem 1.10

Let $\mathcal{P}_n = \{\sum_{i=0}^{n} a_i x^i \colon a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n.

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

(a) $D\colon \mathcal{P}_n \to \mathcal{P}_n,\ p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of p)

(b) $D\colon \mathcal{P}_n \to \mathcal{P}_{n-1},\ p(x) \mapsto \frac{dp(x)}{dx}$

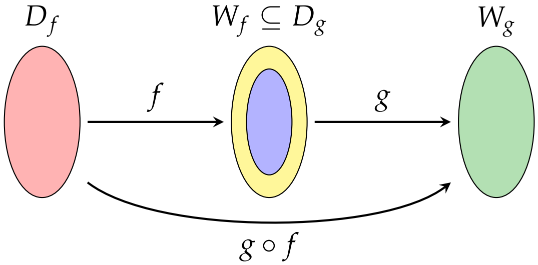
(c) $D\colon \mathcal{P}_n \to \mathcal{P}_{n-2},\ p(x) \mapsto \frac{dp(x)}{dx}$

Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.
Function

$$g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function**.
(read: “g composed with f”, “g circle f”, or “g after f”)



Inverse Map

If $f\colon D_f \to W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.

Thus we get a map

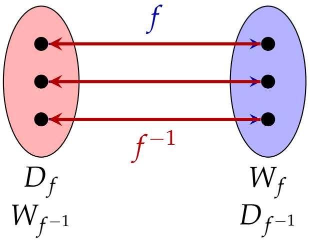
$$f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

which is called the **inverse map** of f.

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\ .$$

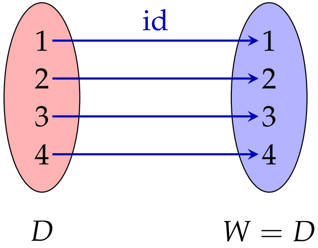
Inverse Map



Identity

The most elementary function is the **identity map** id, which maps its argument to itself, i.e.,

$$\mathrm{id}\colon D \to W = D,\ x \mapsto x$$



Identity

The identity map has a similar role for compositions of functions as 1 has for multiplications of numbers:

$$f \circ \mathrm{id} = f \qquad \text{and} \qquad \mathrm{id} \circ f = f$$

Moreover,

$$f^{-1} \circ f = \mathrm{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}\colon W_f \to W_f$$

Real-valued Functions

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$f : \mathbb{R} \to \mathbb{R}, \; x \mapsto f(x)$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.

Summary

- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan’s law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity

Chapter 2

Terms

Term

A mathematical expression like

$$B = R \cdot \frac{q^n - 1}{q^n(q-1)} \qquad \text{or} \qquad (x+1)(x-1) = x^2 - 1$$

contains symbols which denote mathematical objects.
These symbols and compositions of symbols are called **terms**.

Terms can be

- **numbers**,
- **constants** (symbols, which represent *fixed* values),
- **variables** (which are placeholders for *arbitrary* values), and
- **compositions** *of terms*.

Domain

We have to take care that a term may not be defined for some values of its variables.

- $\frac{1}{x-1}$ is only defined for $x \in \mathbb{R} \setminus \{1\}$.
- $\sqrt{x+1}$ is only defined for $x \geq -1$.

The set of values for which a term is defined is called the **domain** of the *term*.

Sigma Notation

Sums with many terms that can be generated by some rule can be represented in a compact form called *summation* or **sigma notation**.

$$\sum_{n=1}^{6} a_n = a_1 + a_2 + \cdots + a_6$$

a_n ... formula for the terms

n ... index of summation

1 ... first value of index n

6 ... last value of n

The expression is read as the "sum of a_n as n goes from 1 to 6."

First and last value can (and often are) given by *symbols*.

Sigma Notation

The sum of the first 10 integers greater than 2 can be written in both notations as

$$\sum_{i=1}^{10}(2+i) = (2+1) + (2+2) + (2+3) + \cdots + (2+10)$$

The sigma notation can be seen as a convenient shortcut of the long expression on the r.h.s.

It also avoids ambiguity caused by the *ellipsis* "..." that might look like an IQ test rather than an exact mathematical expression.

Sigma Notation – Rules

Keep in mind that all the *usual rules* for multiplication and addition (associativity, commutativity, distributivity) apply:

- $\sum_{i=1}^{n} a_i = \sum_{k=1}^{n} a_k$
- $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$
- $\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$

Sigma Notation – Rules

Simplify

$$\sum_{i=2}^{n}(a_i + b_i)^2 - \sum_{j=2}^{n} a_j^2 - \sum_{k=2}^{n} b_k^2$$

Solution:

$$\begin{aligned}
&= \sum_{i=2}^{n}(a_i + b_i)^2 - \sum_{i=2}^{n} a_i^2 - \sum_{i=2}^{n} b_i^2 \\
&= \sum_{i=2}^{n} \left((a_i + b_i)^2 - a_i^2 - b_i^2\right) \\
&= \sum_{i=2}^{n} 2\, a_i b_i \\
&= 2 \sum_{i=2}^{n} a_i b_i
\end{aligned}$$

Problem 2.1

Which of the following expressions is equal to the summation

$$\sum_{i=2}^{10} 5(i+3) \text{ ?}$$

(a) $5(2+3+4+\ldots+9+10+3)$

(b) $5(2+3+3+3+4+3+5+3+6+3+\ldots+10+3)$

(c) $5(2+3+4+\ldots+9+10)+5\cdot 3$

(d) $5(2+3+4+\ldots+9+10)+9\cdot 5\cdot 3$

Problem 2.2

Compute and simplify:

(a) $$\sum_{i=0}^{5} a^i b^{5-i}$$

(b) $$\sum_{i=1}^{5} (a_i - a_{i+1})$$

(c) $$\sum_{i=1}^{n} (a_i - a_{i+1})$$

Remark: The sum in (c) is a so called *telescoping sum*.

Problem 2.3

Simplify the following summations:

(a) $$\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 - \sum_{k=1}^{n} (a_k - b_k)^2$$

(b) $$\sum_{i=1}^{n} (a_i b_{n-i+1} - a_{n-i+1} b_i)$$

(c) $$\sum_{i=1}^{n} (x_i + y_i)^2 + \sum_{j=1}^{n} (x_j - y_j)^2$$

(d) $$\sum_{j=0}^{n-1} x_j - \sum_{i=1}^{n} x_i$$

Problem 2.4

Arithmetic mean (“average”)

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

and *variance*

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

are important location and dispersion parameters in statistics.

Variance σ^2 can be computed by means of formula

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2$$

which requires to read data (x_i) only once.

Problem 2.4 / 2

Verify

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

for

(a) $n = 2$,

(b) $n = 3$,

(c) $n \geq 2$ arbitrary.

Hint: Show that

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 - \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) = 0.$$

Problem 2.5

Let μ be the "true" value of a metric variate and $\{x_i\}$ the results of some measurement with stochastic errors. Then

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

is called the *mean square error* of sample $\{x_i\}$.

Verify:

$$MSE = \sigma^2 + (\bar{x} - \mu)^2$$

i.e., the MSE is the sum of

- the variance of the measurement (*dispersion*), and
- the squared deviation of the sample mean from μ (*bias*).

Absolute value

The **absolute value** (or *modulus*) $|x|$ of a number x is its distance from origin 0 on the number line:

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

$|5| = 5$ and $|-3| = -(-3) = 3$.

We have

$$|x| \cdot |y| = |x \cdot y|$$

Problem 2.6

Find simple equivalent formulas for the following expressions without using absolute values:

(a) $|x^2 + 1|$

(b) $|x| \cdot x^3$

Find simpler expressions by means of absolute values:

(c) $\begin{cases} x^2, & \text{for } x \geq 0, \\ -x^2, & \text{for } x < 0. \end{cases}$

(d) $\begin{cases} x^\alpha, & \text{for } x > 0, \\ -(-x)^\alpha & \text{for } x < 0, \end{cases}$ for some fixed $\alpha \in \mathbb{R}$.

Power

The n-th **power** of x is defined by

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

- x is the **basis**, and
- n is the **exponent** of x^n.

Expression x^n is read as "x raised to the n-th power", "x raised to the power of n", or "the n-th power of x".

Example: $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

For *negative* exponents we define:

$$x^{-n} = \frac{1}{x^n}$$

Root

A number y is called the

- ***n*-th root** $\sqrt[n]{x}$ of x, if $y^n = x$.

Computing the n*-th root* can be seen as the inverse operation of computing a power.

We just write $\sqrt{x}$ for the *square root* $\sqrt[2]{x}$.

Beware:
Symbol $\sqrt[n]{x}$ is used for the **positive** (real) root of x.

If we need the negative square root of 2 we have to write $\boxed{-\sqrt{2}}$.

Powers with Rational Exponents

Powers with *rational exponents* are defined by

$$x^{\frac{1}{m}} = \sqrt[m]{x} \qquad \text{for } m \in \mathbb{Z} \text{ and } x \geq 0;$$

and

$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad \text{for } m, n \in \mathbb{Z} \text{ and } x \geq 0.$$

Important:
For non-integer exponents the basis must be non-negative!

Powers x^α can also be generalized for $\alpha \in \mathbb{R}$.

Calculation Rule for Powers and Roots

$$x^{-n} = \frac{1}{x^n} \qquad x^0 = 1 \qquad (x \neq 0)$$

$$x^{n+m} = x^n \cdot x^m \qquad x^{\frac{1}{m}} = \sqrt[m]{x} \qquad (x \geq 0)$$

$$x^{n-m} = \frac{x^n}{x^m} \qquad x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad (x \geq 0)$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}} \qquad (x \geq 0)$$

$$(x^n)^m = x^{n \cdot m}$$

Important!

0^0 is *not* defined!

Computations with Powers and Roots

- $\sqrt[3]{5^6} = (5^6)^{\frac{1}{3}} = 5^{(6\cdot\frac{1}{3})} = 5^{\frac{6}{3}} = 5^2 = 25$
- $(\sqrt[3]{5})^6 = (5^{\frac{1}{3}})^6 = 5^{(\frac{1}{3}\cdot 6)} = 5^{\frac{6}{3}} = 5^2 = 25$
- $5^{4-3} = 5^1 = 5$
- $5^{4-3} = \frac{5^4}{5^3} = \frac{625}{125} = 5$
- $5^{2-2} = 5^0 = 1$
- $5^{2-2} = \frac{5^2}{5^2} = \frac{25}{25} = 1$

Computations with Powers and Roots

- $$\frac{(x \cdot y)^4}{x^{-2}y^3} = x^4\, y^4\, x^{-(-2)}\, y^{-3} = x^6\, y$$

- $$\frac{(2\,x^2)^3\,(3y)^{-2}}{(4\,x^2\,y)^2\,(x^3\,y)} = \frac{2^3\,x^{2\cdot 3}\,3^{-2}\,y^{-2}}{4^2\,x^{2\cdot 2}\,y^2\,x^3\,y} = \frac{\frac{8}{9}\,x^6\,y^{-2}}{16\,x^7\,y^3}$$

  $$= \tfrac{1}{18}\,x^{6-7}\,y^{-2-3} = \tfrac{1}{18}\,x^{-1}y^{-5} = \frac{1}{18\,x\,y^5}$$

- $$\left(3\,x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\right)^3 = 3^3\,x^{\frac{3}{3}}\,y^{-\frac{12}{3}} = 27\,x\,y^{-4} = \frac{27\,x}{y^4}$$

Sources of Errors

Important!

- $-x^2$ is **not** equal to $(-x)^2$!
- $(x+y)^n$ is **not** equal to $x^n + y^n$!
- $x^n + y^n$ can**not** be simplified (in general)!

Problem 2.7

Simplify the following expressions:

(a) $$\frac{(xy)^{\frac{1}{3}}}{x^{\frac{1}{6}}y^{\frac{2}{3}}}$$

(b) $$\frac{1}{(\sqrt{x})^{-\frac{3}{2}}}$$

(c) $$\left(\frac{|x|^{\frac{1}{3}}}{|x|^{\frac{1}{6}}}\right)^6$$

Monomial

A **monomial** is a real number, variable, or product of variables raised to positive integer powers.

The **degree** of a monomial is the sum of all exponents of the variables (including a possible implicit exponent 1).

$6\,x^2$ is a monomial of degree 2.

$3\,x^3\,y$ and $x\,y^2\,z$ are monomials of degree 4.

$\sqrt{x}$ and $\frac{2}{3\,x\,y^2}$ are *not* monomials.

Polynomial

A **polynomial** is a sum of (one or more) monomials.

The **degree** of a polynomial is the maximum degree among its monomials.

$4\,x^2\,y^3 - 2\,x^3\,y + 4\,x + 7\,y$ is a polynomial of degree 5.

Polynomials in *one variable* (in sigma notation):

$$P(x) = \sum_{i=0}^{n} a_i\,x^i = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

where x is the variable and $a_i \in \mathbb{R}$ are constants.

Problem 2.8

Which of the following expressions are monomials or polynomials? What is their degree?

Assume that x, y, and z are variables and all other symbols represent constants.

(a) x^2

(b) $x^{2/3}$

(c) $2x^2 + 3xy + 4y^2$

(d) $(2x^2 + 3xy + 4y^2)\,(x^2 - z^2)$

(e) $(x - a)(y - b)(z + 1)$

(f) $x\sqrt{y} - \sqrt{x}y$

(g) $ab + c$

(h) $x\sqrt{a} - \sqrt{b}y$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

is called the **binomial coefficient** (read: "n choose k") and

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

denotes the **factorial** of n (read: "n-factorial").

For convenience we set $\boxed{0! = 1}$.

Binomial Coefficient

Note:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

Computation:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot \qquad 1}$$

$\binom{n}{k}$ is the number of ways to choose an (unordered) subset of k elements from a fixed set of n elements.

Binomial Theorem

- $(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = x^2 + 2xy + y^2$

- $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

  $= x^3 + 3x^2y + 3xy^2 + y^3$

Problem 2.9

Compute

(a) $(x+y)^4$

(b) $(x+y)^5$

by means of the binomial theorem.

Problem 2.10

Show by means of the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hint: Use $1 + 1 = 2$.

Multiplication

The product of two polynomials of degree n and m, resp., is a polynomial of degree $n + m$.

$$\begin{aligned}(2\,x^2 + 3\,x - 5) \cdot (x^3 - 2\,x + 1) = \\ &= 2\,x^2 \cdot x^3 + 2\,x^2 \cdot (-2\,x) + 2\,x^2 \cdot 1 \\ &\quad + 3\,x \cdot x^3 + 3\,x \cdot (-2\,x) + 3\,x \cdot 1 \\ &\quad + (-5) \cdot x^3 + (-5) \cdot (-2\,x) + (-5) \cdot 1 \\ &= 2\,x^5 + 3\,x^4 - 9\,x^3 - 4\,x^2 + 13\,x - 5\end{aligned}$$

Problem 2.11

Simplify the following expressions:

(a) $(x+h)^2 - (x-h)^2$

(b) $(a+b)c - (a+bc)$

(c) $(A-B)(A^2+AB+B^2)$

(d) $(x+y)^4 - (x-y)^4$

Division

Polynomials can be **divided** similarly to the division of integers.

$$
\begin{array}{rl}
 & (x^3 + x^2 \quad + 0\,x - 2) : (x - 1) = \mathbf{x^2 + 2\,x + 2} \\
\mathbf{x^2} \cdot (x-1) \longrightarrow & \underline{x^3 - x^2} \\
 & \quad 2\,x^2 + 0\,x \\
\mathbf{2\,x} \cdot (x-1) \longrightarrow & \quad \underline{2\,x^2 - 2\,x} \\
 & \qquad 2\,x - 2 \\
\mathbf{2} \cdot (x-1) \longrightarrow & \qquad \underline{2\,x - 2} \\
 & \qquad\qquad 0
\end{array}
$$

We thus yield $x^3 + x^2 - 2 = (x-1) \cdot (x^2 + 2\,x + 2)$.

If the *divisor* is not a factor of the *dividend*, then we obtain a *remainder*.

Factorization

The process of expressing a polynomial as the product of polynomials of smaller degree (**factor**) is called **factorization**.

$$2x^2 + 4xy + 8xy^3 = 2x \cdot (x + 2y + 4y^3)$$

$$x^2 - y^2 = (x + y) \cdot (x - y)$$

$$x^2 - 1 = (x + 1) \cdot (x - 1)$$

$$x^2 + 2xy + y^2 = (x + y) \cdot (x + y) = (x + y)^2$$

$$x^3 + y^3 = (x + y) \cdot (x^2 - xy + y^2)$$

These products can be easily verified by multiplying their factors.

Factorization

The factorization

$$(x^2 - y^2) = (x + y) \cdot (x - y)$$

or equivalently

$$(x - y) = (\sqrt{x} + \sqrt{y}) \cdot (\sqrt{x} - \sqrt{y})$$

can be very useful.

Memorize it!

The “Ausmultiplizierreflex”

Important!

Factorizing a polynomial is often *very hard*
while multiplying its factors is fast and easy.
(The RSA public key encryption is based on this idea.)

Beware!

A factorized expression contains more information than their expanded counterpart.

In my experience many students have an
acquired “Ausmultiplizierreflex”:

Instantaneously (and **without thinking**) they *multiply* all factors
(which often turns a simple problem into a difficult one).

The “Ausmultiplizierreflex”

Suppress your “Ausmultiplizierreflex”!

Think first and multiply factors only when it seems to be useful!

Linear Term

A polynomial of *degree* 1 is called a **linear term**.

- $a + b\,x + y + a\,c$ is a linear term in x and y,
  if a, b and c are constants.
- $xy + x + y$ is not linear, as xy has degree 2.

Linear Factor

A factor (polynomial) of degree 1 is called a **linear factor**.

A polynomial in one variable x with root x_1 has linear factor $(x - x_1)$.

If a polynomial in x of degree n,

$$P(x) = \sum_{i=0}^{n} a_i \, x^i$$

has n real roots $x_1, x_2, \ldots, x_n$, then it can be written as the product of the n linear factors $(x - x_i)$:

$$P(x) = a_n \prod_{i=1}^{n} (x - x_i)$$

(This is a special case of the Fundamental Theorem of Algebra.)

Problem 2.12

(a) Give a polynomial in x of degree 4 with roots -1, 2, 3, and 4.

(b) What is the set of all such polynomials?

(c) Can such a polynomial have other roots?

Rational Term

A **rational term** is one of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials called **numerator** and **denominator**, resp.

Alternatively one can write $P(x)/Q(x)$.

The domain of a rational term is $\mathbb{R}$ without the roots of the denominator.

$\frac{x^2+x-4}{x^3+5}$ is a rational term with domain $\mathbb{R} \setminus \{-\sqrt[3]{5}\}$.

Beware! The expression $\frac{0}{0}$ is not defined.

Calculation Rule for Fractions and Rational Terms

Let $b, c, e \neq 0$.

$\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$	*Reduce*
$\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$	*Expand*
$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$	*Multiplying*
$\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e}$	*Dividing*
$\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e}$	*Compound fraction*

Calculation Rule for Fractions and Rational Terms

Let $b, c \neq 0$.

$$\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$$ *Addition with common denominator*

$$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$$ *Addition*

Very important! *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!

Calculation Rule for Fractions and Rational Terms

- $$\frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x-1$$

- $$\frac{4\,x^3+2\,x^2}{2\,x\,y} = \frac{2\,x^2(2\,x+1)}{2\,x\,y} = \frac{x(2\,x+1)}{y}$$

- $$\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{(x+1)^2+(x-1)^2}{(x-1)(x+1)}$$
  $$= \frac{x^2+2\,x+1+x^2-2\,x+1}{(x-1)(x+1)} = 2\,\frac{x^2+1}{x^2-1}$$

Calculation Rule for Fractions and Rational Terms

I want to stress at this point that there happen *a lot of mistakes* in calculations that involve rational terms.

The following examples of such fallacies are collected from students' exams.

Sources of Errors

Very Important! *Really!*

$\dfrac{a+c}{b+c}$ is **not** equal to $\dfrac{a}{b}$

$\dfrac{x}{a}+\dfrac{y}{b}$ is **not** equal to $\dfrac{x+y}{a+b}$

$\dfrac{a}{b+c}$ is **not** equal to $\dfrac{a}{b}+\dfrac{a}{c}$

$$\frac{x+2}{y+2} \neq \frac{x}{y} \qquad \frac{1}{2}+\frac{1}{3} \neq \frac{1}{5}.$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2}+\frac{1}{y^2}$$

Problem 2.13

Simplify the following expressions:

(a) $\dfrac{1}{1+x}+\dfrac{1}{x-1}$

(b) $\dfrac{s}{st^2-t^3}-\dfrac{1}{s^2-st}-\dfrac{1}{t^2}$

(c) $\dfrac{\frac{1}{x}+\frac{1}{y}}{xy+xz+y(z-x)}$

(d) $\dfrac{\frac{x+y}{y}}{\frac{x-y}{x}}+\dfrac{\frac{x+y}{x}}{\frac{x-y}{y}}$

Problem 2.14

Factorize and reduce the fractions:

(a) $\dfrac{1-x^2}{1-x}$

(b) $\dfrac{1+x^2}{1-x}$

(c) $\dfrac{x^3-x^4}{1-x}$

(d) $\dfrac{x^3-x^5}{1-x}$

(e) $\dfrac{x^2-x^6}{1-x}$

(f) $\dfrac{1-x^3}{1-x}$

Problem 2.15

Simplify the following expressions:

(a) $y(xy + x + 1) - \dfrac{x^2y^2 - 1}{x - \frac{1}{y}}$

(b) $\dfrac{\frac{x^2+y}{2x+1}}{\frac{2xy}{2x+y}}$

(c) $\dfrac{\frac{a}{x} - \frac{b}{x+1}}{\frac{a}{x+1} + \frac{b}{x}}$

(d) $\dfrac{2x^2y - 4xy^2}{x^2 - 4y^2} + \dfrac{x^2}{x + 2y}$

Problem 2.16

Simplify the following expressions:

(a) $\dfrac{x^{\frac{1}{4}} - y^{\frac{1}{3}}}{x^{\frac{1}{8}} + y^{\frac{1}{6}}}$

(b) $\dfrac{\sqrt{x} - 4}{x^{\frac{1}{4}} - 2}$

(c) $\dfrac{\frac{2}{x^{-\frac{1}{7}}}}{x^{-\frac{7}{2}}}$

Problem 2.17

Simplify the following expressions:

(a) $$\frac{(\sqrt{x}+y)^{\frac{1}{3}}}{x^{\frac{1}{6}}}$$

(b) $$\frac{1}{3\sqrt{x}-1}\cdot\frac{1}{1+\frac{1}{3\sqrt{x}}}\cdot\frac{1}{\sqrt{x}}$$

(c) $$\frac{(xy)^{\frac{1}{6}}-3}{(xy)^{\frac{1}{3}}-9}$$

(d) $$\frac{x-y}{\sqrt{x}-\sqrt{y}}$$

Exponential Function

- **Power function**

  $(0, \infty) \to (0, \infty),\ x \mapsto x^{\alpha}$ for some *fixed exponent* $\alpha \in \mathbb{R}$

- **Exponential function**

  $\mathbb{R} \to (0, \infty),\ x \mapsto a^{x}$ for some *fixed basis* $a \in (0, \infty)$

Exponent and Logarithm

A number y is called the **logarithm** to basis a, if $a^y = x$.
The logarithm is the *exponent of a number to basis a*.
We write

$$y = \log_a(x) \qquad \Leftrightarrow \qquad x = a^y$$

Important logarithms:

- **natural logarithm $\ln(x)$** with basis $e = 2.7182818\ldots$
  (sometimes called *Euler's number*)
- **common logarithm $\lg(x)$** with basis 10
  (sometimes called decadic or decimal logarithm)

Exponent and Logarithm

- $\log_{10}(100) = 2$, as $10^2 = 100$
- $\log_{10}\left(\frac{1}{1000}\right) = -3$, as $10^{-3} = \frac{1}{1000}$
- $\log_2(8) = 3$, as $2^3 = 8$
- $\log_{\sqrt{2}}(16) = 8$, as $\sqrt{2}^8 = 2^{8/2} = 2^4 = 16$

Calculations with Exponent and Logarithm

Conversion formula:

$$a^x = e^{x \ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$\log_2(123) = \frac{\ln(123)}{\ln(2)} \approx \frac{4.812184}{0.6931472} = 6.942515$$

$$3^7 = e^{7 \ln(3)}$$

Implicit Basis

Important:
Often one can see $\log(x)$ without an explicit basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- In *mathematics*: *natural* logarithm
  financial mathematics, programs like R, *Mathematica*, Maxima, ...

- In *applied sciences*: *common* logarithm
  economics, pocket calculator, Excel, ...

Calculation Rules for Exponent and Logarithm

$a^{x+y} = a^x \cdot a^y$	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
$a^{x-y} = \frac{a^x}{a^y}$	$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
$(a^x)^y = a^{x \cdot y}$	$\log_a(x^\beta) = \beta \cdot \log_a(x)$
$(a \cdot b)^x = a^x \cdot b^x$	
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$
$a^0 = 1$	$\log_a(1) = 0$

$\log_a(x)$ has (as real-valued function) domain $x > 0$!

Problem 2.18

Compute without a calculator:

(a) $\log_2(2)$

(b) $\log_2(4)$

(c) $\log_2(16)$

(d) $\log_2(0)$

(e) $\log_2(1)$

(f) $\log_2\left(\frac{1}{4}\right)$

(g) $\log_2\left(\sqrt{2}\right)$

(h) $\log_2\left(\frac{1}{\sqrt{2}}\right)$

(i) $\log_2(-4)$

Problem 2.19

Compute without a calculator:

(a) $\log_{10}(300)$

(b) $\log_{10}(3^{10})$

Use $\log_{10}(3) = 0.47712$.

Problem 2.20

Compute (simplify) without a calculator:

(a) $0.01^{-\log_{10}(100)}$

(b) $\log_{\sqrt{5}}\left(\frac{1}{25}\right)$

(c) $10^{3\log_{10}(3)}$

(d) $\frac{\log_{10}(200)}{\log_{\frac{1}{\sqrt{7}}}(49)}$

(e) $\log_8\left(\frac{1}{512}\right)$

(f) $\log_{\frac{1}{3}}(81)$

Problem 2.21

Represent the following expression in the form $y = A\,e^{cx}$
(i.e., determine A and c):

(a) $y = 10^{x-1}$

(b) $y = 4^{x+2}$

(c) $y = 3^x 5^{2x}$

(d) $y = 1.08^{x-\frac{x}{2}}$

(e) $y = 0.9 \cdot 1.1^{\frac{x}{10}}$

(f) $y = \sqrt{q}\,2^{x/2}$

Summary

- sigma notation
- absolute value
- powers and roots
- monomials and polynomials
- binomial theorem
- multiplication, factorization, and linear factors
- trap door “Ausmultiplizierreflex”
- fractions, rational terms and many fallacies
- exponent and logarithm

Chapter 3

Equations and Inequalities

Equation

We get an **equation** by **equating** two terms.

$$\boxed{\text{l.h.s.} = \text{r.h.s.}}$$

- **Domain**:
  Intersection of domains of all involved terms restricted to a feasible region (e.g., non-negative numbers).

- **Solution set**:
  Set of all objects from the domain that solve the equation(s).

Transform into Equivalent Equation

Idea:
The equation is transformed into an *equivalent* but *simpler* equation, i.e., one with the *same solution* set.

- *Add* or *subtract* a number or term on both sides of the equation.
- *Multiply* or *divide* by a **non-zero** number or term on both sides of the equation.
- Take the *logarithm* or *antilogarithm* on both sides.

A useful strategy is to *isolate* the unknown quantity on one side of the equation.

Sources of Errors

Beware!
These operations may change the *domain* of the equation.
This may or may not alter the solution set.

In particular this happens if a rational term is reduced or expanded by a factor that contains the unknown.

Important!
Verify that both sides are *strictly positive* before taking the *logarithm*.

Beware!
Any term that contains the unknown may *vanish* (become 0).

- Multiplication may result in an *additional* but *invalid* “solution”.
- Division may *eliminate* a *valid* solution.

Non-equivalent Domains

Equation

$$\frac{(x-1)(x+1)}{x-1} = 1$$

can be transformed into the seemingly equivalent equation

$$x + 1 = 1$$

by reducing the rational term by factor $(x-1)$.

However, the latter has domain $\mathbb{R}$
while the given equation has domain $\mathbb{R} \setminus \{1\}$.

Fortunately, the solution set $L = \{0\}$ remains unchanged by this transformation.

Multiplication

By multiplication of

$$\frac{x^2 + x - 2}{x - 1} = 1$$

by $(x - 1)$ we get

$$x^2 + x - 2 = x - 1$$

with solution set $L = \{-1, 1\}$.

However, $x = 1$ is not in the domain of $\frac{x^2 + x - 2}{x - 1}$ and thus not a valid solution of our equation.

Division

If we divide both sides of equation

$$(x-1)(x-2)=0 \qquad \text{(solution set } L=\{1,2\}\text{)}$$

by $(x-1)$ we obtain equation

$$x-2=0 \qquad \text{(solution set } L=\{2\}\text{)}$$

Thus solution $x=1$ has been “lost” by this division.

Division

Important!
We need a *case-by-case* analysis when we divide by some term that contains an unknown:

Case 1: Division is *allowed* (the divisor is **non-zero**).

Case 2: Division is *forbidden* (the divisor is **zero**).

Find all solutions of $(x-1)(x-2)=0$:

Case $x-1 \neq 0$:
By division we get equation $x-2=0$ with solution $x_1 = 2$.

Case $x-1=0$:
This implies solution $x_2 = 1$.

We shortly will discuss a better method for finding roots.

Division

System of two equations in two unknowns

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

Addition and division in the first equation yields:

$$x\,y = x \;\rightsquigarrow\; y = \frac{x}{x} = 1$$

Substituting into the second equation then gives $x = \pm 1$.
Seemingly solution set: $L = \{(-1,1),(1,1)\}$.

However: Division is only allowed if $x \neq 0$.

$x = 0$ also satisfies the first equation (for every y).

Correct solution set: $L = \{(-1,1),(1,1),(0,\sqrt{2}),(0,-\sqrt{2})\}$.

Factorization

Factorizing a term can be a suitable method for finding roots (points where a term vanishes).

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

The first equation $\quad x\,y - x = x \cdot (y-1) = 0 \quad$ implies

$$x = 0 \quad \textit{or} \quad y - 1 = 0 \quad \text{(or both).}$$

Case $x = 0$: $\quad y = \pm\sqrt{2}$

Case $y - 1 = 0$: $\quad y = 1$ and $x = \pm 1$.

Solution set $L = \{(-1,1), (1,1), (0,\sqrt{2}), (0,-\sqrt{2})\}$.

Verification

A (seemingly correct) solution can be easily *verified* by **substituting** it into the given equation.

If unsure, verify the correctness of a solution.

Hint for your exams:
If a (homework or exam) problem asks for verification of a given solution, then simply substitute into the equation.
There is no need to solve the equation from scratch.

Linear Equation

Linear equations only contain *linear terms* and can (almost) always be solved.

Express annuity R from the formula for the present value

$$B_n = R \cdot \frac{q^n - 1}{q^n(q-1)} \ .$$

As R is to the power 1 only we have a linear equation which can be solved by dividing by (non-zero) constant $\frac{q^n-1}{q^n(q-1)}$:

$$R = B_n \cdot \frac{q^n(q-1)}{q^n - 1}$$

Equation with Absolute Value

An equation with *absolute value* can be seen as an abbreviation for a *system* of two (or more) equations:

$$|x| = 1 \quad \Leftrightarrow \quad x = 1 \text{ or } -x = 1$$

Find all solutions of $|2x - 3| = |x + 1|$.

Union of the respective solutions of the two equations

$$\begin{aligned} (2x-3) &= (x+1) \qquad \Rightarrow \; x = 4 \\ -(2x-3) &= (x+1) \qquad \Rightarrow \; x = \tfrac{2}{3} \end{aligned}$$

(Equations $-(2x-3) = -(x+1)$ and $(2x-3) = -(x+1)$ are equivalent to the above ones.)

We thus find solution set: $L = \left\{\frac{2}{3}, 4\right\}$.

Problem 3.1

Solve the following equations:

(a) $|x(x-2)| = 1$

(b) $|x+1| = \dfrac{1}{|x-1|}$

(c) $\left|\dfrac{x^2-1}{x+1}\right| = 2$

Equation with Exponents

Equations where the unknown is an exponent can (sometimes) be solved by taking the logarithm:

- Isolate the term with the unknown on one side of the equation.
- Take the **logarithm** on both sides.

Solve equation $2^x = 32$.
By taking the logarithm we obtain

$$\begin{aligned} & 2^x = 32 \\ \Leftrightarrow \quad & \ln(2^x) = \ln(32) \\ \Leftrightarrow \quad & x \ln(2) = \ln(32) \\ \Leftrightarrow \quad & x = \frac{\ln(32)}{\ln(2)} = 5 \end{aligned}$$

Solution set: $L = \{5\}$.

Equation with Exponents

Compute the term n of a loan over K monetary units and accumulation factor q from formula

$$X = K \cdot q^n \frac{q-1}{q^n - 1}$$

for installment X.

$$\begin{aligned}
X &= K \cdot q^n \tfrac{q-1}{q^n-1} && \big| \cdot (q^n - 1) \\
X\,(q^n - 1) &= K\,q^n\,(q-1) && \big| -K\,q^n\,(q-1) \\
q^n\,(X - K(q-1)) - X &= 0 && \big| +X \\
q^n\,(X - K(q-1)) &= X && \big| : (X - K(q-1)) \\
q^n &= \tfrac{X}{X-K(q-1)} && \big| \ln \\
n\,\ln(q) &= \ln(X) - \ln(X - K(q-1)) && \big| : \ln(q) \\
n &= \tfrac{\ln(X) - \ln(X-K(q-1))}{\ln(q)}
\end{aligned}$$

Equation with logarithms

Equations which contain (just) the logarithm of the unknown can (sometimes) be solved by taking the antilogarithm.

We get solution of $\ln(x+1) = 0$ by:

$$\begin{aligned} & \ln(x+1) = 0 \\ \Leftrightarrow \quad & e^{\ln(x+1)} = e^0 \\ \Leftrightarrow \quad & x+1 = 1 \\ \Leftrightarrow \quad & x = 0 \end{aligned}$$

Solution set: $L = \{0\}$.

Problem 3.2

Solve the following equations:

(a) $2^x = 3^{x-1}$

(b) $3^{2-x} = 4^{\frac{x}{2}}$

(c) $2^x 5^{2x} = 10^{x+2}$

(d) $2 \cdot 10^{x-2} = 0.1^{3x}$

(e) $\frac{1}{2^{x+1}} = 0.2^x 10^4$

(f) $(3^x)^2 = 4 \cdot 5^{3x}$

Problem 3.3

Function

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

is called the *hyperbolic cosine*.
Find all solutions of

$$\cosh(x) = a$$

Hint: Use auxiliary variable $y = e^x$. Then the equation simplifies to $(y + \frac{1}{y})/2 = a$.

Problem 3.4

Solve the following equation:

$$\ln\left(x^2\left(x-\frac{7}{4}\right)+\left(\frac{x}{4}+1\right)^2\right)=0$$

Equation with Powers

An Equation that contains **only one** power of the unknown which in addition has *integer* degree can be solved by **calculating roots**.

Important!

- Take care that the equation may not have a (unique) solution (in $\mathbb{R}$) if the power has *even* degree.
- If its degree is *odd*, then the solution always exists and is unique (in $\mathbb{R}$).

The solution set of $x^2 = 4$ is $L = \{-2, 2\}$.

Equation $x^2 = -4$ does not have a (real) solution, $L = \emptyset$.

The solution set of $x^3 = -8$ is $L = \{-2\}$.

Equation with Roots

We can solve an **equation with roots** by squaring or taking a power of both sides.

We get the solution of $\sqrt[3]{x-1} = 2$ by taking the third power:

$$\sqrt[3]{x-1} = 2 \quad \Leftrightarrow \quad x - 1 = 2^3 \quad \Leftrightarrow \quad x = 9$$

Square Root

Beware!

Squaring an equation with square roots may create *additional* but *invalid* solutions (cf. multiplying with possible negative terms).

Squaring “non-equality” $-3 \neq 3$ yields equality $(-3)^2 = 3^2$.

Beware!

The domain of an equation with roots often is just a subset of $\mathbb{R}$.

For roots with even root degree the *radicand* must not be negative.

Important!

Always *verify* solutions of equations with roots!

Square Root

Solve equation $\sqrt{x-1} = 1 - \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 - \sqrt{x-4} && \big|^2 \\
x - 1 &= 1 - 2\cdot\sqrt{x-4} + (x-4) && \big| - x + 3 \quad \big| : 2 \\
1 &= -\sqrt{x-4} && \big|^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

However, substitution gives $\sqrt{5-1} = 1 - \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 0$,
which is **false**. Thus we get solution set $L = \emptyset$.

Square Root

Solve equation $\sqrt{x-1} = 1 + \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 + \sqrt{x-4} && |^2 \\
x - 1 &= 1 + 2 \cdot \sqrt{x-4} + (x-4) && | - x + 3 \quad | : 2 \\
1 &= \sqrt{x-4} && |^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

Now, verification yields $\sqrt{5-1} = 1 + \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 2$,
which is a **true** statement. Thus we get non-empty solution $L = \{5\}$.

Problem 3.5

Solve the following equations:

(a) $\sqrt{x+3} = x+1$

(b) $\sqrt{x-2} = \sqrt{x+1} - 1$

Quadratic Equation

A **quadratic equation** is one of the form

$$a\,x^2 + b\,x + c = 0 \qquad \text{Solution:} \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or in standard form

$$x^2 + p\,x + q = 0 \qquad \text{Solution:} \quad x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Roots of Polynomials

Quadratic equations are a special case of **algebraic equations** (*polynomial equations*)

$$P_n(x) = 0$$

where $P_n(x)$ is a polynomial of degree n.

There exist closed form solutions for algebraic equations of degree 3 (*cubic equations*) and 4, resp. However, these are rather tedious.

For polynomials of degree 5 or higher no general formula does exist.

Roots of Polynomials

A polynomial equation can be solved by reducing its degree recursively.

1. Search for a root x_1 of $P_n(x)$
   (e.g. by trial and error, by means of Vieta's formulas,
   or by means of Newton's method)

2. We obtain linear factor $(x - x_1)$ of $P_n(X)$.

3. By division $P_n(x) : (x - x_1)$
   we get a polynomial $P_{n-1}(x)$ of degree $n - 1$.

4. If $n - 1 = 2$, solve the resulting quadratic equation.

   Otherwise goto Step 1.

Roots of Polynomials

Find all solutions of

$$x^3 - 6x^2 + 11x - 6 = 0.$$

By educated guess we find solution $x_1 = 1$.

Division by the linear factor $(x - 1)$ yields

$$(x^3 - 6x^2 + 11x - 6) : (x - 1) = x^2 - 5x + 6$$

Quadratic equation $x^2 - 5x + 6 = 0$ has solutions $x_2 = 2$ and $x_3 = 3$.

The solution set is thus $L = \{1, 2, 3\}$.

Roots of Products

A *product* of two (or more) terms $f(x) \cdot g(x)$ is zero if and only if *at least one* factor is zero:

$$f(x) = 0 \quad \textit{or} \quad g(x) = 0 \quad \text{(or both).}$$

Equation $x^2 \cdot (x-1) \cdot e^x = 0$ is satisfied if

- $x^2 = 0 \quad (\Rightarrow x = 0)$, *or*
- $x - 1 = 0 \quad (\Rightarrow x = 1)$, *or*
- $e^x = 0$ (no solution).

Thus we have solution set $L = \{0, 1\}$.

Roots of Products

Important!

If a polynomial is already factorized one should resist to expand this expression.

The roots of polynomial

$$(x-1)\cdot(x+2)\cdot(x-3)=0$$

are obviously 1, -2 and 3.

Roots of the expanded expression

$$x^3-2x^2-5x+6=0$$

are hard to find.

Problem 3.6

Compute all roots and decompose into linear factors:

(a) $f(x) = x^2 + 4x + 3$

(b) $f(x) = 3x^2 - 9x + 2$

(c) $f(x) = x^3 - x$

(d) $f(x) = x^3 - 2x^2 + x$

(e) $f(x) = (x^2 - 1)(x - 1)^2$

Problem 3.7

Solve with respect to x and with respect to y:

(a) $xy + x - y = 0$

(b) $3xy + 2x - 4y = 1$

(c) $x^2 - y^2 + x + y = 0$

(d) $x^2y + xy^2 - x - y = 0$

(e) $x^2 + y^2 + 2xy = 4$

(f) $9x^2 + y^2 + 6xy = 25$

(g) $4x^2 + 9y^2 = 36$

(h) $4x^2 - 9y^2 = 36$

(i) $\sqrt{x} + \sqrt{y} = 1$

Problem 3.8

Solve with respect to x and with respect to y:

(a) $xy^2 + yx^2 = 6$

(b) $xy^2 + (x^2 - 1)y - x = 0$

(c) $\frac{x}{x+y} = \frac{y}{x-y}$

(d) $\frac{y}{y+x} = \frac{y-x}{y+x^2}$

(e) $\frac{1}{y-1} = \frac{y+x}{2y+1}$

(f) $\frac{yx}{y+x} = \frac{1}{y}$

(g) $(y + 2x)^2 = \frac{1}{1+x} + 4x^2$

(h) $y^2 - 3xy + (2x^2 + x - 1) = 0$

(i) $\frac{y}{x+2y} = \frac{2x}{x+y}$

Problem 3.9

Find constants a, b and c such that the following equations hold for all x in the corresponding domains:

(a) $$\frac{x}{1+x} - \frac{2}{2-x} = -\frac{2a + bx + cx^2}{2 + x - x^2}$$

(b) $$\frac{x^2 + 2x}{x+2} - \frac{x^2 + 3}{x+3} = \frac{a(x-b)}{x+c}$$

Inequalities

We get an **inequality** by comparing two terms by means of one of the "inequality" symbols

- $\leq$ (less than or equal to),
- $<$ (less than),
- $>$ (greater than),
- $\geq$ (greater than or equal to):

$$\text{l.h.s.} \leq \text{r.h.s.}$$

The inequality is called **strict** if equality does not hold.

Solution set of an inequality is the set of all numbers in its domain that satisfy the inequality.
Usually this is an (open or closed) interval or union of intervals.

Transform into Equivalent Inequality

Idea:
The inequality is transformed into an *equivalent* but *simpler* inequality.
Ideally we try to isolate the unknown on one side of the inequality.

Beware!
If we *multiply* an inequality by some **negative** number, then the *direction* of the inequality symbol is *reverted*.
Thus we need a case analysis:

- Case: term is **greater** than zero:
  Direction of inequality symbol is *not revert*.
- Case: term is **less** than zero:
  Direction of inequality symbol *is revert*.
- Case: term is **equal** to zero:
  Multiplication or division is *forbidden*!

Transform into Equivalent Inequality

Find all solutions of $\dfrac{2x-1}{x-2} \leq 1$.

We multiply inequality by $(x-2)$.

- Case $x - 2 > 0 \Leftrightarrow x > 2$:
  We find $2x - 1 \leq x - 2 \Leftrightarrow x \leq -1$,
  a contradiction to our assumption $x > 2$.

- Case $x - 2 < 0 \Leftrightarrow x < 2$:
  The inequality symbol is reverted, and
  we find $2x - 1 \geq x - 2 \Leftrightarrow x \geq -1$.
  Hence $x < 2$ and $x \geq -1$.

- Case $x - 2 = 0 \Leftrightarrow x = 2$:
  not in domain of inequality.

Solution set is the interval $L = [-1, 2)$.

Sources of Errors

Inequalities with polynomials *cannot* be directly solved by transformations.

Important!
One **must not** simply replace the equality sign "$=$" in the formula for quadratic equations by an inequality symbols.

We want to find all solutions of

$$x^2 - 3\,x + 2 \leq 0$$

Invalid approach: $x_{1,2} \leq \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2}$

and thus $x \leq 1$ (and $x \leq 2$) which would imply "solution" set $L = (-\infty, 1]$.

However, $0 \in L$ but violates the inequality as $2 \not\leq 0$.

Inequalities with Polynomials

1. Move all terms on the l.h.s. and obtain an expression of the form $T(x) \leq 0$ (and $T(x) < 0$, resp.).

2. Compute all roots $x_1 < \ldots < x_k$ of $T(x)$,
   i.e., solve equation $T(x) = 0$ as we would with any polynomial as described above.

3. These roots decompose the *domain* into intervals I_j.
   These are open if the inequality is *strict* (with $<$ or $>$), and closed otherwise.

   In each of these intervals the inequality now holds
   either in *all* **or** in *none* of its points.

4. Select some point $z_j \in I_j$ which is not on the boundary.
   If z_j satisfies the corresponding **strict** inequality,
   then I_j belongs to the solution set, else none of its points.

Inequalities with Polynomials

Find all solutions of

$$x^2 - 3\,x + 2 \leq 0\,.$$

The solutions of $x^2 - 3\,x + 2 = 0$ are $x_1 = 1$ and $x_2 = 2$.

We obtain three intervals and check by means of three points (0, $\frac{3}{2}$, and 3) whether the inequality is satisfied in each of these:

$(-\infty, 1]$	not satisfied:	$0^2 - 3 \cdot 0 + 2 = 2 \not\leq 0$
$[1, 2]$	satisfied:	$\left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 2 = -\frac{1}{4} < 0$
$[2, \infty)$	not satisfied:	$3^2 - 3 \cdot 3 + 2 = 2 \not\leq 0$

Solution set is $L = [1, 2]$.

Continuous Terms

The above principle also works for inequalities where all terms are **continuous**.

If there is any point where $T(x)$ is *not continuous*, then we also have to use this point for decomposing the domain into intervals.

Furthermore, we have to take care when the domain of the inequality is a union of two or more disjoint intervals.

Continuous Terms

Find all solutions of inequality

$$\frac{x^2 + x - 3}{x - 2} \geq 1 \,.$$

Its domain is the union of two intervals: $(-\infty, 2) \cup (2, \infty)$.

We find for the solutions of the equation $\dfrac{x^2 + x - 3}{x - 2} = 1$:

$$\frac{x^2 + x - 3}{x - 2} = 1 \;\Leftrightarrow\; x^2 + x - 3 = x - 2 \;\Leftrightarrow\; x^2 - 1 = 0$$

and thus $x_1 = -1$, $x_2 = 1$.
So we get four intervals:

$(-\infty, -1]$, $[-1, 1]$, $[1, 2)$ and $(2, \infty)$.

Continuous Terms

We check by means of four points whether the inequality holds in these intervals:

$(-\infty,-1]$	not satisfied:	$\frac{(-2)^2+(-2)-3}{(-2)-2} = \frac{1}{4} \not> 1$
$[-1,1]$	satisfied:	$\frac{0^2-0-3}{0-2} = \frac{3}{2} > 1$
$[1,2)$	not satisfied:	$\frac{1.5^2+1.5-3}{1.5-2} = -\frac{3}{2} \not> 1$
$(2,\infty)$	satisfied:	$\frac{3^2+3-3}{3-2} = 9 > 1$

Solution set is $L = [-1,1] \cup (2,\infty)$.

Inequalities with Absolute Values

Inequalities with *absolute values* can be solved by the above procedure.

However, we also can see such an inequality as a system of two (or more) inequalities:

$$|x| < 1 \qquad \Leftrightarrow \qquad x < 1 \text{ and } x > -1$$

$$|x| > 1 \qquad \Leftrightarrow \qquad x > 1 \text{ or } x < -1$$

Problem 3.10

Solve the following inequalities:

(a) $x^3 - 2x^2 - 3x \geq 0$

(b) $x^3 - 2x^2 - 3x > 0$

(c) $x^2 - 2x + 1 \leq 0$

(d) $x^2 - 2x + 1 \geq 0$

(e) $x^2 - 2x + 6 \leq 1$

Problem 3.11

Solve the following inequalities:

(a) $7 \leq |12x + 1|$

(b) $\dfrac{x+4}{x+2} < 2$

(c) $\dfrac{3(4-x)}{x-5} \leq 2$

(d) $25 < (-2x + 3)^2 \leq 50$

(e) $42 \leq |12x + 6| < 72$

(f) $5 \leq \dfrac{(x+4)^2}{|x+4|} \leq 10$

Summary

- equations and inequalities
- domain and solution set
- transformation into equivalent problem
- possible errors with multiplication and division
- equations with powers and roots
- equations with polynomials and absolute values
- roots of polynomials
- equations with exponents and logarithms
- method for solving inequalities

Chapter 4

Sequences and Series

Sequences

A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$\boxed{a\colon \mathbb{N} \to \mathbb{R},\ n \mapsto a_n}$$

Sequences are denoted by $(a_n)_{n=1}^{\infty}$ or just (a_n) for short.

An alternative notation used in literature is $\langle a_n \rangle_{n=1}^{\infty}$.

Sequences

Sequences can be defined

- by **enumerating** of its terms,
- by a **formula**, or
- by **recursion**.
  Each term is determined by its predecessor(s).

Enumeration: $(a_n) = (1, 3, 5, 7, 9, \ldots)$

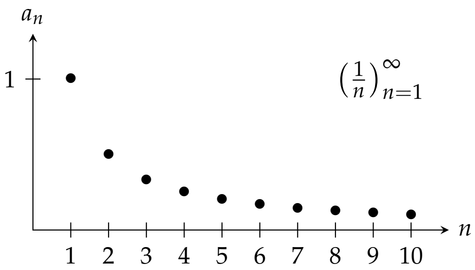
Formula: $(a_n) = (2\,n - 1)$

Recursion: $a_1 = 1, \;\; a_{n+1} = a_n + 2$

Graphical Representation

A sequence (a_n) can by represented

(1) by drawing tuples (n, a_n) in the plane, or



(2) by drawing points on the number line.

Properties

Properties of a sequence (a_n):

Property	Definition	
monotonically increasing	$a_{n+1} \geq a_n$	for all $n \in \mathbb{N}$
monotonically decreasing	$a_{n+1} \leq a_n$	
alternating	$a_{n+1} \cdot a_n < 0$	i.e. the sign changes
bounded	$\lvert a_n \rvert \leq M$	for some $M \in \mathbb{R}$

Sequence $\left(\frac{1}{n}\right)$ is

- monotonically decreasing
- bounded, as for all $n \in \mathbb{N}$, $\lvert a_n \rvert = \lvert 1/n \rvert \leq M = 1$
  (we could also choose $M = 1000$)
- but *not* alternating.

Problem 4.1

Draw the first 10 elements of the following sequences.
Which of these sequences are monotone, alternating, or bounded?

(a) $\left(n^2\right)_{n=1}^{\infty}$

(b) $\left(n^{-2}\right)_{n=1}^{\infty}$

(c) $\left(\sin(\pi/n)\right)_{n=1}^{\infty}$

(d) $a_1 = 1,\ \ a_{n+1} = 2a_n$

(e) $a_1 = 1,\ \ a_{n+1} = -\frac{1}{2}a_n$

Series

The sum of the first n terms of sequence $(a_i)_{i=1}^{\infty}$

$$s_n = \sum_{i=1}^{n} a_i$$

is called the n-th **partial sum** of the *sequence*.

The sequence (s_n) of all partial sums is called the **series** of the sequence.

The series of sequence $(a_i) = (2\,i - 1)$ is

$$(s_n) = \left(\sum_{i=1}^{n} (2\,i - 1) \right) = (1, 4, 9, 16, 25, \dots) = (n^2)\ .$$

Problem 4.2

Compute the first 5 partial sums of the following sequences:

(a) $2n$

(b) $\frac{1}{2+n}$

(c) $2^{n/10}$

Arithmetic Sequence

Formula and recursion:

$$a_n = a_1 + (n-1) \cdot d \qquad a_{n+1} = a_n + d$$

Differences of consecutive terms are constant:

$$a_{n+1} - a_n = d$$

Each term is the *arithmetic mean* of its neighboring terms:

$$a_n = \tfrac{1}{2}(a_{n+1} + a_{n-1})$$

Arithmetic series:

$$s_n = \tfrac{n}{2}(a_1 + a_n)$$

Geometric Sequence

Formula and recursion:

$$a_n = a_1 \cdot q^{n-1} \qquad a_{n+1} = a_n \cdot q$$

Ratios of consecutive terms are constant:

$$\frac{a_{n+1}}{a_n} = q$$

Each term is the *geometric mean* of its neighboring terms:

$$a_n = \sqrt{a_{n+1} \cdot a_{n-1}}$$

Geometric series:

$$s_n = a_1 \cdot \frac{q^n - 1}{q - 1} \qquad \text{for } q \neq 1$$

Sources of Errors

Indices of sequences may also start with 0 (instead of 1).

Beware!
Formulæ then are slightly changed.

Arithmetic sequence:

$$a_n = a_0 + n \cdot d \qquad \text{and} \qquad s_n = \tfrac{n+1}{2}(a_0 + a_n)$$

Geometric sequence:

$$a_n = a_0 \cdot q^n \qquad \text{and} \qquad s_n = a_0 \cdot \frac{q^{n+1} - 1}{q - 1} \quad \text{(for } q \neq 1\text{)}$$

Problem 4.3

We are given a geometric sequence (a_n) with $a_1 = 2$ and relative change 0.1, i.e., each term of the sequence is increased by 10% compared to its predecessor.
Give formula and term a_7.

Problem 4.4

Compute the first 10 partial sums of the arithmetic series for

(a) $a_1 = 0$ and $d = 1$,

(b) $a_1 = 1$ and $d = 2$.

Problem 4.5

Compute $\sum_{n=1}^{N} a_n$ for

(a) $N = 7$ and $a_n = 3^{n-2}$

(b) $N = 7$ and $a_n = 2(-1/4)^n$

Applications of Geometric Sequences

See your favorite book /course on finance and accounting.

Summary

- sequence
- formula and recursion
- series and partial sums
- arithmetic and geometric sequence

Chapter 5

Real Functions

Real Function

Real functions are maps where both *domain* and *codomain* are (unions of) intervals in $\mathbb{R}$.

Often only function terms are given but neither domain nor codomain. Then domain and codomain are implicitly given as following:

- *Domain* of the function is the largest *sensible* subset of the domain of the function terms (i.e., where the terms are defined).
- *Codomain* is the image (range) of the function

$$f(D) = \{y \mid y = f(x) \text{ for an } x \in D_f\} .$$

Implicit Domain

Production function $f(x) = \sqrt{x}$ is an abbreviation for

$$f\colon [0, \infty) \to [0, \infty),\ x \mapsto f(x) = \sqrt{x}$$

(There are no negative amounts of goods.
Moreover, $\sqrt{x}$ is not real for $x < 0$.)

Its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is an abbreviation for

$$f'\colon (0, \infty) \to (0, \infty),\ x \mapsto f'(x) = \frac{1}{2\sqrt{x}}$$

(Note the open interval $(0, \infty)$; $\frac{1}{2\sqrt{x}}$ is not defined for $x = 0$.)

Problem 5.1

Give the largest possible domain of the following functions?

(a) $h(x) = \frac{x-1}{x-2}$

(b) $D(p) = \frac{2p+3}{p-1}$

(c) $f(x) = \sqrt{x-2}$

(d) $g(t) = \frac{1}{\sqrt{2t-3}}$

(e) $f(x) = 2 - \sqrt{9 - x^2}$

(f) $f(x) = 1 - x^3$

(g) $f(x) = 2 - |x|$

Problem 5.2

Give the largest possible domain of the following functions?

(a) $f(x) = \frac{|x-3|}{x-3}$

(b) $f(x) = \ln(1 + x)$

(c) $f(x) = \ln(1 + x^2)$

(d) $f(x) = \ln(1 - x^2)$

(e) $f(x) = \exp(-x^2)$

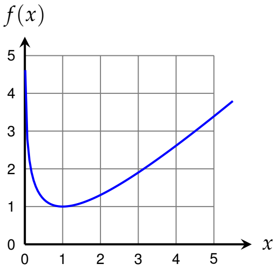
(f) $f(x) = (e^x - 1)/x$

Graph of a Function

Each tuple $(x, f(x))$ corresponds to a point in the xy-plane.
The set of all these points forms a curve called the **graph** of function f.

$$\mathcal{G}_f = \{(x,y) \mid x \in D_f,\ y = f(x)\}$$

Graphs can be used to visualize functions.
They allow to detect many properties of the given function.



$$f(x) = x - \ln(x)$$

How to Draw a Graph

1. Get an idea about the possible shape of the graph. One should be able to sketch graphs of elementary functions by heart.

2. Find an appropriate range for the x-axis.
   (It should show a characteristic detail of the graph.)

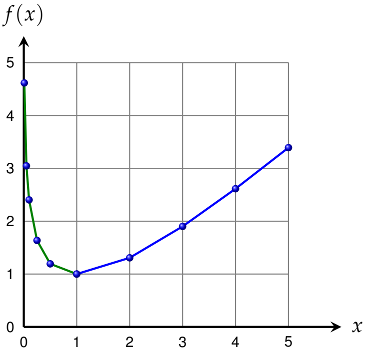
3. Create a table of function values and draw the corresponding points into the xy-plane.

   If known, use characteristic points like local extrema or inflection points.

4. Check if the curve can be constructed from the drawn points.
   If not add adapted points to your table of function values.

5. Fit the curve of the graph through given points in a proper way.

How to Draw a Graph



Graph of function

$$f(x) = x - \ln x$$

Table of values:

x	$f(x)$
0	ERROR
1	1
2	1.307
3	1.901
4	2.614
5	3.391
0.5	1.193
0.25	1.636
0.1	2.403
0.05	3.046

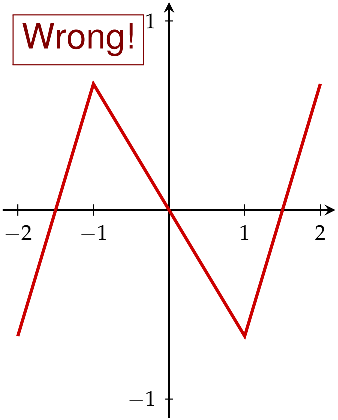
Sources of Errors

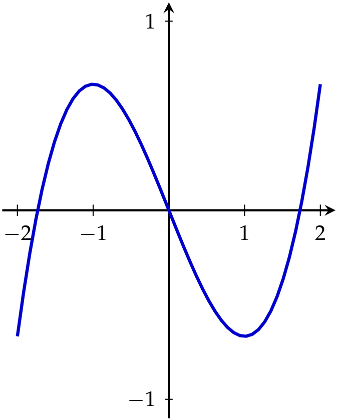
Most frequent errors when drawing function graphs:

- **Table of values is too small:**
  It is not possible to construct the curve from the computed function values.
- **Important points are ignored:**
  Ideally extrema and inflection points should be known and used.
- **Range for x and y-axes not suitable:**
  The graph is tiny or important details vanish in the “noise” of handwritten lines (or pixel size in case of a computer program).

Sources of Errors

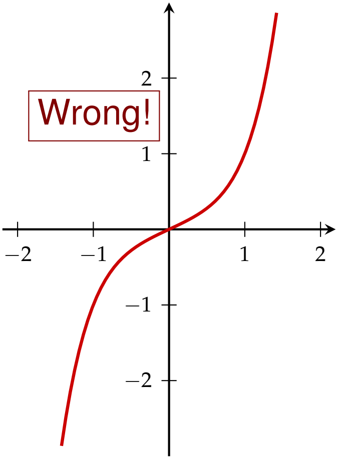
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:

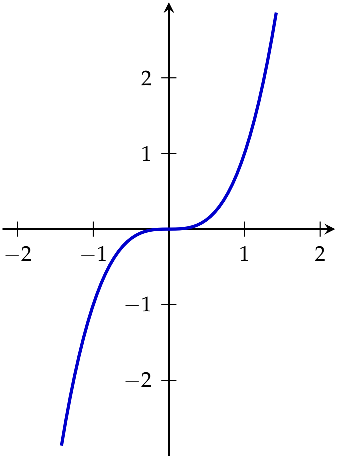




Sources of Errors

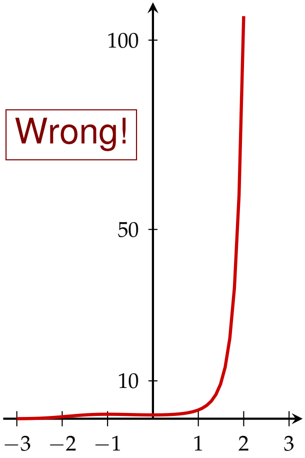
Graph of $f(x) = x^3$ has slope 0 in $x = 0$:

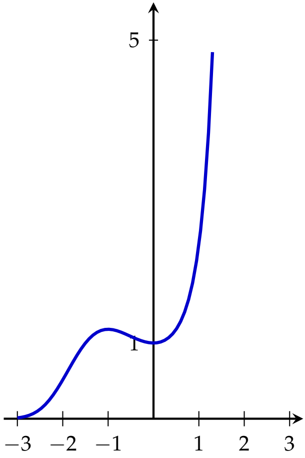




Sources of Errors

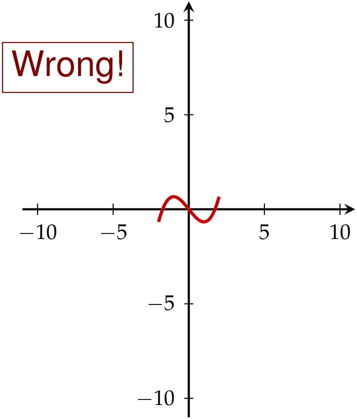
Function $f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$ has a local maximum in $x = -1$:

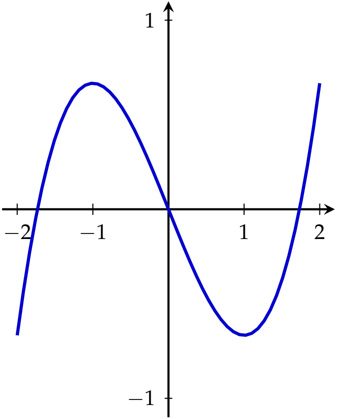




Sources of Errors

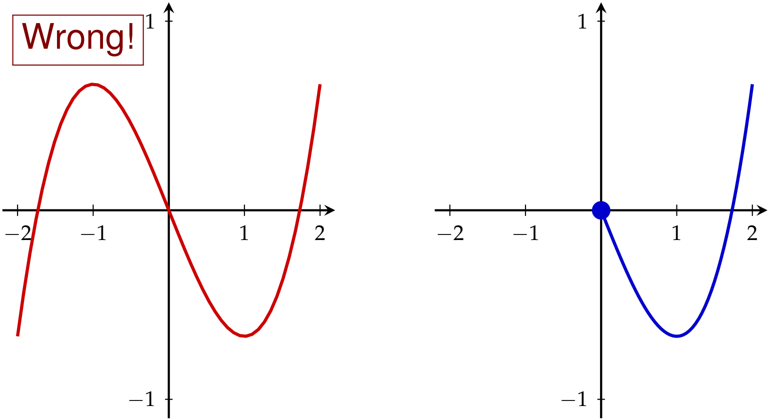
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:





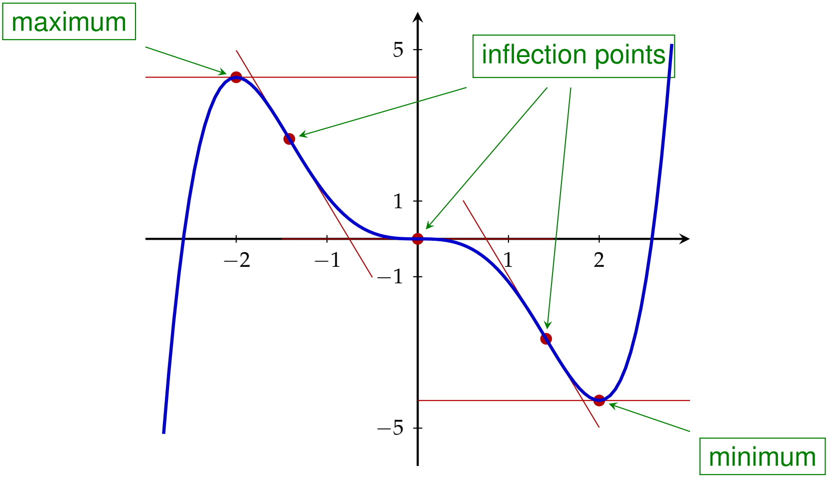
Sources of Errors

Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[0, 2]$: (not in $[-2, 2]$!)



Extrema and Inflection Points

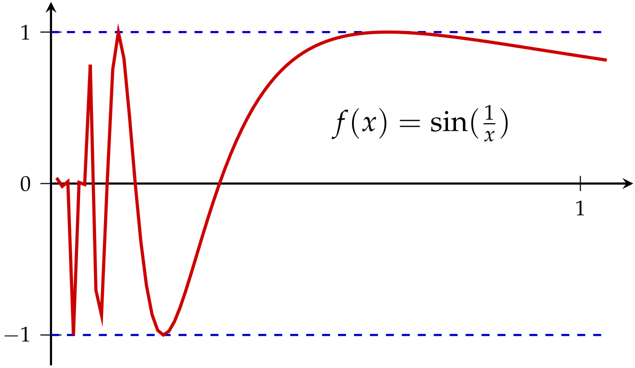
Graph of function $f(x) = \frac{1}{15}(3x^5 - 20x^3)$:



Sources of Errors

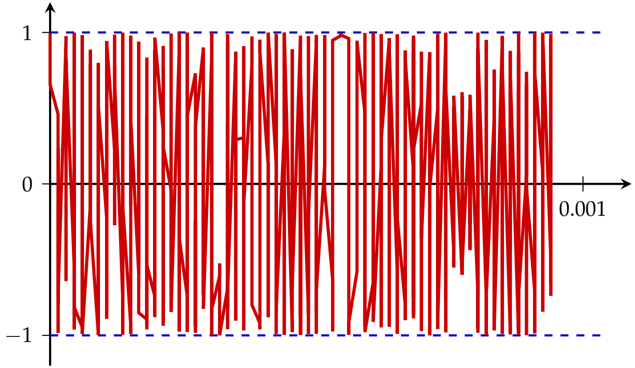
It is *important* that one already has an *idea of the shape* of the function graph **before** drawing the curve.

Even a graph drawn by means of a computer program can differ significantly from the true curve.



Sources of Errors

$$f(x) = \sin(\tfrac{1}{x})$$



Sketch of a Function Graph

Often a **sketch** of the graph is sufficient. Then the exact function values are not so important. Axes may not have scales.

However, it is important that the sketch clearly shows all characteristic details of the graph (like extrema or important function values).

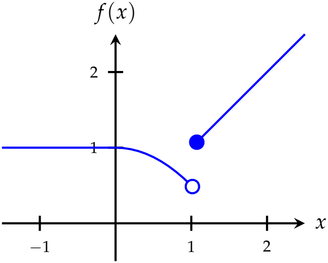
Sketches can also be drawn like a caricature:
They stress prominent parts and properties of the function.

Piece-wise Defined Functions

The function term can be defined differently in subintervals of the domain.

At the boundary points of these subintervals we have to mark which points belong to the graph and which do not:

- $\bullet$ (belongs) and $\circ$ (does not belong).



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1, \\ x, & \text{for } x \ge 1. \end{cases}$$

Problem 5.3

Draw the graph of function

$$f(x) = -x^4 + 2x^2$$

in interval $[-2, 2]$.

Problem 5.4

Draw the graph of function

$$f(x) = e^{-x^4+2x^2}$$

in interval $[-2, 2]$.

Problem 5.5

Draw the graph of function

$$f(x) = \frac{x-1}{|x-1|}$$

in interval $[-2, 2]$.

Problem 5.6

Draw the graph of function

$$f(x) = \sqrt{|1 - x^2|}$$

in interval $[-2, 2]$.

Bijectivity

Recall that each argument has exactly one image and that the number of preimages of an element in the codomain can vary.
Thus we can characterize maps by their possible number of preimages.

- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Also recall that a function has an *inverse* if and only if it is one-to-one and onto (i.e., *bijective*).

A Simple Horizontal Test

How can we determine whether a real function is one-to-one or onto?
I.e., how many preimage may a $y \in W_f$ have?

(1) Draw the graph of the given function.

(2) Mark some $y \in W$ on the y-axis and draw a line parallel to the x-axis (*horizontal*) through this point.

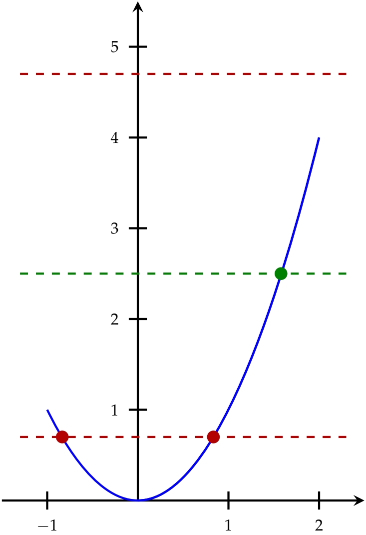
(3) The number of intersection points of horizontal line and graph coincides with the number of preimages of y.

(4) Repeat Steps (2) and (3) for a *representative* set of y-values.

(5) Interpretation: If all horizontal lines intersect the graph in

- **(a)** *at most one* point, then f is **one-to-one**;
- **(b)** *at least one* point, then f is **onto**;
- **(c)** *exactly one* point, then f is **bijective**.

Example



$f\colon [-1,2] \to \mathbb{R},\ x \mapsto x^2$

- is not one-to-one;
- is not onto.

$f\colon [0,2] \to \mathbb{R},\ x \mapsto x^2$

- is one-to-one;
- is not onto.

$f\colon [0,2] \to [0,4],\ x \mapsto x^2$

- is one-to-one and onto.

Beware! *Domain* and *codomain* are part of the function!

Problem 5.7

Draw the graphs of the following functions and determine whether these functions are one-to-one or onto (or both).

(a) $f\colon [-2,2] \to \mathbb{R},\ x \mapsto 2\,x + 1$

(b) $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ x \mapsto \frac{1}{x}$

(c) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$

(d) $f\colon [2,6] \to \mathbb{R},\ x \mapsto (x-4)^2 - 1$

(e) $f\colon [2,6] \to [-1,3],\ x \mapsto (x-4)^2 - 1$

(f) $f\colon [4,6] \to [-1,3],\ x \mapsto (x-4)^2 - 1$

Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.

$$g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function**.
(read: "g composed with f", "g circle f", or "g after f")

Let $\quad g\colon \mathbb{R} \to [0,\infty),\ x \mapsto g(x) = x^2,$
$\quad f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = 3x - 2.$

Then $\quad (g \circ f)\colon \mathbb{R} \to [0,\infty),$
$\quad x \mapsto (g \circ f)(x) = g(f(x)) = g(3x-2) = (3x-2)^2$

and $\quad (f \circ g)\colon \mathbb{R} \to \mathbb{R},$
$\quad x \mapsto (f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 2$

Problem 5.8

Let $f(x) = x^2 + 2x - 1$ and $g(x) = 1 + |x|^{\frac{3}{2}}$.

Compute

(a) $(f \circ g)(4)$

(b) $(f \circ g)(-9)$

(c) $(g \circ f)(0)$

(d) $(g \circ f)(-1)$

Problem 5.9

Determine $f \circ g$ and $g \circ f$.
What are the domains of f, g, $f \circ g$ and $g \circ f$?

(a) $f(x) = x^2, \quad g(x) = 1 + x$

(b) $f(x) = \sqrt{x} + 1, \quad g(x) = x^2$

(c) $f(x) = \frac{1}{x+1}, \quad g(x) = \sqrt{x} + 1$

(d) $f(x) = 2 + \sqrt{x}, \quad g(x) = (x-2)^2$

(e) $f(x) = x^2 + 2, \quad g(x) = x - 3$

(f) $f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{1}{x}$

(g) $f(x) = \ln(x), \quad g(x) = \exp(x^2)$

(h) $f(x) = \ln(x-1), \quad g(x) = x^3 + 1$

Inverse Function

If $f\colon D_f \to W_f$ is a **bijection**, then there exists a so called **inverse function**

$$f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

with the property

$$f^{-1} \circ f = \mathrm{id} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}$$

We get the function term of the inverse by *interchanging* the roles of *argument* x and *image* y.

Example

We get the term for the inverse function by expressing x as function of y

We need the inverse function of

$$y = f(x) = 2x - 1$$

By rearranging we obtain

$$y = 2x - 1 \quad \Leftrightarrow \quad y + 1 = 2x \quad \Leftrightarrow \quad \frac{1}{2}(y+1) = x$$

Thus the term of the inverse function is $\quad f^{-1}(y) = \frac{1}{2}(y+1)$.

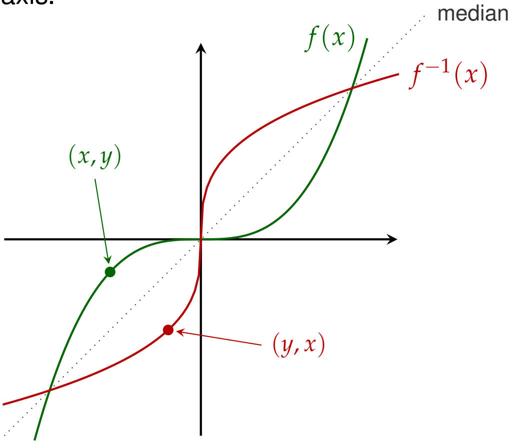
Arguments are usually denoted by x. So we write

$$f^{-1}(x) = \frac{1}{2}(x+1)\ .$$

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

Geometric Interpretation

Interchanging of x and y corresponds to reflection across the median between x and y-axis.



(Graph of function $f(x) = x^3$ and its inverse.)

Problem 5.10

Find the inverse function of

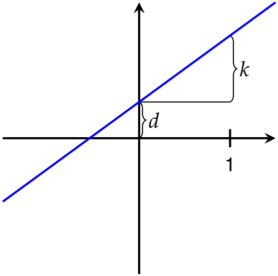
$$f(x) = \ln(1 + x)$$

Draw the graphs of f and f^{-1}.

Linear Function and Absolute Value

- **Linear function**

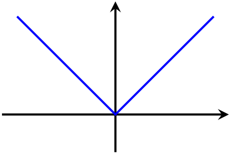
$$f(x) = k\,x + d$$



k … **slope**
d … **intercept**

- **Absolute value** (or *modulus*)

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$



Problem 5.11

Draw the graph of function

$$f(x) = 2x + 1$$

in interval $[-2, 2]$.

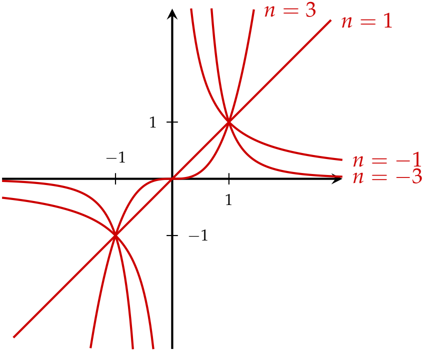
Hint: Two points and a ruler are sufficient.

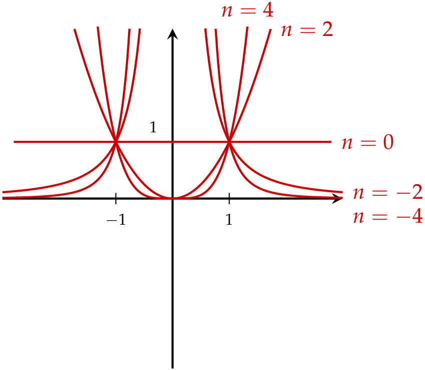
Power Function

Power function with integer exponents:

$$f\colon x \mapsto x^n, \quad n \in \mathbb{Z}$$

$$D = \begin{cases} \mathbb{R} & \text{for } n \geq 0 \\ \mathbb{R} \setminus \{0\} & \text{for } n < 0 \end{cases}$$



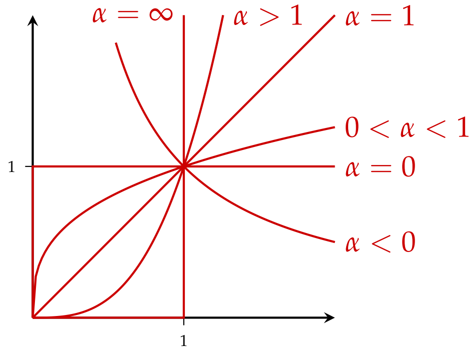


Power Function

Power function with *real* exponents:

$$f\colon x \mapsto x^{\alpha} \qquad \alpha \in \mathbb{R}$$

$$D = \begin{cases} [0,\infty) & \text{for } \alpha \geq 0 \\ (0,\infty) & \text{for } \alpha < 0 \end{cases}$$



Problem 5.12

Draw (sketch) the graph of power function

$$f(x) = x^n$$

in interval $[0, 2]$ for

$$n = -4, -2, -1, -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1, 2, 3, 4\ .$$

Polynomial and Rational Functions

- **Polynomial** of degree n:

$$f(x) = \sum_{k=0}^{n} a_k\, x^k$$

$a_i \in \mathbb{R}$, for $i = 1, \ldots, n$, $\quad a_n \neq 0$.

- **Rational Function:**

$$D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}$$

$p(x)$ and $q(x)$ are polynomials
$D = \mathbb{R} \setminus \{\text{roots of } q\}$

Problem 5.13

Draw (sketch) the graphs of the following functions in interval $[-2, 2]$:

(a) $f(x) = \dfrac{x}{x^2 + 1}$

(b) $f(x) = \dfrac{x}{x^2 - 1}$

(c) $f(x) = \dfrac{x^2}{x^2 + 1}$

(d) $f(x) = \dfrac{x^2}{x^2 - 1}$

Exponential Function

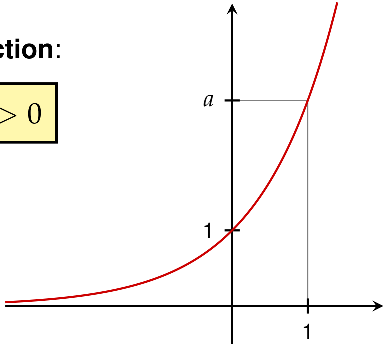
- **Exponential function**:

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \exp(x) = e^x$$

$e = 2{,}7182818\ldots$ Euler's number

- **Generalized exponential function**:

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto a^x \qquad a > 0$$



Problem 5.14

Draw (sketch) the graph of the following functions:

(a) $f(x) = e^x$

(b) $f(x) = 3^x$

(c) $f(x) = e^{-x}$

(d) $f(x) = e^{x^2}$

(e) $f(x) = e^{-x^2}$

(f) $f(x) = e^{-1/x^2}$

(g) $\cosh(x) = (e^x + e^{-x})/2$

(h) $\sinh(x) = (e^x - e^{-x})/2$

Logarithm Function

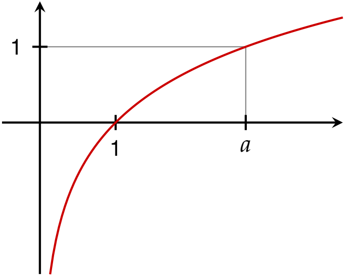
- **Logarithm**:

  Inverse of *exponential function*.

  $$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log(x) = \ln(x)$$

- **Generalized Logarithm** to basis a:

  $$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log_a(x)$$



Problem 5.15

Draw (sketch) the graph of the following functions:

(a) $f(x) = \ln(x)$

(b) $f(x) = \ln(x+1)$

(c) $f(x) = \ln\left(\frac{1}{x}\right)$

(d) $f(x) = \log_{10}(x)$

(e) $f(x) = \log_{10}(10x)$

(f) $f(x) = (\ln(x))^2$

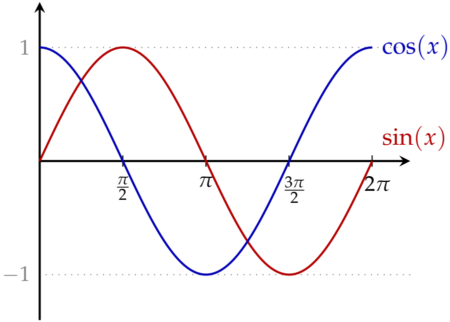
Trigonometric Functions

- **Sine**:

$$\mathbb{R} \to [-1,1],\ x \mapsto \sin(x)$$

- **Cosine**:

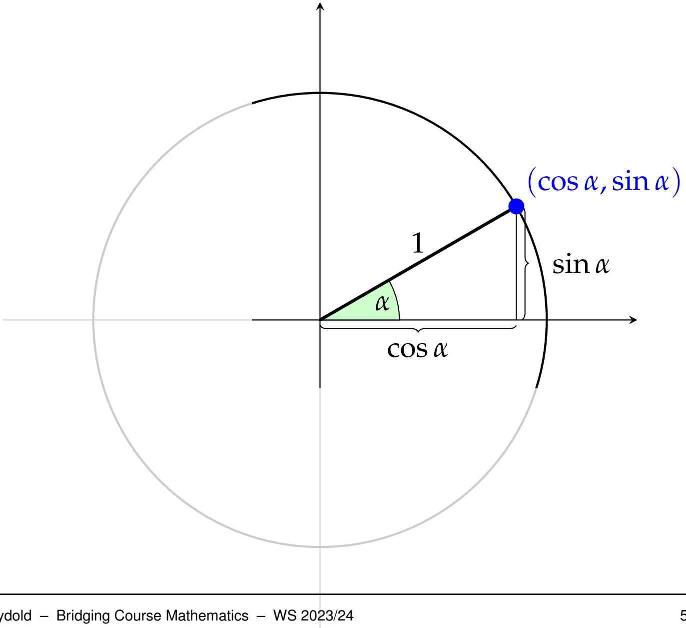
$$\mathbb{R} \to [-1,1],\ x \mapsto \cos(x)$$



Beware!

These functions use **radian** for their arguments, i.e., angles are measured by means of the length of arcs on the unit circle and not by degrees. A right angle then corresponds to $x = \pi/2$.

Sine and Cosine



Sine and Cosine

Important formulas:

Periodic: For all $k \in \mathbb{Z}$,

$$\sin(x + 2k\pi) = \sin(x)$$
$$\cos(x + 2k\pi) = \cos(x)$$

Relation between $\sin$ and $\cos$:

$$\sin^2(x) + \cos^2(x) = 1$$

Problem 5.16

Assign the following functions to the graphs $\boxed{1}$ – $\boxed{18}$:

(a) $f(x) = x^2$

(b) $f(x) = \frac{x}{x+1}$

(c) $f(x) = \frac{1}{x+1}$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = x^3 - 3x^2 + 2x$

(f) $f(x) = \sqrt{|2x - x^2|}$

(g) $f(x) = -x^2 - 2x$

(h) $f(x) = (x^3 - 3x^2 + 2x)\text{sgn}(1 - x) + 1$
(sgn$(x) = 1$ if $x \geq 0$ and -1 otherwise.)

Problem 5.16 / 2

(i) $f(x) = e^x$

(j) $f(x) = e^{x/2}$

(k) $f(x) = e^{2x}$

(l) $f(x) = 2^x$

(m) $f(x) = \ln(x)$

(n) $f(x) = \log_{10}(x)$

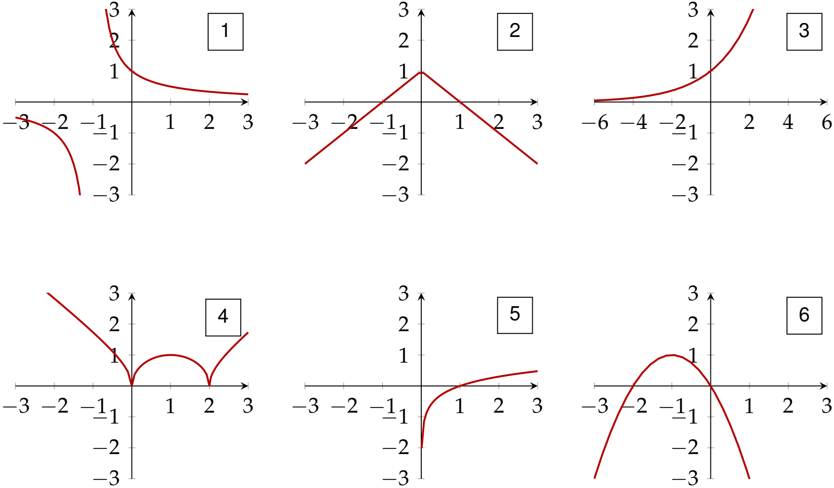
(o) $f(x) = \log_2(x)$

(p) $f(x) = \sqrt{4 - x^2}$

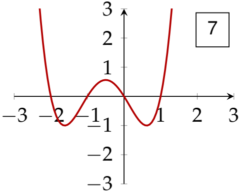
(q) $f(x) = 1 - |x|$

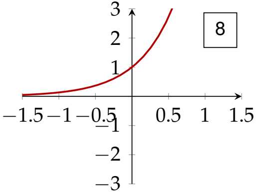
(r) $f(x) = \prod_{k=-1}^{2}(x + k)$

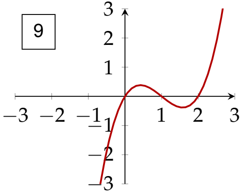
Problem 5.16 / 3

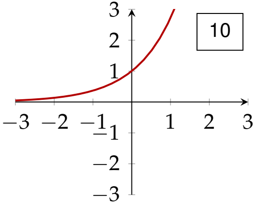


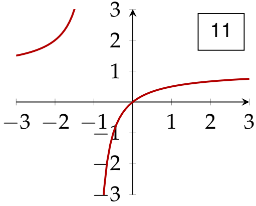
Problem 5.16 / 4

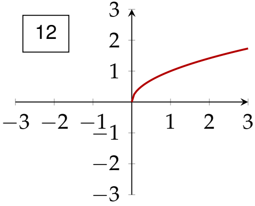




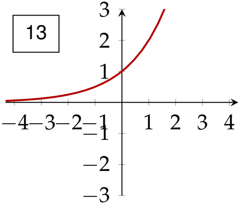


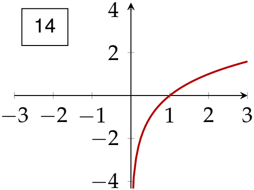


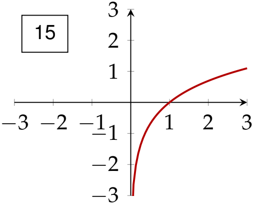


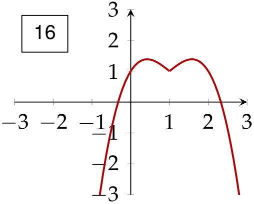


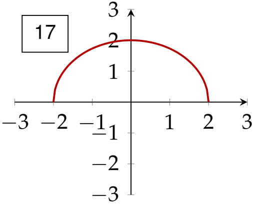
Problem 5.16 / 5

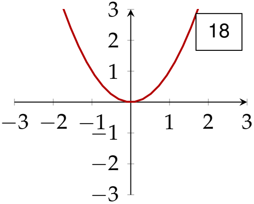












Multivariate Function

A **function of several variables** (or **multivariate function**) is a function with more than one argument which evaluates to a real number.

$$f : \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$$

Arguments x_i are the **variables** of function f.

$$f(x, y) = \exp(-x^2 - 2y^2)$$

is a bivariate function in variables x and y.

$$p(x_1, x_2, x_3) = x_1^2 + x_1 x_2 - x_2^2 + 5x_1 x_3 - 2x_2 x_3$$

is a function in the three variables x_1, x_2, and x_3.

Graphs of Bivariate Functions

Bivariate functions (i.e., of *two* variables) can be visualized by its graph:

$$\mathcal{G}_f = \{(x, y, z) \mid z = f(x, y) \text{ for } x, y \in \mathbb{R}\}$$

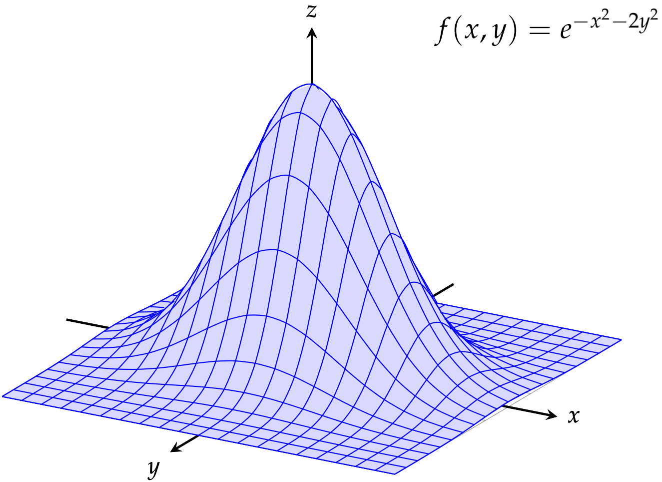
It can be seen as the two-dimensional *surface* of a three-dimensional landscape.

The notion of *graph* exists analogously for functions of three or more variables.

$$\mathcal{G}_f = \{(\mathbf{x}, y) \mid y = f(\mathbf{x}) \text{ for an } \mathbf{x} \in \mathbb{R}^n\}$$

However, it can hardly be used to visualize such functions.

Graphs of Bivariate Functions



Contour Lines of Bivariate Functions

Let $c \in \mathbb{R}$ be fixed. Then the set of all points (x, y) in the real plane with $f(x, y) = c$ is called **contour line** of function f.

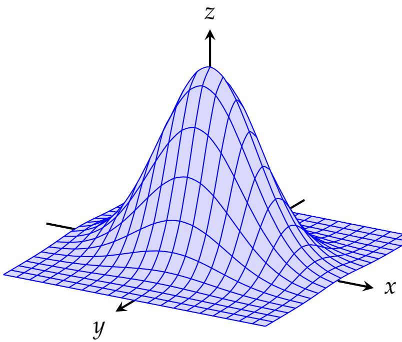
Function f is constant on each of its contour lines.

Other names:

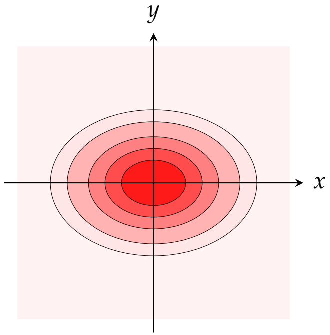
- *Indifference curve*
- *Isoquant*
- *Level set* (is a generalization of a contour line for functions of any number of variables.)

A collection of contour lines can be seen as a kind of “hiking map” for the “landscape” of the function.

Contour Lines of Bivariate Functions



graph



contour lines

$$f(x,y) = e^{-x^2-2y^2}$$

Problem 5.17

In a simplistic model we are given utility function U of a household w.r.t. two complementary goods (e.g. left and right shoes):

$$U(x_1, x_2) = \sqrt{\min\{x_1, x_2\}}, \quad x_1, x_2 \geq 0.$$

(a) Sketch the graph of U.

(b) Sketch the contour lines for $U = U_0 = 1$ and $U = U_1 = 2$.

Indifference Curves

Indifference curves are determined by an equation

$$F(x, y) = 0$$

We can (try to) draw such curves by expressing one of the variables as function of the other one
(i.e., solve the equation w.r.t. one of the two variables).

So we may get an univariate function. The graph of this function coincides with the indifference curve.

We then draw the graph of this univariate function by the method described above.

Cobb-Douglas-Function

We want to draw indifference curve

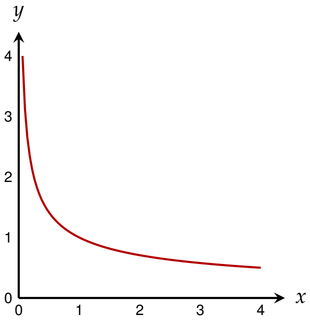
$$x^{\frac{1}{3}} y^{\frac{2}{3}} = 1\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$x = \frac{1}{y^2}$$

Alternatively we can express y by x:

$$y = \frac{1}{\sqrt{x}}$$



CES-Function

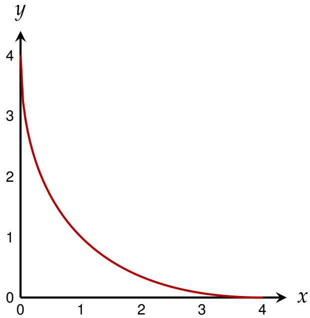
We want to draw indifference curve

$$\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 = 4\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$y = \left(2 - x^{\frac{1}{2}}\right)^2$$

(Take care about the domain of this curve!)



Problem 5.18

Draw the following indifference curves:

(a) $x + y^2 - 1 = 0$

(b) $x^2 + y^2 - 1 = 0$

(c) $x^2 - y^2 - 1 = 0$

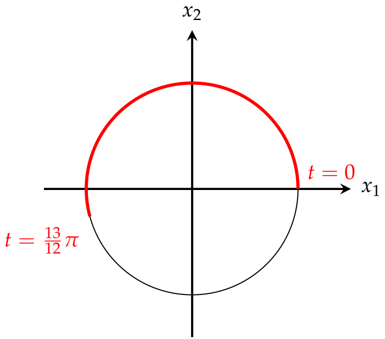
Paths

A function

$$s\colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto s(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}$$

is called a **path** in $\mathbb{R}^n$.
Variable t is often interpreted as *time*.

$$[0, \infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$



Problem 5.19

Sketch the graphs of the following paths:

(a) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

(b) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$

(c) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t\ \cos(2\pi t) \\ t\ \sin(2\pi t) \end{pmatrix}$

Vector-valued Function

Generalized vector-valued function:

$$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}$$

- Univariate functions:
  $\mathbb{R} \to \mathbb{R},\ x \mapsto y = x^2$
- Multivariate functions:
  $\mathbb{R}^2 \to \mathbb{R},\ \mathbf{x} \mapsto y = x_1^2 + x_2^2$
- Paths:
  $[0,1) \to \mathbb{R}^n,\ s \mapsto (s, s^2)^t$
- Linear maps:
  $\mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x}$ $\qquad$ $\mathbf{A} \dots m \times n$-Matrix

Summary

- real functions
- implicit domain
- graph of a function
- sources of errors
- piece-wise defined functions
- one-to-one and onto
- function composition
- inverse function
- elementary functions
- multivariate functions
- paths
- vector-valued functions

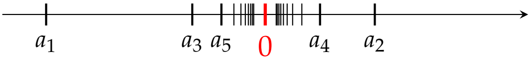
Chapter 6

Limits

Limit of a Sequence

Consider the following sequence of numbers

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \frac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \ldots\right)$$



The terms of this sequence *tend* to 0 with increasing n.
We say that sequence (a_n) **converges** to 0.

We write

$$(a_n) \to 0 \quad \text{or} \quad \lim_{n\to\infty} a_n = 0$$

(read: “limit of a_n for n tends to ∞”)

Limit of a Sequence / Definition

Definition:
A number $a \in \mathbb{R}$ is a **limit** of sequence (a_n), if there *exists an* N *for every interval* $(a - \varepsilon, a + \varepsilon)$ such that $a_n \in (a - \varepsilon, a + \varepsilon)$ for all $n \geq N$; i.e., all terms following a_N are contained in this interval.

Equivalent Definition: A sequence (a_n) converges to **limit** $a \in \mathbb{R}$ if there *exists an* N *for every* $\varepsilon > 0$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

[Mathematicians like to use ε for a very small positive number.]

A sequence that has a *limit* is called **convergent**.
It **converges** to its limit.

It can be shown that a limit of a sequence is *unique*ly defined *(if it exists)*.

A sequence *without* a limit is called **divergent**.

Limit of a Sequence / Example

Sequence

$$\big(a_n\big)_{n=1}^{\infty} = \big((-1)^n \tfrac{1}{n}\big)_{n=1}^{\infty} = \big(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\big)$$

has limit $a = 0$.

For example, if we set $\varepsilon = 0.3$, then all terms following a_4 are contained in interval $(a - \varepsilon, a + \varepsilon)$.

If we set $\varepsilon = \frac{1}{1\,000\,000}$, then all terms starting with the 1 000 001-st term are contained in the interval.

Thus

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0 \,.$$

Limit of a Sequence / Example

Sequence $\big(a_n\big)_{n=1}^{\infty} = \big(\frac{1}{2^n}\big)_{n=1}^{\infty} = \big(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\big)$ converges to 0:

$$\lim_{n\to\infty} a_n = 0$$

Sequence $\big(b_n\big)_{n=1}^{\infty} = \big(\frac{n-1}{n+1}\big)_{n=1}^{\infty} = \big(0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots\big)$ is convergent:

$$\lim_{n\to\infty} b_n = 1$$

Sequence $\big(c_n\big)_{n=1}^{\infty} = \big((-1)^n\big)_{n=1}^{\infty} = \big(-1, 1, -1, 1, -1, 1, \dots\big)$
is divergent.

Sequence $\big(d_n\big)_{n=1}^{\infty} = \big(2^n\big)_{n=1}^{\infty} = \big(2, 4, 8, 16, 32, \dots\big)$ is divergent,
but tends to ∞. By abuse of notation we write:

$$\lim_{n\to\infty} d_n = \infty$$

Limits of Important Sequences

$$\lim_{n\to\infty} n^a = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a = 0 \\ \infty & \text{for } a > 0 \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{for } |q| < 1 \\ 1 & \text{for } q = 1 \\ \infty & \text{for } q > 1 \\ \nexists & \text{for } q \leq -1 \end{cases}$$

$$\lim_{n\to\infty} \frac{n^a}{q^n} = \begin{cases} 0 & \text{for } |q| > 1 \\ \infty & \text{for } 0 < q < 1 \\ \nexists & \text{for } -1 < q < 0 \end{cases} \qquad (|q| \notin \{0,1\})$$

Rules for Limits

Let $\left(a_n\right)_{n=1}^{\infty}$ and $\left(b_n\right)_{n=1}^{\infty}$ be convergent sequences with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, resp., and let $\left(c_n\right)_{n=1}^{\infty}$ be a bounded sequence.
Then

(1)	$\lim\limits_{n\to\infty} (k \cdot a_n + d) = k \cdot a + d$	
(2)	$\lim\limits_{n\to\infty} (a_n + b_n) = a + b$	
(3)	$\lim\limits_{n\to\infty} (a_n \cdot b_n) = a \cdot b$	
(4)	$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{a}{b}$	for $b \neq 0$
(5)	$\lim\limits_{n\to\infty} (a_n \cdot c_n) = 0$	provided $a = 0$
(6)	$\lim\limits_{n\to\infty} a_n^k = a^k$	

Rules for Limits

$$\lim_{n\to\infty}\left(2+\frac{3}{n^2}\right) = 2+3\underbrace{\lim_{n\to\infty} n^{-2}}_{=0} = 2+3\cdot 0 = 2$$

$$\lim_{n\to\infty}(2^{-n}\cdot n^{-1}) = \lim_{n\to\infty}\frac{n^{-1}}{2^n} = 0$$

$$\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2-\frac{3}{n^2}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2-\frac{3}{n^2}\right)} = \frac{1}{2}$$

$$\lim_{n\to\infty}\underbrace{\sin(n)}_{\text{bounded}}\cdot\underbrace{\frac{1}{n^2}}_{\to 0} = 0$$

Rules for Limits / Rational Terms

Important!
When we apply these rules we have to take care that we never obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0 \cdot \infty$.

These expressions are **not defined**!

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \frac{\lim_{n\to\infty} 3n^2+1}{\lim_{n\to\infty} n^2-1} = \frac{\infty}{\infty} \quad \text{(not defined)}$$

Trick: Reduce the fraction by the *largest power* in its **denominator**.

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{3+n^{-2}}{1-n^{-2}} = \frac{\lim_{n\to\infty} 3+n^{-2}}{\lim_{n\to\infty} 1-n^{-2}} = \frac{3}{1} = 3$$

Euler's Number

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182818284590\ldots$$

This limit is very important in many applications including finance (continuous compounding).

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{1}{n/x}\right)^n \\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} && \left(m=\frac{n}{x}\right) \\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = e^x
\end{aligned}$$

Problem 6.1

Compute the following limits:

(a) $\lim_{n\to\infty} \left(7 + \left(\frac{1}{2}\right)^n\right)$

(b) $\lim_{n\to\infty} \left(\frac{2n^3 - 6n^2 + 3n - 1}{7n^3 - 16}\right)$

(c) $\lim_{n\to\infty} \left(n^2 - (-1)^n n^3\right)$

(d) $\lim_{n\to\infty} \left(\frac{n^2 + 1}{n + 1}\right)$

(e) $\lim_{n\to\infty} \left(\frac{n \bmod 10}{(-2)^n}\right)$

$a \bmod b$ is the remainder after integer division, e.g., $17 \bmod 5 = 2$ and $12 \bmod 4 = 0$.

Problem 6.2

Compute the following limits:

(a) $\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$

(b) $\lim\limits_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$

(c) $\lim\limits_{n\to\infty}\left(1+\frac{1}{nx}\right)^{n}$

Limit of a Function

What happens with the value of a function f, if the argument x tends to some value x_0 (which need not belong to the domain of f)?

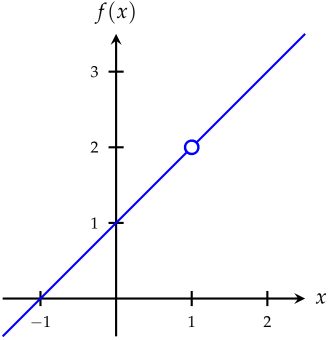
Function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

is not defined in $x = 1$.

By factorizing and reducing we get function

$$g(x) = x + 1 = \begin{cases} f(x), & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$



Limit of a Function

Suppose we approach argument $x_0 = 1$.

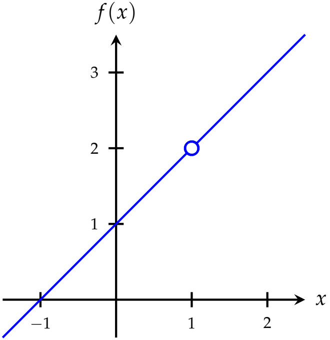
Then the value of function $f(x) = \dfrac{x^2 - 1}{x - 1}$ tends to 2.

We say:

$f(x)$ **converges** to 2 when x tends to 1

and write:

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$$



Limit of a Function

Formal **definition**:
If sequence $\left(f(x_n)\right)_{n=1}^{\infty}$ of function values converges to number y_0 for every convergent sequence $\left(x_n\right)_{n=1}^{\infty} \to x_0$ of arguments,
then y_0 is called the **limit** of f as x approaches x_0.

We write

$$\lim_{x \to x_0} f(x) = y_0 \quad \text{or} \quad f(x) \to y_0 \text{ for } x \to x_0$$

x_0 need not belong to the domain of f.
y_0 need not belong to the codomain of f.

Rules for Limits

Rules for limits of functions are analogous to rules for sequences.

Let $\lim_{x\to x_0} f(x) = a$ and $\lim_{x\to x_0} g(x) = b$.

(1) $\lim_{x\to x_0} (c\cdot f(x) + d) = c\cdot a + d$

(2) $\lim_{x\to x_0} (f(x) + g(x)) = a + b$

(3) $\lim_{x\to x_0} (f(x)\cdot g(x)) = a\cdot b$

(4) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{a}{b}$ for $b \neq 0$

(5) $\lim_{x\to x_0} (f(x))^k = a^k$ for $k \in \mathbb{N}$

How to Find Limits?

The following recipe is suitable for "simple" functions:

1. Draw the graph of the function.

2. Mark x_0 on the x-axis.

3. Follow the graph with your pencil until we reach x_0 starting from *right* of x_0.

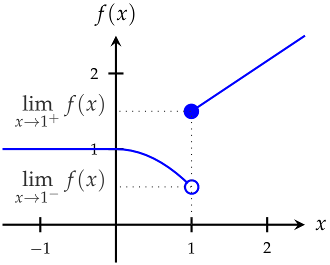
4. The y-coordinate of your pencil in this point is then the so called **right-handed limit** of f as x approaches x_0 (from above):
$\lim_{x\to x_0^+} f(x)$. (Other notations: $\lim_{x\downarrow x_0} f(x)$ or $\lim_{x\searrow x_0} f(x)$)

5. Analogously we get the **left-handed limit** of f as x approaches x_0 (from below): $\lim_{x\to x_0^-} f(x)$.

6. If both limits *coincide*, then the limit exists and we have

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x)$$

How to Find Limits?

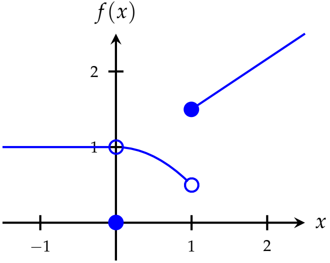


$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

$0.5 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 1.5$
i.e., the limit of f at $x_0 = 1$ does not exist.

The limits at other points, however, do exist,
e.g. $\lim_{x\to 0} f(x) = 1$.

How to Find Limits?



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

The only difference is to above is the function value at $x_0 = 0$.
Nevertheless, the limit does exist:

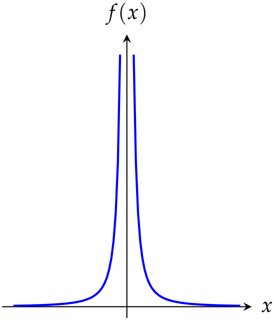
$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x) \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 1 \,.$$

Unbounded Function

It may happen that $f(x)$ tends to ∞ (or $-\infty$) if x tends to x_0.

We then write (by abuse of notation):

$$\lim_{x \to x_0} f(x) = \infty$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Limit as $x \to \infty$

By abuse of language we can define the *limit* analogously for $x_0 = \infty$ and $x_0 = -\infty$, resp.

Limit

$$\lim_{x \to \infty} f(x)$$

exists, if $f(x)$ converges whenever x tends to infinity.

$$\lim_{x \to \infty} \frac{1}{x^2} = 0 \quad \text{and} \quad \lim_{x \to -\infty} \frac{1}{x^2} = 0$$

Problem 6.3

Draw the graph of function

$$f(x) = \begin{cases} -\frac{x^2}{2}, & \text{for } x \leq -2, \\ x+1, & \text{for } -2 < x < 2, \\ \frac{x^2}{2}, & \text{for } x \geq 2. \end{cases}$$

and determine $\lim_{x \to x_0^+} f(x)$, $\lim_{x \to x_0^-} f(x)$, and $\lim_{x \to x_0} f(x)$
for $x_0 = -2$, 0 and 2:

$$\lim_{x \to -2^+} f(x) \qquad \lim_{x \to -2^-} f(x) \qquad \lim_{x \to -2} f(x)$$

$$\lim_{x \to 0^+} f(x) \qquad \lim_{x \to 0^-} f(x) \qquad \lim_{x \to 0} f(x)$$

$$\lim_{x \to 2^+} f(x) \qquad \lim_{x \to 2^-} f(x) \qquad \lim_{x \to 2} f(x)$$

Problem 6.4

Determine the following left-handed and right-handed limits:

(a) $\lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^+} f(x)$

for $f(x) = \begin{cases} 1, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$

(b) $\lim_{x \to 0^-} \frac{1}{x}$

$\lim_{x \to 0^+} \frac{1}{x}$

(c) $\lim_{x \to 1^-} x$

$\lim_{x \to 1^+} x$

Problem 6.5

Determine the following limits:

(a) $\lim_{x \to \infty} \frac{1}{x+1}$

(b) $\lim_{x \to 0} x^2$

(c) $\lim_{x \to \infty} \ln(x)$

(d) $\lim_{x \to 0} \ln|x|$

(e) $\lim_{x \to \infty} \frac{x+1}{x-1}$

Problem 6.6

Determine

(a) $\lim_{x \to 1^+} \frac{x^{3/2} - 1}{x^3 - 1}$

(b) $\lim_{x \to -2^-} \frac{\sqrt{|x^2 - 4|^2}}{x + 2}$

(c) $\lim_{x \to 0^-} \lfloor x \rfloor$

(d) $\lim_{x \to 1^+} \frac{x - 1}{\sqrt{x - 1}}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Problem 6.7

Determine

(a) $\lim_{x\to 2^+} \frac{2x^2-3x-2}{|x-2|}$

(b) $\lim_{x\to 2^-} \frac{2x^2-3x-2}{|x-2|}$

(c) $\lim_{x\to -2^+} \frac{|x+2|^{3/2}}{2+x}$

(d) $\lim_{x\to 1^-} \frac{x+1}{x^2-1}$

(e) $\lim_{x\to -7^+} \frac{2|x+7|}{x^2+4x-21}$

Problem 6.8

Compute

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = x^n$, for $n \in \mathbb{N}$.

L'Hôpital's Rule

Suppose we want to compute

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

and find

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \qquad \text{(or } = \pm\infty\text{)}$$

However, expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are not defined.

(You must not reduce the fraction by 0 or ∞!)

L'Hôpital's Rule

If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ (or $= \infty$ or $= -\infty$), then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

Assumption: f and g are differentiable in x_0.

This formula is called **l'Hôpital's rule** (also written as *l'Hospital's rule*).

L'Hôpital's Rule

$$\lim_{x\to 2} \frac{x^3 - 7\,x + 6}{x^2 - x - 2} = \lim_{x\to 2} \frac{3\,x^2 - 7}{2\,x - 1} = \frac{5}{3}$$

$$\lim_{x\to\infty} \frac{\ln\, x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2\,x} = \lim_{x\to\infty} \frac{1}{2\,x^2} = 0$$

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{1 - (1+x)^{-1}}{2\,x} = \lim_{x\to 0} \frac{(1+x)^{-2}}{2} = \frac{1}{2}$$

L'Hôpital's Rule

L'Hôpital's rule can be applied iteratively:

$$\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Problem 6.9

Compute the following limits:

(a) $$\lim_{x\to 4} \frac{x^2 - 2\,x - 8}{x^3 - 2\,x^2 - 11\,x + 12}$$

(b) $$\lim_{x\to -1} \frac{x^2 - 2\,x - 8}{x^3 - 2\,x^2 - 11\,x + 12}$$

(c) $$\lim_{x\to 2} \frac{x^3 - 5\,x^2 + 8\,x - 4}{x^3 - 3\,x^2 + 4}$$

(d) $$\lim_{x\to 0} \frac{1 - \cos(x)}{x^2}$$

(e) $$\lim_{x\to 0^+} x\,\ln(x)$$

(f) $$\lim_{x\to \infty} x\,\ln(x)$$

Problem 6.10

If we apply l'Hôpital's rule on the following limit we obtain

$$\lim_{x \to 1} \frac{x^3 + x^2 - x - 1}{x^2 - 1} = \lim_{x \to 1} \frac{3\,x^2 + 2\,x - 1}{2\,x} = \lim_{x \to 1} \frac{6\,x + 2}{2} = 4\,.$$

However, the correct value for the limit is 2.

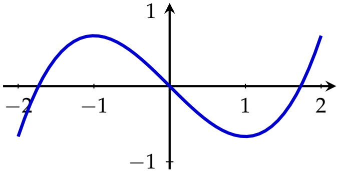
Why does l'Hôpital's rule not work for this problem?

How do you get the correct value?

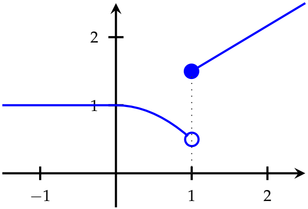
Continuous Functions

We may observe that we can draw the graph of a function *without removing the pencil from paper*. We call such functions **continuous**.

For some other functions we *have to remove* the pencil. At such points the function has a **jump discontinuity**.



continuous



jump discontinuity at $x = 1$

Continuous Functions

Formal **Definition:**

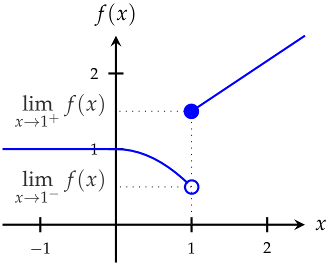
Function $f\colon D \to \mathbb{R}$ is called **continuous** *at* $x_0 \in D$, if

1. $\lim_{x \to x_0} f(x)$ exists, and
2. $\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$.

The function is called **continuous** if it is continuous *at all* points of its domain.

Note that continuity is a *local* property of a function.

Discontinuous Function



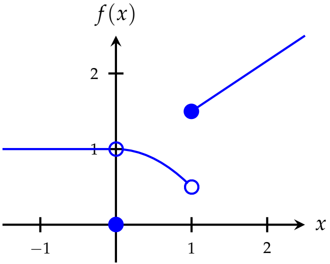
$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Not continuous in $x = 1$ as $\lim_{x\to 1} f(x)$ does not exist.

So f is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{1\}$.
For example at $x = 0$, $\lim_{x\to 0} f(x)$ does exist and $\lim_{x\to 0} f(x) = 1 = f(0)$.

Discontinuous Function



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Not continuous in all $x = 0$, either.

$\lim_{x \to 0} f(x) = 1$ does exist but $\lim_{x \to 0} f(x) \neq f(0)$.

So f is not a continuous function.

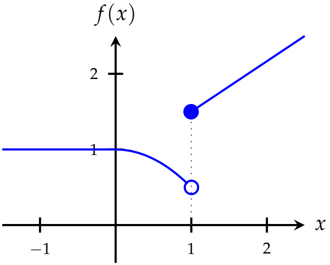
However, it is still continuous in all $x \in \mathbb{R} \setminus \{0, 1\}$.

Recipe for "Nice" Functions

(1) Draw the graph of the given function.

(2) At all points of the *domain*, where we *have to remove* the pencil from paper the function is *not continuous*.

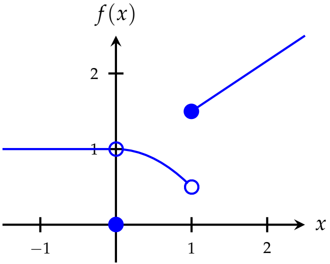
(3) At all other points of the domain (where we need not remove the pencil) the function is *continuous*.



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1, \\ \frac{x}{2} + 1, & \text{for } x \geq 1. \end{cases}$$

f is continuous *except* at point $x = 1$.

Discontinuous Function



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Function f is continuous *except* at points $x = 0$ and $x = 1$.

Problem 6.11

Draw the graph of function

$$f(x) = \begin{cases} -\frac{x^2}{2}, & \text{for } x \leq -2, \\ x+1, & \text{for } -2 < x < 2, \\ \frac{x^2}{2}, & \text{for } x \geq 2. \end{cases}$$

and compute $\lim_{x \to x_0^+} f(x)$, $\lim_{x \to x_0^-} f(x)$, and $\lim_{x \to x_0} f(x)$
for $x_0 = -2, 0$, and 2.

Is function f continuous at these points?

Problem 6.12

Determine the left and right-handed limits of function

$$f(x) = \begin{cases} x^2 + 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -x^2 - 1, & \text{for } x < 0. \end{cases}$$

at $x_0 = 0$.

Is function f continuous at this point?

Is function f differentiable at this point?

Problem 6.13

Is function

$$f(x) = \begin{cases} x + 1, & \text{for } x \leq 1, \\ \frac{x}{2} + \frac{3}{2}, & \text{for } x > 1, \end{cases}$$

continuous at $x_0 = 1$?

Is it differentiable at $x_0 = 1$?

Compute the limit of f at $x_0 = 1$.

Problem 6.14

Sketch the graphs of the following functions.
Which of these are continuous (on its domain)?

(a) $D = \mathbb{R}$, $f(x) = x$

(b) $D = \mathbb{R}$, $f(x) = 3x + 1$

(c) $D = \mathbb{R}$, $f(x) = e^{-x} - 1$

(d) $D = \mathbb{R}$, $f(x) = |x|$

(e) $D = \mathbb{R}^+$, $f(x) = \ln(x)$

(f) $D = \mathbb{R}, f(x) = \lfloor x \rfloor$

(g) $D = \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{for } x \le 0, \\ x + 1, & \text{for } 0 < x \le 2, \\ x^2, & \text{for } x > 2. \end{cases}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Problem 6.15

Sketch the graph of

$$f(x) = \frac{1}{x}\ .$$

Is it continuous?

Problem 6.16

Determine a value for h, such that function

$$f(x) = \begin{cases} x^2 + 2hx, & \text{for } x \leq 2, \\ 3x - h, & \text{for } x > 2, \end{cases}$$

is continuous.

Limits of Continuous Functions

If function f is known to be *continuous*, then its limit $\lim_{x \to x_0} f(x)$ exists for all $x_0 \in D_f$ and we obviously find

$$\lim_{x \to x_0} f(x) = f(x_0) \, .$$

Polynomials are always continuous. Hence

$$\lim_{x \to 2} 3x^2 - 4x + 5 = 3 \cdot 2^2 - 4 \cdot 2 + 5 = 9 \, .$$

Summary

- limit of a sequence
- limit of a function
- convergent and divergent
- Euler's number
- rules for limits
- l'Hôpital's rule
- continuous functions

Chapter 7

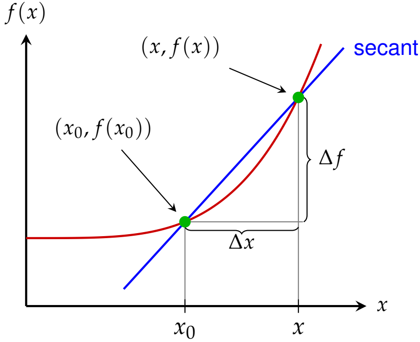
Derivatives

Difference Quotient

Let $f\colon \mathbb{R} \to \mathbb{R}$ be some function. Then the ratio

$$\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}$$

is called **difference quotient**.



Differential Quotient

If the *limit*

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, then function f is called **differentiable** at x_0. This limit is then called **differential quotient** or **(first) derivative** of function f at x_0.

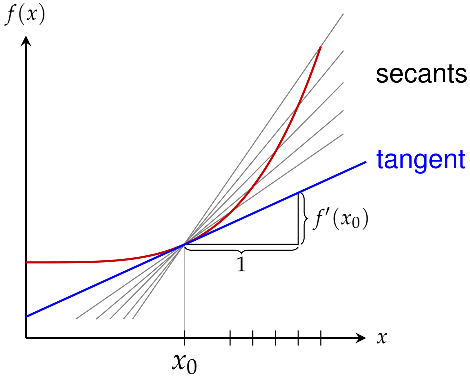
We write

$$f'(x_0) \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}$$

Function f is called *differentiable*, if it is differentiable at each point of its domain.

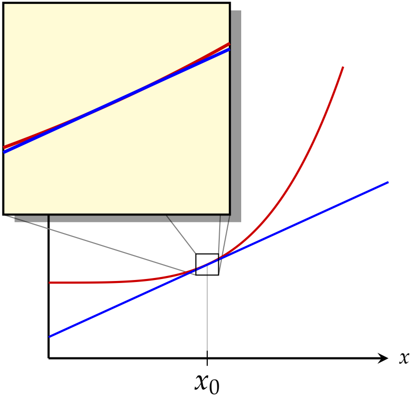
Slope of Tangent

- The differential quotient gives the *slope of the tangent* to the graph of function $f(x)$ at x_0.



Marginal Function

- Instantaneous change of function f.
- “Marginal function” (as in *marginal utility*)



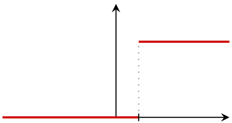
Existence of Differential Quotient

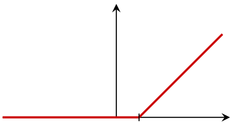
Function f is differentiable at all points, where we can draw the tangent (with finite slope) uniquely to the graph.

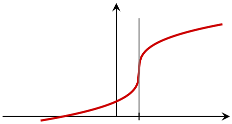
Function f is *not* differentiable at all points where this is not possible.

In particular these are

- jump discontinuities
- "kinks" in the graph of the function
- vertical tangents







Computation of the Differential Quotient

We can compute a differential quotient by determining the limit of the difference quotient.

Let $f(x) = x^2$. The we find for the first derivative

$$\begin{aligned} f'(x_0) &= \lim_{h\to 0} \frac{(x_0+h)^2 - {x_0}^2}{h} \\ &= \lim_{h\to 0} \frac{{x_0}^2 + 2\,x_0\,h + h^2 - {x_0}^2}{h} \\ &= \lim_{h\to 0} \frac{2\,x_0\,h + h^2}{h} = \lim_{h\to 0} (2\,x_0 + h) \\ &= 2\,x_0 \end{aligned}$$

Problem 7.1

Draw (sketch) the graphs of the following functions.
At which points are these function differentiable?

(a) $f(x) = 2x + 2$

(b) $f(x) = 3$

(c) $f(x) = |x|$

(d) $f(x) = \sqrt{|x^2 - 1|}$

(e) $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{for } x \leq -1, \\ x, & \text{for } -1 < x \leq 1, \\ \frac{1}{2}x^2, & \text{for } x > 1. \end{cases}$

(f) $f(x) = \begin{cases} 2 + x, & \text{for } x \leq -1, \\ x^2, & \text{for } x > -1. \end{cases}$

Derivative of a Function

Function

$$f' \colon D \to \mathbb{R},\ x \mapsto f'(x) = \left.\frac{df}{dx}\right|_x$$

is called the **first derivative** of function f.
Its domain D is the set of all points where the differential quotient (i.e., the limit of the difference quotient) exists.

Derivatives of Elementary Functions

$f(x)$	$f'(x)$
c	0
x^α	$\alpha \cdot x^{\alpha-1}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Computation Rules for Derivatives

- $(c \cdot f(x))' = c \cdot f'(x)$
- $(f(x) + g(x))' = f'(x) + g'(x)$ Summation rule
- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ Product rule
- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ Chain rule
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$ Quotient rule

Computation Rules for Derivatives

$$\left(3\,x^3+2\,x-4\right)'=3\cdot 3\cdot x^2+2\cdot 1-0=9\,x^2+2$$

$$\left(e^x\cdot x^2\right)'=\ (e^x)'\cdot x^2+e^x\cdot\left(x^2\right)'=e^x\cdot x^2+e^x\cdot 2\,x$$

$$\left((3\,x^2+1)^2\right)'=2\,(3\,x^2+1)\cdot 6\,x$$

$$\left(\sqrt{x}\right)'=\ \left(x^{\frac{1}{2}}\right)'=\tfrac{1}{2}\cdot x^{-\frac{1}{2}}=\tfrac{1}{2\sqrt{x}}$$

$$(a^x)'=\ \left(e^{\ln(a)\cdot x}\right)'=e^{\ln(a)\cdot x}\cdot\ln(a)=a^x\ \ln(a)$$

$$\left(\frac{1+x^2}{1-x^3}\right)'=\frac{2x\cdot(1-x^3)-(1+x^2)\cdot 3x^2}{(1-x^3)^2}$$

Higher Order Derivatives

We can compute derivatives of the derivative of a function.

Thus we obtain the

- **second derivative** $f''(x)$ of function f,
- **third derivative** $f'''(x)$, etc.,
- n**-th derivative** $f^{(n)}(x)$.

Other notations:

- $f''(x) = \frac{d^2 f}{dx^2}(x) = \left(\frac{d}{dx}\right)^2 f(x)$
- $f^{(n)}(x) = \frac{d^n f}{dx^n}(x) = \left(\frac{d}{dx}\right)^n f(x)$

Higher Order Derivatives

The first five derivatives of function

$$f(x) = x^4 + 2x^2 + 5x - 3$$

are

$$\begin{aligned}
f'(x) &= (x^4 + 2x^2 + 5x - 3)' = 4x^3 + 4x + 5 \\
f''(x) &= (4x^3 + 4x + 5)' = 12x^2 + 4 \\
f'''(x) &= (12x^2 + 4)' = 24x \\
f^{\text{IV}}(x) &= (24x)' = 24 \\
f^{\text{V}}(x) &= 0
\end{aligned}$$

Problem 7.2

Compute the first and second derivative of the following functions:

(a) $f(x) = 4x^4 + 3x^3 - 2x^2 - 1$

(b) $f(x) = e^{-\frac{x^2}{2}}$

(c) $f(x) = \exp\left(-\frac{x^2}{2}\right)$

(d) $f(x) = \frac{x+1}{x-1}$

Problem 7.3

Compute the first and second derivative of the following functions:

(a) $f(x) = \dfrac{1}{1 + x^2}$

(b) $f(x) = \dfrac{1}{(1 + x)^2}$

(c) $f(x) = x \ln(x) - x + 1$

(d) $f(x) = \ln(|x|)$

Problem 7.4

Compute the first and second derivative of the following functions:

(a) $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

(b) $f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

(c) $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

(d) $f(x) = \cos(1 + x^2)$

Problem 7.5

Derive the quotient rule by means of product rule and chain rule.

Marginal Change

We can estimate the derivative $f'(x_0)$ approximately by means of the difference quotient with *small* change Δx:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx \frac{\Delta f}{\Delta x}$$

Vice verse we can estimate the change Δf of f for *small* changes Δx approximately by the first derivative of f:

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

Beware:

- $f'(x_0) \cdot \Delta x$ is a *linear function* in Δx.
- It is the *best possible* approximation of f by a linear function *around* x_0.
- This approximation is useful only for *"small"* values of Δx.

Differential

Approximation

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

becomes exact if Δx (and thus Δf) becomes *infinitesimally small*.
We then write dx and df instead of Δx and Δf, resp.

$$\boxed{df = f'(x_0)\, dx}$$

Symbols df and dx are called the **differentials** of function f and the independent variable x, resp.

Differential

Differential df can be seen as a linear function in dx.
We can use it to compute f approximately around x_0.

$$f(x_0 + dx) \approx f(x_0) + df$$

Let $f(x) = e^x$.

Differential of f at point $x_0 = 1$:
$df = f'(1)\,dx = e^1\,dx$

Approximation of $f(1.1)$ by means of this differential:
$\Delta x = (x_0 + dx) - x_0 = 1.1 - 1 = 0.1$

$f(1.1) \approx f(1) + df = e + e \cdot 0.1 \approx 2.99$

Exact value: $f(1.1) = 3.004166\ldots$

Problem 7.6

Let $f(x) = \dfrac{\ln(x)}{x}$.

Compute $\Delta f = f(3.1) - f(3)$ approximately by means of the differential at point $x_0 = 3$.

Compare your approximation to the exact value.

Elasticity

The first derivative of a function gives *absolute* rate of change of f at x_0. Hence it depends on the scales used for argument and function values.

However, often *relative* rates of change are more appropriate.

We obtain *scale invariance* and *relative* rate of changes by

$$\frac{\text{change of function value relative to value of function}}{\text{change of argument relative to value of argument}}$$

and thus

$$\lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)-f(x)}{f(x)}}{\frac{\Delta x}{x}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \cdot \frac{x}{f(x)} = f'(x) \cdot \frac{x}{f(x)}$$

Elasticity

The expression

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)}$$

is called the **elasticity** of f at point x.

Let $f(x) = 3\,e^{2x}$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{6\,e^{2x}}{3\,e^{2x}} = 2\,x$$

Let $f(x) = \beta\,x^{\alpha}$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{\beta\,\alpha\,x^{\alpha-1}}{\beta\,x^{\alpha}} = \alpha$$

Elasticity II

The relative rate of change of f can be expressed as

$$\ln(f(x))' = \frac{f'(x)}{f(x)}$$

What happens if we compute the derivative of $\ln(f(x))$ w.r.t. $\ln(x)$?

Let $\quad v = \ln(x) \quad \Leftrightarrow \quad x = e^v$

Derivation by means of the chain rule yields:

$$\frac{d(\ln(f(x)))}{d(\ln(x))} = \frac{d(\ln(f(e^v)))}{dv} = \frac{f'(e^v)}{f(e^v)}\, e^v = \frac{f'(x)}{f(x)}\, x = \varepsilon_f(x)$$

$$\boxed{\varepsilon_f(x) = \frac{d(\ln(f(x)))}{d(\ln(x))}}$$

Elasticity II

We can use the chain rule *formally* in the following way:

Let

- $u = \ln(y)$,
- $y = f(x)$,
- $x = e^v \quad \Leftrightarrow \quad v = \ln(x)$

Then we find

$$\frac{d(\ln f)}{d(\ln x)} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = \frac{1}{y} \cdot f'(x) \cdot e^v = \frac{f'(x)}{f(x)}\, x$$

Elastic Functions

A Function f is called

- **elastic** in x, if $\left| \varepsilon_f(x) \right| > 1$
- **1-elastic** in x, if $\left| \varepsilon_f(x) \right| = 1$
- **inelastic** in x, if $\left| \varepsilon_f(x) \right| < 1$

For elastic functions we then have:
The value of the function changes *relatively* faster than the value of the argument.

Function $f(x) = 3\,e^{2x}$ is [$\varepsilon_f(x) = 2\,x$]

- 1-elastic, for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$;
- inelastic, for $-\frac{1}{2} < x < \frac{1}{2}$;
- elastic, for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

Source of Errors

Beware!
Function f is elastic if the **absolute value** of the *elasticity* is greater than 1.

Elastic Demand

Let $q(p)$ be an *elastic* demand function, where p is the price.
We have: $p > 0$, $q > 0$, and $q' < 0$ (q is decreasing). Hence

$$\varepsilon_q(p) = p \cdot \frac{q'(p)}{q(p)} < -1$$

What happens to the revenue (= price $\times$ selling)?

$$\begin{aligned} u'(p) &= (p \cdot q(p))' = 1 \cdot q(p) + p \cdot q'(p) \\ &= q(p) \cdot (1 + \underbrace{p \cdot \frac{q'(p)}{q(p)}}_{=\varepsilon_q < -1}) \\ &< 0 \end{aligned}$$

In other words, the revenue decreases if we raise prices.

Problem 7.7

Compute the regions where the following functions are elastic, 1-elastic and inelastic, resp.

(a) $g(x) = x^3 - 2\,x^2$

(b) $h(x) = \alpha\,x^\beta, \quad \alpha, \beta \neq 0$

Problem 7.8

Which of the following statements are correct?
Suppose function $y = f(x)$ is elastic in its domain.

(a) If x changes by one unit, then the change of y is greater than one unit.

(b) If x changes by one percent, then the relative change of y is greater than one percent.

(c) The relative rate of change of y is larger than the relative rate of change of x.

(d) The larger x is the larger will be y.

Partial Derivative

We investigate the rate of change of function $f(x_1, \ldots, x_n)$, when variable x_i changes and the other variables remain fixed.
Limit

$$\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(\ldots, x_i + \Delta x_i, \ldots) - f(\ldots, x_i, \ldots)}{\Delta x_i}$$

is called the (first) **partial derivative** of f w.r.t. x_i.

Other notations for partial derivative $\frac{\partial f}{\partial x_i}$:

- $f_{x_i}(\mathbf{x})$ (derivative w.r.t. variable x_i)
- $f_i(\mathbf{x})$ (derivative w.r.t. the i-th variable)
- $f_i'(\mathbf{x})$ (i-th component of the gradient)

Computation of Partial Derivatives

We obtain partial derivatives $\frac{\partial f}{\partial x_i}$ by applying the rules for *univariate* functions for variable x_i while we treat *all other* variables *as constants*.

First partial derivatives of

$$f(x_1, x_2) = \sin(2\,x_1) \cdot \cos(x_2)$$

$$f_{x_1} = 2 \cdot \cos(2\,x_1) \cdot \underbrace{\cos(x_2)}_{\text{treated as constant}}$$

$$f_{x_2} = \underbrace{\sin(2\,x_1)}_{\text{treated as constant}} \cdot (-\sin(x_2))$$

Higher Order Partial Derivatives

We can compute partial derivatives of partial derivatives analogously to their univariate counterparts and obtain **higher order partial derivatives**:

$$\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x})$$

Other notations for partial derivative $\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x})$:

- $f_{x_i x_k}(\mathbf{x})$ (derivative w.r.t. variables x_i and x_k)
- $f_{ik}(\mathbf{x})$ (derivative w.r.t. the i-th and k-th variable)
- $f''_{ik}(\mathbf{x})$ (component of the Hessian matrix with index ik)

Higher Order Partial Derivatives

If all second order partial derivatives exists and are *continuous*, then the order of differentiation does not matter (Schwarz's theorem):

$$\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_k}(\mathbf{x})$$

Remark: Practically all differentiable functions in economic models have this property.

Higher Order Partial Derivatives

Compute the first and second order partial derivatives of

$$f(x,y) = x^2 + 3\,x\,y$$

First order partial derivatives:

$$f_x = 2\,x + 3\,y \qquad f_y = 0 + 3\,x$$

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & \qquad f_{xy} &= \boxed{3} \\ f_{yx} &= \boxed{3} & \qquad f_{yy} &= 0 \end{aligned}$$

Problem 7.9

Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$

(b) $f(x,y) = x\,y$

(c) $f(x,y) = x^2 + y^2$

(d) $f(x,y) = x^2\,y^2$

(e) $f(x,y) = x^\alpha\,y^\beta, \quad \alpha, \beta > 0$

Problem 7.10

Compute the first and second order partial derivatives of

$$f(x,y) = \exp(x^2 + y^2)$$

at point $(0,0)$.

Problem 7.11

Compute the first and second order partial derivatives of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

Gradient

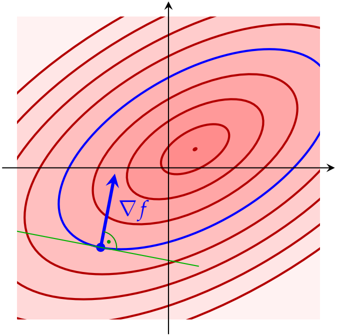
We collect all *first order partial derivatives* into a (row) vector which is called the **gradient** at point $\mathbf{x}$.

$$\boxed{\nabla f(\mathbf{x}) = (f_{x_1}(\mathbf{x}), \dots, f_{x_n}(\mathbf{x}))}$$

- read: "gradient of f" or "nabla f".
- Other notation: $f'(\mathbf{x})$
- Alternatively the gradient can also be a column vector.
- The gradient is the analog of the first derivative of univariate functions.

Properties of the Gradient

- The gradient of f always points in the direction of *steepest ascent*.
- Its length is equal to the slope at this point.
- The gradient is *normal* (i.e. in right angle) to the corresponding *contour line* (level set).



Gradient

Compute the gradient of

$$f(x, y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (3, 2)$.

$$\begin{aligned} f_x &= 2\,x + 3\,y \\ f_y &= 0 + 3\,x \end{aligned}$$

$$\nabla f(\mathbf{x}) = (2x + 3y,\ 3x)$$

$$\nabla f(3, 2) = (12, 9)$$

Problem 7.12

Compute the gradients of the following functions at point $(1, 1)$:

(a) $f(x, y) = x + y$

(b) $f(x, y) = x\,y$

(c) $f(x, y) = x^2 + y^2$

(d) $f(x, y) = x^2\,y^2$

(e) $f(x, y) = x^\alpha\,y^\beta, \quad \alpha, \beta > 0$

Problem 7.13

Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

Hessian Matrix

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be two times differentiable. Then matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \dots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \dots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \dots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

is called the **Hessian matrix** of f at $\mathbf{x}$.

- The Hessian matrix is symmetric, i.e., $f_{x_ix_k}(\mathbf{x}) = f_{x_kx_i}(\mathbf{x})$.
- Other notation: $f''(\mathbf{x})$
- The Hessian matrix is the analog of the second derivative of univariate functions.

Gradient

Compute the Hessian matrix of

$$f(x, y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (1, 2)$.

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & \qquad f_{xy} &= 3 \\ f_{yx} &= 3 & \qquad f_{yy} &= 0 \end{aligned}$$

Hessian matrix:

$$\mathbf{H}_f(x, y) = \begin{pmatrix} f_{xx}(x, y) & f_{xy}(x, y) \\ f_{yx}(x, y) & f_{yy}(x, y) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} = \mathbf{H}_f(1, 2)$$

Problem 7.14

Compute the Hessian matrix of the following functions at point $(1, 1)$:

(a) $f(x, y) = x + y$

(b) $f(x, y) = x\, y$

(c) $f(x, y) = x^2 + y^2$

(d) $f(x, y) = x^2\, y^2$

(e) $f(x, y) = x^\alpha\, y^\beta, \quad \alpha, \beta > 0$

Problem 7.15

Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

Jacobian Matrix

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$

The $m \times n$ matrix

$$D\mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is called the **Jacobian matrix** of $\mathbf{f}$ at point $\mathbf{x}_0$.

It is the generalization of *derivatives* (and gradients) for vector-valued functions.

Jacobian Matrix

- $f(\mathbf{x}) = f(x_1, x_2) = \exp(-x_1^2 - x_2^2)$

  $$D f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = \nabla f(\mathbf{x})$$
  $$= \left(-2\,x_1\, \exp(-x_1^2 - x_2^2), -2\,x_2\, \exp(-x_1^2 - x_2^2)\right)$$

- $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2) = \begin{pmatrix} f_1(x_1, x_2) \\ f_2(x_1, x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \end{pmatrix}$

  $$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2\,x_1 & 2\,x_2 \\ 2\,x_1 & -2\,x_2 \end{pmatrix}$$

- $\mathbf{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

  $$D\mathbf{s}(t) = \begin{pmatrix} \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$

Chain Rule

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g}\colon \mathbb{R}^m \to \mathbb{R}^k$. Then

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x})$$

$$\mathbf{f}(x,y) = \begin{pmatrix} e^x \\ e^y \end{pmatrix} \qquad \mathbf{g}(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$$

$$\mathbf{f}'(x,y) = \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \qquad \mathbf{g}'(x,y) = \begin{pmatrix} 2\,x & 2\,y \\ 2\,x & -2\,y \end{pmatrix}$$

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 2\,e^x & 2\,e^y \\ 2\,e^x & -2\,e^y \end{pmatrix} \cdot \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix}$$

$$= \begin{pmatrix} 2\,e^{2x} & 2\,e^{2y} \\ 2\,e^{2x} & -2\,e^{2y} \end{pmatrix}$$

Example – Indirect Dependency

Let $f(x_1, x_2, t)$ where $x_1(t)$ and $x_2(t)$ also depend on t.
What is the total derivative of f w.r.t. t?

Chain rule:
Let $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \\ t \end{pmatrix}$

$$\begin{aligned}
\frac{df}{dt} &= (f \circ \mathbf{x})'(t) = f'(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\
&= \nabla f(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} = (f_{x_1}(\mathbf{x}(t)), f_{x_2}(\mathbf{x}(t)), f_t(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} \\
&= f_{x_1}(\mathbf{x}(t)) \cdot x_1'(t) + f_{x_2}(\mathbf{x}(t)) \cdot x_2'(t) + f_t(\mathbf{x}(t)) \\
&= f_{x_1}(x_1, x_2, t) \cdot x_1'(t) + f_{x_2}(x_1, x_2, t) \cdot x_2'(t) + f_t(x_1, x_2, t)
\end{aligned}$$

Problem 7.16

Let

$$f(x,y) = x^2 + y^2 \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix} .$$

Compute the derivative of the composite functions

(a) $h = f \circ \mathbf{g}$, and

(b) $\mathbf{p} = \mathbf{g} \circ f$

by means of the chain rule.

Problem 7.17

Let $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^3 - x_2 \\ x_1 - x_2^3 \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ x_1 \end{pmatrix}$.

Compute the derivatives of the composite functions

(a) $\mathbf{g} \circ \mathbf{f}$, and

(b) $\mathbf{f} \circ \mathbf{g}$

by means of the chain rule.

Problem 7.18

Let $Q(K, L, t)$ be a production function, where $L = L(t)$ and $K = K(t)$ also depend on time t. Compute the total derivative $\frac{dQ}{dt}$ by means of the chain rule.

Summary

- difference quotient and differential quotient
- differential quotient and derivative
- derivatives of elementary functions
- differentiation rules
- higher order derivatives
- total differential
- elasticity
- partial derivatives
- gradient and Hessian matrix
- Jacobian matrix and chain rule

Chapter 8

Monotone, Convex and Extrema

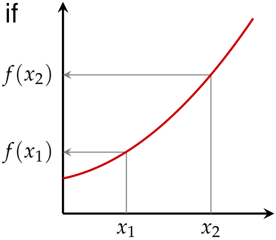
Monotone Functions

Function f is called **monotonically increasing**, if

$$x_1 \leq x_2 \;\Rightarrow\; f(x_1) \leq f(x_2)$$

It is called *strictly monotonically increasing*, if

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)$$

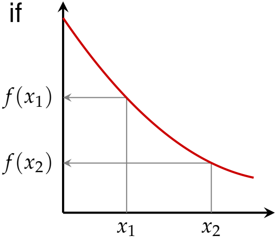


Function f is called **monotonically decreasing**, if

$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

It is called *strictly monotonically decreasing*, if

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) > f(x_2)$$



Monotone Functions

For differentiable functions we have

f monotonically increasing	$\Leftrightarrow$	$f'(x) \geq 0$	for all $x \in D_f$
f monotonically decreasing	$\Leftrightarrow$	$f'(x) \leq 0$	for all $x \in D_f$

f strictly monotonically increasing	$\Leftarrow$	$f'(x) > 0$	for all $x \in D_f$
f strictly monotonically decreasing	$\Leftarrow$	$f'(x) < 0$	for all $x \in D_f$

Function $f\colon (0, \infty),\ x \mapsto \ln(x)$ is strictly monotonically increasing, as

$$f'(x) = (\ln(x))' = \frac{1}{x} > 0 \quad \text{for all } x > 0$$

Locally Monotone Functions

A function f can be monotonically increasing in some interval and decreasing in some other interval.

For *continuously* differentiable functions (i.e., when $f'(x)$ is continuous) we can use the following procedure:

1. Compute first derivative $f'(x)$.
2. Determine all roots of $f'(x)$.
3. We thus obtain intervals where $f'(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f'(x_i)$.

Locally Monotone Functions

In which region is function $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ monotonically increasing?

We have to solve inequality $f'(x) \geq 0$:

1. $f'(x) = 6\,x^2 - 24\,x + 18$
2. Roots: $\quad x^2 - 4\,x + 3 = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 3$
3. Obtain 3 intervals: $\quad (-\infty, 1]$, $[1, 3]$, and $[3, \infty)$
4. Sign of $f'(x)$ at appropriate points in each interval:
   $f'(0) = 3 > 0$, $f'(2) = -1 < 0$, and $f'(4) = 3 > 0$.
5. $f'(x)$ cannot change sign in each interval:
   $f'(x) \geq 0$ in $(\,-\infty, 1]$ and $[3, \infty)$.

Function $f(x)$ is monotonically increasing in $(-\infty, 1]$ and in $[3, \infty)$.

Monotone and Inverse Function

If f is *strictly monotonically increasing*, then

$$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$$

immediately implies

$$x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$$

That is, f is one-to-one.

So if f is onto and strictly monotonically increasing (or decreasing), then f is invertible.

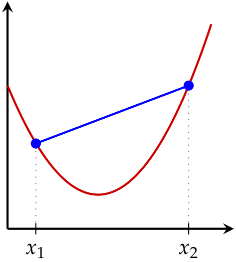
Convex and Concave Functions

Function f is called **convex**, if its domain D_f is an interval and

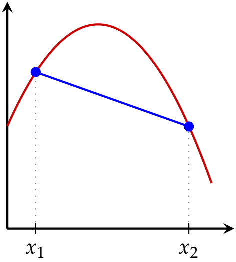
$$f((1-h)\,x_1 + h\,x_2) \le (1-h)\,f(x_1) + h\,f(x_2)$$

for all $x_1, x_2 \in D_f$ and all $h \in [0,1]$. It is called **concave**, if

$$f((1-h)\,x_1 + h\,x_2) \ge (1-h)\,f(x_1) + h\,f(x_2)$$



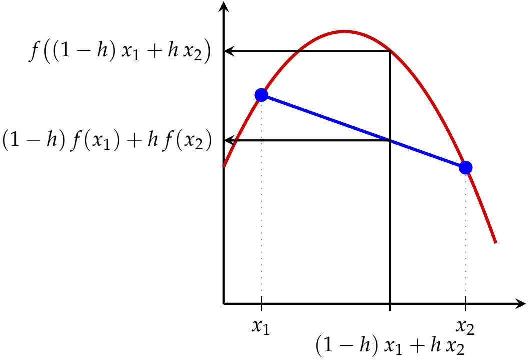
convex



concave

Concave Function

$$f\big((1-h)\,x_1 + h\,x_2\big) \;\geq\; (1-h)\,f(x_1) + h\,f(x_2)$$

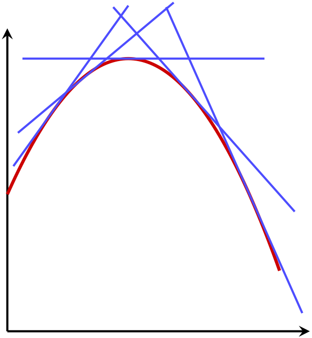


Secant below graph of function

Convex and Concave Functions

For two times differentiable functions we have

$$f \text{ convex} \quad \Leftrightarrow \quad f''(x) \geq 0 \quad \text{for all } x \in D_f$$
$$f \text{ concave} \quad \Leftrightarrow \quad f''(x) \leq 0 \quad \text{for all } x \in D_f$$



$f'(x)$ is
monotonically decreasing,

thus $f''(x) \leq 0$

Strictly Convex and Concave Functions

Function f is called **strictly convex**, if its domain D_f is an interval and

$$f((1-h)\,x_1 + h\,x_2) < (1-h)\,f(x_1) + h\,f(x_2)$$

for all $x_1, x_2 \in D_f$, $x_1 \neq x_2$ and all $h \in (0,1)$.

It is called **strictly concave**, if its domain D_f is an interval and

$$f((1-h)\,x_1 + h\,x_2) > (1-h)\,f(x_1) + h\,f(x_2)$$

For two times differentiable functions we have

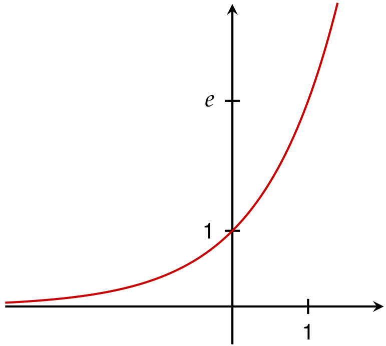
$$\begin{array}{llll} f \text{ strictly convex} & \Leftarrow & f''(x) > 0 & \text{for all } x \in D_f \\ f \text{ strictly concave} & \Leftarrow & f''(x) < 0 & \text{for all } x \in D_f \end{array}$$

Convex Function

Exponential function:

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x > 0 \quad \text{for all } x \in \mathbb{R}$$

$\exp(x)$ is (strictly) convex.

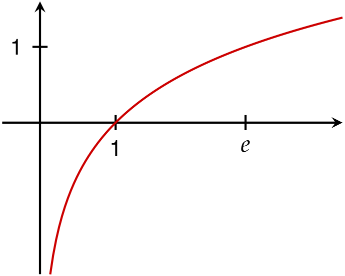


Concave Function

Logarithm function: $(x > 0)$

$$f(x) = \ln(x)$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2} < 0 \quad \text{for all } x > 0$$

$\ln(x)$ is (strictly) concave.



Locally Convex Functions

A function f can be convex in some interval and concave in some other interval.

For two times *continuously* differentiable functions (i.e., when $f''(x)$ is continuous) we can use the following procedure:

1. Compute second derivative $f''(x)$.
2. Determine all roots of $f''(x)$.
3. We thus obtain intervals where $f''(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f''(x_i)$.

Locally Concave Function

In which region is $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ concave?

We have to solve inequality $f''(x) \leq 0$.

1. $f''(x) = 12\,x - 24$
2. Roots: $12\,x - 24 = 0 \quad \Rightarrow \quad x = 2$
3. Obtain 2 intervals: $(-\infty, 2]$ and $[2, \infty)$
4. Sign of $f''(x)$ at appropriate points in each interval:
   $f''(0) = -24 < 0$ and $f''(4) = 24 > 0$.
5. $f''(x)$ cannot change sign in each interval: $f''(x) \leq 0$ in $(\,-\infty, 2]$

Function $f(x)$ is concave in $(-\infty, 2]$.

Problem 8.1

Determine whether the following functions are concave or convex (or neither).

(a) $\exp(x)$

(b) $\ln(x)$

(c) $\log_{10}(x)$

(d) x^{α} for $x > 0$ for an $\alpha \in \mathbb{R}$.

Problem 8.2

In which region is function

$$f(x) = x^3 - 3x^2 - 9x + 19$$

monotonically increasing or decreasing?
In which region is it convex or concave?

Problem 8.3

In which region the following functions monotonically increasing or decreasing?
In which region is it convex or concave?

(a) $f(x) = x\, e^{x^2}$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \dfrac{1}{x^2 + 1}$

Problem 8.4

Function

$$f(x) = b\, x^{1-a}\,, \qquad 0 < a < 1,\, b > 0,\, x \geq 0$$

is an example of a *production function*.
Production functions usually have the following properties:

(1) $f(0) = 0, \quad \lim_{x\to\infty} f(x) = \infty$

(2) $f'(x) > 0, \quad \lim_{x\to\infty} f'(x) = 0$

(3) $f''(x) < 0$

(a) Verify these properties for the given function.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

Problem 8.5

Function

$$f(x) = b \ln(ax+1) \,, \qquad a, b > 0,\ x \geq 0$$

is an example of a utility function.
Utility functions have the same properties as production functions.

(a) Verify the properties from Problem 8.4.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

Problem 8.6

Use the definition of convexity and show that $f(x) = x^2$ is strictly convex.

Hint: Show that inequality $\left(\frac{1}{2}x + \frac{1}{2}y\right)^2 - \left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) < 0$ holds for all $x \neq y$.

Problem 8.7

Show:

If $f(x)$ is a two times differentiable concave function, then $g(x) = -f(x)$ convex.

Problem 8.8

Show:

If $f(x)$ is a concave function, then $g(x) = -f(x)$ convex.
You may not assume that f is differentiable.

Problem 8.9

Let $f(x)$ and $g(x)$ be two differentiable concave functions.
Show that

$$h(x) = \alpha f(x) + \beta g(x) , \qquad \text{for } \alpha, \beta > 0,$$

is a concave function.

What happens, if $\alpha > 0$ and $\beta < 0$?

Problem 8.10

Sketch the graph of a function $f\colon [0,2] \to \mathbb{R}$ with the properties:

- continuous,
- monotonically decreasing,
- strictly concave,
- $f(0) = 1$ and $f(1) = 0$.

In addition find a particular term for such a function.

Problem 8.11

Suppose we relax the condition *strict concave* into *concave* in Problem 8.10.
Can you find a much simpler example?

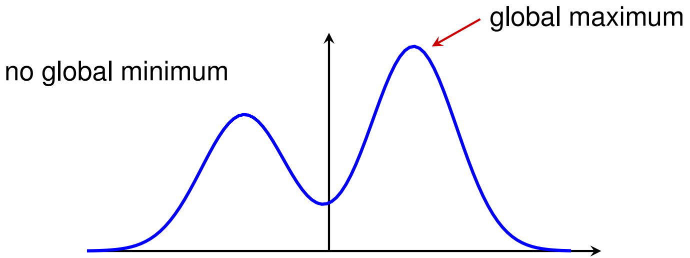
Global Extremum (Optimum)

A point x^* is called **global maximum** (*absolute maximum*) of f, if for all $x \in D_f$,

$$f(x^*) \geq f(x) \, .$$

A point x^* is called **global minimum** (*absolute minimum*) of f, if for all $x \in D_f$,

$$f(x^*) \leq f(x) \, .$$



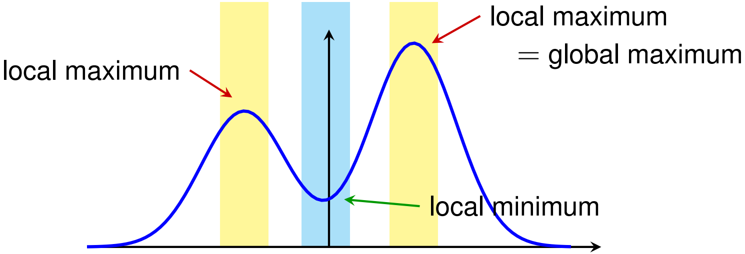
Local Extremum (Optimum)

A point x_0 is called **local maximum** (*relative* maximum) of f,
if for all x in some *neighborhood* of x_0,

$$f(x_0) \geq f(x) \; .$$

A point x_0 is called **local minimum** (*relative minimum*) of f,
if for all x in some neighborhood of x_0,

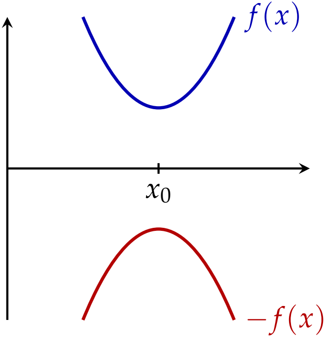
$$f(x_0) \leq f(x) \; .$$



Minima and Maxima

Notice!
Every minimization problem can be transformed into a maximization problem (and vice versa).

Point x_0 is a minimum of $f(x)$, if and only if x_0 is a maximum of $-f(x)$.



Critical Point

At a (local) maximum or minimum the first derivative of the function must vanish (i.e., must be equal 0).

A point x_0 is called a **critical point** (or *stationary point*) of function f, if

$$f'(x_0) = 0$$

Necessary condition for differentiable functions:

Each extremum of f is a critical point of f.

Global Extremum

Sufficient condition:

Let x_0 be a critical point of f.
If f is **concave** then x_0 is a **global maximum** of f.
If f is *convex* then x_0 is a *global minimum* of f.

If f is **strictly** concave (or convex), then the extremum is *unique*.

Global Extremum

Let $f(x) = e^x - 2\,x$.

Function f is strictly convex:

$$f'(x) = e^x - 2$$
$$f''(x) = e^x \quad > 0 \quad \textbf{for all } x \in \mathbb{R}$$

Critical point:

$$f'(x) = e^x - 2 = 0 \quad \Rightarrow \quad x_0 = \ln 2$$

$x_0 = \ln 2$ is the (unique) global minimum of f.

Local Extremum

A point x_0 is a **local maximum** (or *local minimum*) of f, if

- x_0 is a **critical point** of f,
- f is **locally concave** (and *locally convex*, resp.) around x_0.

Local Extremum

Sufficient condition for two times differentiable functions:

Let x_0 be a critical point of f. Then

- $f''(x_0) < 0 \quad \Rightarrow \quad x_0$ is local maximum
- $f''(x_0) > 0 \quad \Rightarrow \quad x_0$ is local minimum

It is sufficient to evaluate $f''(x)$ at the critical point x_0.
(In opposition to the condition for global extrema.)

Necessary and Sufficient

We want to explain two important concepts using the example of local minima.

Condition "$f'(x_0) = 0$" is **necessary** for a local minimum:

Every local minimum must have this properties.
However, not every point with such a property is a local minimum (e.g. $x_0 = 0$ in $f(x) = x^3$).
Stationary points are *candidates* for local extrema.

Condition "$f'(x_0) = 0$ *and* $f''(x_0) > 0$" is **sufficient** for a local minimum.

If it is satisfied, then x_0 is a local minimum.
However, there are local minima where this condition is not satisfied (e.g. $x_0 = 0$ in $f(x) = x^4$).
If it is *not* satisfied, we cannot draw *any conclusion*.

Procedure for Local Extrema

Sufficient condition
for local extrema of a differentiable function in *one* variable:

1. Compute $f'(x)$ and $f''(x)$.

2. Find all roots x_i of $f'(x_i) = 0$ (critical points).

3. If $f''(x_i) < 0 \quad \Rightarrow \quad x_i$ is a *local maximum*.

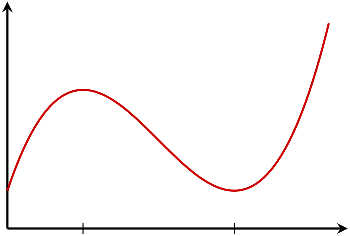
If $f''(x_i) > 0 \quad \Rightarrow \quad x_i$ is a *local minimum*.

If $f''(x_i) = 0 \quad \Rightarrow$ *no conclusion possible*!

Local Extrema

Find all local extrema of

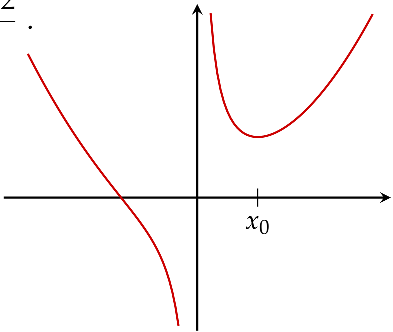
$$f(x) = \frac{1}{12}\,x^3 - x^2 + 3\,x + 1$$



1. $f'(x) = \frac{1}{4}\,x^2 - 2\,x + 3$,
   $f''(x) = \frac{1}{2}\,x - 2$.
2. $\frac{1}{4}\,x^2 - 2\,x + 3 = 0$
   has roots
   $x_1 = 2$ and $x_2 = 6$.
3. $f''(2) = -1 \quad \Rightarrow \quad x_1$ is a local maximum.
   $f''(6) = 1 \quad \Rightarrow \quad x_2$ is a local minimum.

Sources of Errors

Find all global minima of $f(x) = \dfrac{x^3+2}{3x}$.



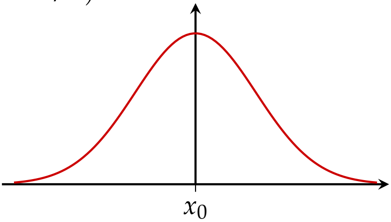
1. $f'(x) = \frac{2(x^3-1)}{3x^2}$,
   $f''(x) = \frac{2x^3+4}{3x^3}$.
2. critical point at $x_0 = 1$.
3. $f''(1) = 2 > 0$
   $\Rightarrow$ global minimum **???**

However, looking *just* at $f''(\mathbf{1})$ is not sufficient as we are looking for *global* minima!

Beware! We have to look at $f''(x)$ *at all* $x \in D_f$.
However, $f''(-1) = -\frac{2}{3} < 0$.
Moreover, domain $D = \mathbb{R} \setminus \{0\}$ is not an interval.
So f is not convex and we cannot apply our theorem.

Sources of Errors

Find all global maxima of $f(x) = \exp(-x^2/2)$.



1. $f'(x) = x \exp(-x^2)$,
   $f''(x) = (x^2 - 1) \exp(-x^2)$.

2. critical point at $x_0 = 0$.

3. However,
   $f''(0) = -1 < 0$ but $f''(2) = 2e^{-2} > 0$.
   So f is not concave and thus there cannot be a global maximum.
   Really ???

Beware! We are checking a *sufficient* condition.
Since an assumption does not hold (f is not concave),
we simply **cannot apply** the theorem.
We *cannot* conclude that f does not have a global maximum.

Global Extrema in $[a, b]$

Extrema of $f(x)$ in **closed** interval $[a, b]$.

Procedure for differentiable functions:

(1) Compute $f'(x)$.

(2) Find all stationary points x_i (i.e., $f'(x_i) = 0$).

(3) Evaluate $f(x)$ for all *candidates*:

- all stationary points x_i,
- boundary points a and b.

(4) Largest of these values is **global maximum**,
smallest of these values is **global minimum**.

It is *not* necessary to compute $f''(x_i)$.

Global Extrema in $[a, b]$

Find all *global* extrema of function

$$f\colon [0{,}5; 8{,}5] \to \mathbb{R},\ x \mapsto \frac{1}{12}\,x^3 - x^2 + 3\,x + 1$$

(1) $f'(x) = \frac{1}{4}\,x^2 - 2\,x + 3$.

(2) $\frac{1}{4}\,x^2 - 2\,x + 3 = 0$ has roots $x_1 = 2$ and $x_2 = 6$.

(3)
$$\begin{aligned} f(0.5) &= 2.260 & & \\ f(2) &= 3.667 & & \\ f(6) &= 1.000 & \Rightarrow &\quad \text{global minimum} \\ f(8.5) &= 5.427 & \Rightarrow &\quad \text{global maximum} \end{aligned}$$

(4) $x_2 = 6$ is the global minimum and
$b = 8.5$ is the global maximum of f.

Global Extrema in (a, b)

Extrema of $f(x)$ in **open** interval (a, b) (or $(-\infty, \infty)$).

Procedure for differentiable functions:

(1) Compute $f'(x)$.

(2) Find all stationary points x_i (i.e., $f'(x_i) = 0$).

(3) Evaluate $f(x)$ for all *stationary* points x_i.

(4) Determine $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$.

(5) Largest of these values is **global maximum**,
smallest of these values is **global minimum**.

(6) A global extremum exists **only if** the largest (smallest) value is obtained in a *stationary point*!

Global Extrema in (a, b)

Compute all *global* extrema of

$$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-x^2}$$

(1) $f'(x) = -2x\, e^{-x^2}$.

(2) $f'(x) = -2x\, e^{-x^2} = 0$ has unique root $x_1 = 0$.

(3)
$$\begin{aligned} f(0) = 1 \quad &\Rightarrow \quad \text{global maximum} \\ \lim_{x\to-\infty} f(x) = 0 \quad &\Rightarrow \quad \text{no global minimum} \\ \lim_{x\to\infty} f(x) = 0 \quad & \end{aligned}$$

(4) The function has a global maximum in $x_1 = 0$, but no global minimum.

Existence and Uniqueness

- A function need not have maxima or minima:

  $$f\colon (0,1) \to \mathbb{R},\ x \mapsto x$$

  (Points 1 and -1 are not in domain $(0,1)$.)

- (Global) maxima need not be unique:

  $$f\colon \mathbb{R} \to \mathbb{R}, x \mapsto x^4 - 2\,x^2$$

  has two global minima at -1 and 1.

Problem 8.12

Find all local extrema of the following functions.

(a) $f(x) = e^{-x^2}$

(b) $g(x) = \frac{x^2+1}{x}$

(c) $h(x) = (x-3)^6$

Problem 8.13

Find all global extrema of the following functions.

(a) $f\colon (0,\infty) \to \mathbb{R},\ x \mapsto \frac{1}{x} + x$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x} - x$

(c) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-2x} + 2x$

(d) $f\colon (0,\infty) \to \mathbb{R},\ x \mapsto x - \ln(x)$

(e) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-x^2}$

Problem 8.14

Compute all global maxima and minima of the following functions.

(a) $f(x) = \frac{x^3}{12} - \frac{5}{4}x^2 + 4x - \frac{1}{2}$ in interval $[1, 12]$

(b) $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + 2$ in interval $[-2, 6]$

(c) $f(x) = x^4 - 2\,x^2$ in interval $[-2, 2]$

Summary

- monotonically increasing and decreasing
- convex and concave
- global and local extrema

Chapter 9

Integration

Antiderivative

A function $F(x)$ is called an **antiderivative** (or *primitive*) of function $f(x)$, if

$$F'(x) = f(x)$$

Computation:

Guess and verify

Example: We want the antiderivative of $f(x) = \ln(x)$.

Guess: $F(x) = x\,(\ln(x) - 1)$

Verify: $F'(x) = (x\,(\ln(x) - 1))' =$

$$= 1 \cdot (\ln(x) - 1) + x \cdot \frac{1}{x} = \ln(x)$$

But also: $F(x) = x\,(\ln(x) - 1) + 5$

Antiderivative

The antiderivative is denoted by symbol

$$\boxed{\int f(x)\,dx + c}$$

and is also called the **indefinite integral** of function f. Number c is called **integration constant**.

Unfortunately, there are no *"recipes"* for computing antiderivatives (but tools one can try and which may help).

There are functions where antiderivatives cannot be expressed by means of elementary functions.

E.g., the antiderivative of $\exp(-\frac{1}{2}x^2)$.

Basic Integrals

Integrals of some elementary functions:

$f(x)$	$\int f(x)\,dx$
0	c
x^a	$\frac{1}{a+1} \cdot x^{a+1} + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$

(Table is created by exchanging the columns in our list of derivatives.)

Integration Rules

- **Summation rule**

$$\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

- **Integration by parts**

$$\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$$

- **Integration by substitution**

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(z)\,dz$$

with $z = g(x)$ and $dz = g'(x)\,dx$

Example – Summation Rule

Antiderivative of $f(x) = 4\,x^3 - x^2 + 3\,x - 5$.

$$
\begin{aligned}
\int f(x)\,dx &= \int 4\,x^3 - x^2 + 3\,x - 5\,dx \\
&= 4\int x^3\,dx - \int x^2\,dx + 3\int x\,dx - 5\int dx \\
&= 4\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3\frac{1}{2}x^2 - 5x + c \\
&= x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + c
\end{aligned}
$$

Example – Integration by Parts

Antiderivative of $f(x) = x \cdot e^x$.

$$\int \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'} \, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g} \, dx = x \cdot e^x - e^x + c$$

$$\begin{aligned} f = x \quad &\Rightarrow \quad f' = 1 \\ g' = e^x \quad &\Rightarrow \quad g = e^x \end{aligned}$$

Example – Integration by Parts

Antiderivative of $f(x) = x^2 \cos(x)$.

$$\int \underbrace{x^2}_{f} \cdot \underbrace{\cos(x)}_{g'} \, dx = \underbrace{x^2}_{f} \cdot \underbrace{\sin(x)}_{g} - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_{g} \, dx$$

Integration by parts of the second terms yields:

$$\begin{aligned} \int \underbrace{2x}_{f} \cdot \underbrace{\sin(x)}_{g'} \, dx &= \underbrace{2x}_{f} \cdot \underbrace{(-\cos(x))}_{g} - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_{g} \, dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c \end{aligned}$$

Thus the antiderivative of f is given by

$$\int x^2 \cos(x) \, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$

Example – Integration by Substitution

Antiderivative of $f(x) = 2x \cdot e^{x^2}$.

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)}\,dx = \int \exp(z)\,dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad \Rightarrow \quad dz = g'(x)\,dx = 2x\,dx$$

Integration Rules – Derivation

Integration by parts follows from the product rule for derivatives:

$$\begin{aligned} f(x) \cdot g(x) &= \int (f(x) \cdot g(x))' \, dx = \int \big(f'(x)\, g(x) + f(x)\, g'(x)\big)\, dx \\ &= \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx \end{aligned}$$

Integration by substitution follows from the chain rule:
Let F be an antiderivative of f and let $z = g(x)$. Then

$$\begin{aligned} \int f(z)\, dz = F(z) = F(g(x)) &= \int (F(g(x)))'\, dx \\ &= \int F'(g(x))\, g'(x)\, dx = \int f(g(x))\, g'(x)\, dx \end{aligned}$$

Problem 9.1

Compute the antiderivatives of the following functions by means of integration by parts.

(a) $f(x) = 2x\, e^x$

(b) $f(x) = x^2\, e^{-x}$

(c) $f(x) = x\ \ln(x)$

(d) $f(x) = x^3\ \ln x$

(e) $f(x) = x\,(\ln(x))^2$

(f) $f(x) = x^2\ \sin(x)$

Problem 9.2

Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\int x\, e^{x^2}\, dx$

(b) $\int 2\, x\, \sqrt{x^2 + 6}\, dx$

(c) $\int \frac{x}{3\, x^2 + 4}\, dx$

(d) $\int x\, \sqrt{x + 1}\, dx$

(e) $\int \frac{\ln(x)}{x}\, dx$

Problem 9.3

Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\displaystyle\int \frac{1}{x \ln x}\, dx$

(b) $\displaystyle\int \sqrt{x^3 + 1}\, x^2\, dx$

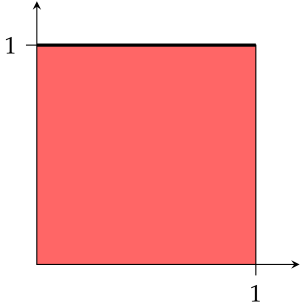
(c) $\displaystyle\int \frac{x}{\sqrt{5 - x^2}}\, dx$

(d) $\displaystyle\int \frac{x^2 - x + 1}{x - 3}\, dx$

(e) $\displaystyle\int x(x-8)^{\frac{1}{2}}\, dx$

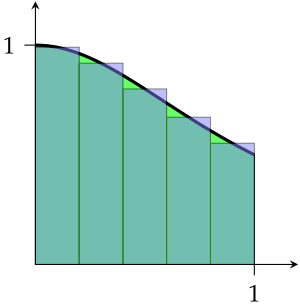
Area

Compute the areas of the given regions.



$f(x) = 1$

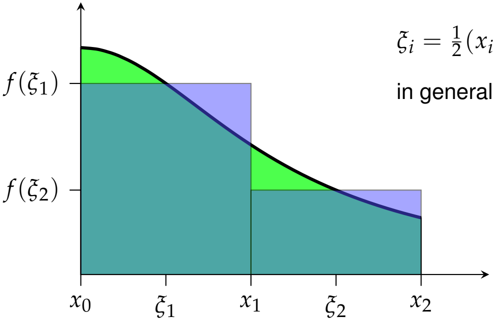
Area: $A = 1$



$f(x) = \frac{1}{1+x^2}$

Approximation
by step function

Riemann Sum



$\xi_i = \frac{1}{2}(x_{i-1} + x_i)$

in general: $\xi_i \in (x_i, x_{i-1})$

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

Riemann Integral

$$I_n = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

is called a **Riemann sum** of f.

It can be shown that in many cases these Riemann sums converge when the length of the longest interval tends to 0.
This limit then is called the **Riemann integral** (or integral for short) of f.

Riemann Integral – Properties

$$\int_a^b (\alpha f(x) + \beta g(x))\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

$$\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$$

$$\int_a^a f(x)\, dx = 0$$

$$\int_a^c f(x)\, dx = \int_a^b f(x)\, dx + \int_b^c f(x)\, dx$$

$$\int_a^b f(x)\, dx \le \int_a^b g(x)\, dx \qquad \text{if } f(x) \le g(x) \text{ for all } x \in [a, b]$$

Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a *continuous* function $f(x)$, then we find

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason $\int_a^b f(x)\,dx$ is called a **definite integral** of f.

Example:

Compute the integral of $f(x) = x^2$ over interval $[0, 1]$.

$$\int_0^1 x^2\,dx = \tfrac{1}{3}\,x^3\Big|_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \frac{1}{3}$$

Integration Rules / (Definite Integrals)

- **Summation rule**

$$\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

- **Integration by parts**

$$\int_a^b f \cdot g'\,dx = f \cdot g\Big|_a^b - \int_a^b f' \cdot g\,dx$$

- **Integration by Substitution**

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(z)\,dz$$

with $z = g(x)$ and $dz = g'(x)\,dx$

Example – Integration by Parts

Compute the definite integral $\int_0^2 x \cdot e^x \, dx$.

$$\int_0^2 \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'} \, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} \Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g} \, dx$$

$$= x \cdot e^x \Big|_0^2 - e^x \Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0)$$

$$= e^2 + 1$$

Note: we also could use our indefinite integral from above,

$$\int_0^2 x \cdot e^x \, dx = (x \cdot e^x - e^x) \Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$

Example – Integration by Substitution

Compute the definite integral $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx$.

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx \quad = \quad \int_1^{\ln(10)} \frac{1}{z}\,dz =$$

$$z = \ln(x) \quad \Rightarrow \quad dz = \frac{1}{x}\,dx$$

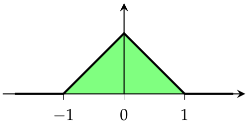
$$= \quad \ln(z)\Big|_1^{\ln(10)} =$$

$$= \quad \ln(\ln(10)) - \ln(1) \approx 0.834$$

Example

Compute $\int_{-2}^{2} f(x)\,dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$



We have

$$\begin{aligned} \int_{-2}^{2} f(x)\,dx &= \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx \\ &= \int_{-2}^{-1} 0\,dx + \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx + \int_{1}^{2} 0\,dx \\ &= \left(x + \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(x - \tfrac{1}{2}x^2\right)\Big|_{0}^{1} \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned}$$

Problem 9.4

Compute the following definite integrals:

(a) $\int_1^4 2x^2 - 1\,dx$

(b) $\int_0^2 3e^x\,dx$

(c) $\int_1^4 3x^2 + 4x\,dx$

(d) $\int_0^{\frac{\pi}{3}} \frac{-\sin(x)}{3}\,dx$

(e) $\int_0^1 \frac{3\,x + 2}{3\,x^2 + 4\,x + 1}\,dx$

Problem 9.5

Compute the following definite integrals by means of antiderivatives:

(a) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

(b) $\displaystyle\int_0^1 x\,(x^2+3)^4\,dx$

(c) $\displaystyle\int_0^2 x\,\sqrt{4-x^2}\,dx$

(d) $\displaystyle\int_1^2 \frac{x}{x^2+1}\,dx$

Problem 9.6

Compute the following definite integrals by means of antiderivatives:

(a) $\int_0^2 x \, \exp\left(-\frac{x^2}{2}\right) \, dx$

(b) $\int_0^3 (x-1)^2 \, x \, dx$

(c) $\int_0^1 x \exp(x) \, dx$

(d) $\int_0^2 x^2 \exp(x) \, dx$

(e) $\int_1^2 x^2 \ln x \, dx$

Problem 9.7

Compute $\int_{-2}^{2} x^2 f(x)\,dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \leq x < 0, \\ 1-x, & \text{for } 0 \leq x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \geq 1. \end{cases}$$

Problem 9.8

Compute $F(x) = \int_{-2}^{x} f(t)\,dt$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

Summary

- antiderivate
- Riemann sum and Riemann integral
- indefinite and definite integral
- Fundamental Theorem of Calculus
- integration rules