Chapter 9

Integration

Antiderivative

A function $F(x)$ is called an **antiderivative** (or *primitive*) of function $f(x)$, if

$$F'(x) = f(x)$$

Computation:

Guess and verify

Example: We want the antiderivative of $f(x) = \ln(x)$.

Guess: $F(x) = x\,(\ln(x) - 1)$

Verify:
$$\begin{aligned} F'(x) &= (x\,(\ln(x) - 1))' = \\ &= 1 \cdot (\ln(x) - 1) + x \cdot \frac{1}{x} = \ln(x) \end{aligned}$$

But also: $F(x) = x\,(\ln(x) - 1) + 5$

Antiderivative

The antiderivative is denoted by symbol

$$\int f(x)\, dx + c$$

and is also called the **indefinite integral** of function f. Number c is called **integration constant**.

Unfortunately, there are no *“recipes”* for computing antiderivatives (but tools one can try and which may help).

There are functions where antiderivatives cannot be expressed by means of elementary functions.

E.g., the antiderivative of $\exp(-\frac{1}{2}x^2)$.

Basic Integrals

Integrals of some elementary functions:

$f(x)$	$\int f(x)\,dx$
0	c
x^a	$\frac{1}{a+1} \cdot x^{a+1} + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln\lvert x \rvert + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$

(Table is created by exchanging the columns in our list of derivatives.)

Integration Rules

- **Summation rule**

$$\int \alpha f(x) + \beta g(x)\, dx = \alpha \int f(x)\, dx + \beta \int g(x)\, dx$$

- **Integration by parts**

$$\int f \cdot g'\, dx = f \cdot g - \int f' \cdot g\, dx$$

- **Integration by substitution**

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(z)\, dz$$

with $z = g(x)$ and $dz = g'(x)\, dx$

Example – Summation Rule

Antiderivative of $f(x) = 4\,x^3 - x^2 + 3\,x - 5$.

$$\begin{aligned}
\int f(x)\,dx &= \int 4\,x^3 - x^2 + 3\,x - 5\,dx \\
&= 4\int x^3\,dx - \int x^2\,dx + 3\int x\,dx - 5\int dx \\
&= 4\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3\frac{1}{2}x^2 - 5x + c \\
&= x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + c
\end{aligned}$$

Example – Integration by Parts

Antiderivative of $f(x) = x \cdot e^x$.

$$\int \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'} \, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g} \, dx = x \cdot e^x - e^x + c$$

$$\begin{aligned} f &= x & \Rightarrow \quad f' &= 1 \\ g' &= e^x & \Rightarrow \quad g &= e^x \end{aligned}$$

Example – Integration by Parts

Antiderivative of $f(x) = x^2 \cos(x)$.

$$\int \underbrace{x^2}_{f} \cdot \underbrace{\cos(x)}_{g'}\, dx = \underbrace{x^2}_{f} \cdot \underbrace{\sin(x)}_{g} - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_{g}\, dx$$

Integration by parts of the second terms yields:

$$\begin{aligned}\int \underbrace{2x}_{f} \cdot \underbrace{\sin(x)}_{g'}\, dx &= \underbrace{2x}_{f} \cdot \underbrace{(-\cos(x))}_{g} - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_{g}\, dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c\end{aligned}$$

Thus the antiderivative of f is given by

$$\int x^2 \cos(x)\, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$

Example – Integration by Substitution

Antiderivative of $f(x) = 2x \cdot e^{x^2}$.

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)} \, dx = \int \exp(z) \, dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad \Rightarrow \quad dz = g'(x) \, dx = 2x \, dx$$

Integration Rules – Derivation

Integration by parts follows from the product rule for derivatives:

$$f(x) \cdot g(x) = \int (f(x) \cdot g(x))' \, dx = \int \big(f'(x)\, g(x) + f(x)\, g'(x)\big) \, dx$$
$$= \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

Integration by substitution follows from the chain rule:
Let F be an antiderivative of f and let $z = g(x)$. Then

$$\int f(z)\, dz = F(z) = F(g(x)) = \int (F(g(x)))' \, dx$$
$$= \int F'(g(x))\, g'(x)\, dx = \int f(g(x))\, g'(x)\, dx$$

Problem 9.1

Compute the antiderivatives of the following functions by means of integration by parts.

(a) $f(x) = 2x\, e^x$

(b) $f(x) = x^2\, e^{-x}$

(c) $f(x) = x\, \ln(x)$

(d) $f(x) = x^3\, \ln x$

(e) $f(x) = x\, (\ln(x))^2$

(f) $f(x) = x^2\, \sin(x)$

Problem 9.2

Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\int x\, e^{x^2}\, dx$

(b) $\int 2\, x\, \sqrt{x^2 + 6}\, dx$

(c) $\int \frac{x}{3\, x^2 + 4}\, dx$

(d) $\int x\, \sqrt{x + 1}\, dx$

(e) $\int \frac{\ln(x)}{x}\, dx$

Problem 9.3

Compute the antiderivatives of the following functions by means of integration by substitution.

(a) $\displaystyle\int \frac{1}{x\,\ln x}\,dx$

(b) $\displaystyle\int \sqrt{x^3+1}\,x^2\,dx$

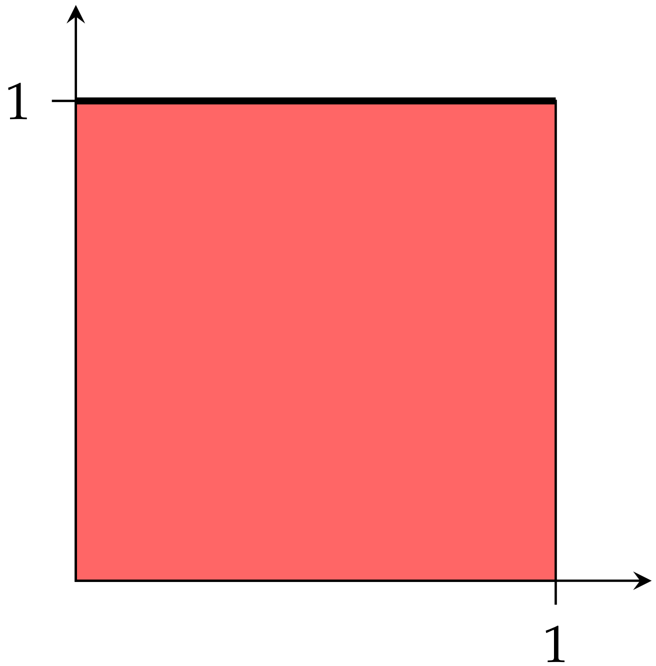
(c) $\displaystyle\int \frac{x}{\sqrt{5-x^2}}\,dx$

(d) $\displaystyle\int \frac{x^2-x+1}{x-3}\,dx$

(e) $\displaystyle\int x(x-8)^{\frac{1}{2}}\,dx$

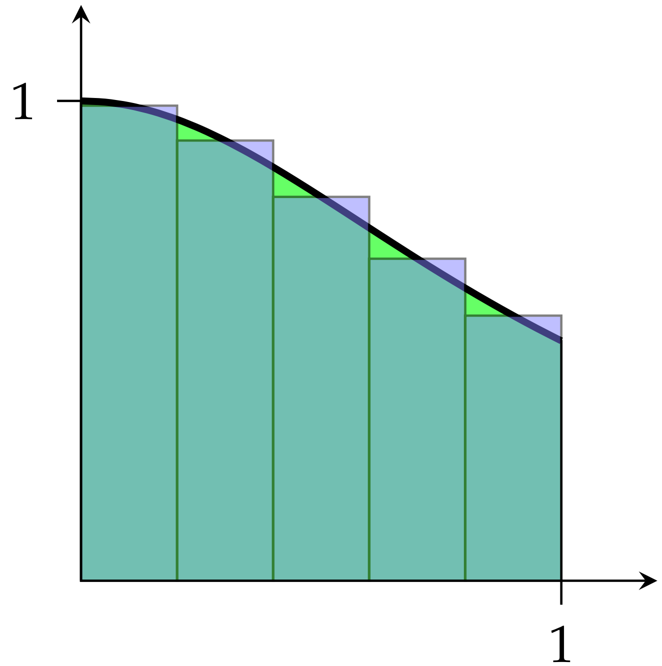
Area

Compute the areas of the given regions.



$f(x) = 1$

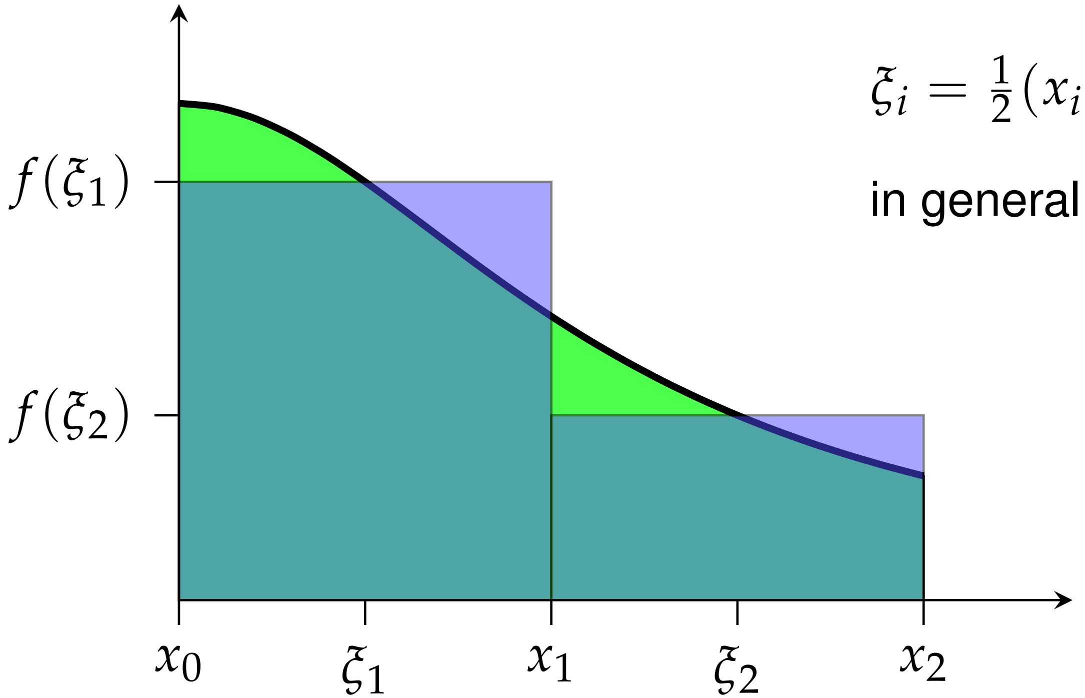
Area: $A = 1$



$f(x) = \frac{1}{1+x^2}$

Approximation
by step function

Riemann Sum



$\xi_i = \frac{1}{2}(x_{i-1} + x_i)$

in general: $\xi_i \in (x_i, x_{i-1})$

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

Riemann Integral

$$I_n = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

is called a **Riemann sum** of f.

It can be shown that in many cases these Riemann sums converge when the length of the longest interval tends to 0.
This limit then is called the **Riemann integral** (or integral for short) of f.

Riemann Integral – Properties

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \qquad \text{if } f(x) \leq g(x) \text{ for all } x \in [a,b]$$

Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a *continuous* function $f(x)$, then we find

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason $\int_a^b f(x)\,dx$ is called a **definite integral** of f.

Example:

Compute the integral of $f(x) = x^2$ over interval $[0, 1]$.

$$\int_0^1 x^2\,dx = \tfrac{1}{3}\,x^3\Big|_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \frac{1}{3}$$

Integration Rules / (Definite Integrals)

- **Summation rule**

$$\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

- **Integration by parts**

$$\int_a^b f \cdot g'\,dx = f \cdot g\Big|_a^b - \int_a^b f' \cdot g\,dx$$

- **Integration by Substitution**

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(z)\,dz$$

with $z = g(x)$ and $dz = g'(x)\,dx$

Example – Integration by Parts

Compute the definite integral $\int_0^2 x \cdot e^x \, dx$.

$$\int_0^2 \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'} \, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} \Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g} \, dx$$

$$= x \cdot e^x \Big|_0^2 - e^x \Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0)$$

$$= e^2 + 1$$

Note: we also could use our indefinite integral from above,

$$\int_0^2 x \cdot e^x \, dx = (x \cdot e^x - e^x) \Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$

Example – Integration by Substitution

Compute the definite integral $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx$.

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx \quad = \quad \int_1^{\ln(10)} \frac{1}{z}\,dz =$$

$$z = \ln(x) \quad \Rightarrow \quad dz = \frac{1}{x}\,dx$$

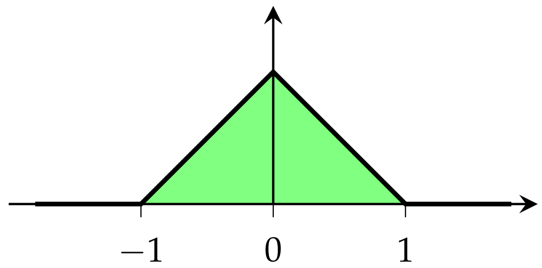
$$= \quad \ln(z)\Big|_1^{\ln(10)} =$$

$$= \quad \ln(\ln(10)) - \ln(1) \approx 0.834$$

Example

Compute $\int_{-2}^{2} f(x)\,dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$



We have

$$\begin{aligned} \int_{-2}^{2} f(x)\,dx &= \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx \\ &= \int_{-2}^{-1} 0\,dx + \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx + \int_{1}^{2} 0\,dx \\ &= \left(x + \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(x - \tfrac{1}{2}x^2\right)\Big|_{0}^{1} \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned}$$

Problem 9.4

Compute the following definite integrals:

(a) $\int_1^4 2x^2 - 1\,dx$

(b) $\int_0^2 3e^x\,dx$

(c) $\int_1^4 3x^2 + 4x\,dx$

(d) $\int_0^{\frac{\pi}{3}} \frac{-\sin(x)}{3}\,dx$

(e) $\int_0^1 \frac{3\,x + 2}{3\,x^2 + 4\,x + 1}\,dx$

Problem 9.5

Compute the following definite integrals by means of antiderivatives:

(a) $\displaystyle\int_1^e \frac{\ln x}{x}\,dx$

(b) $\displaystyle\int_0^1 x\,(x^2+3)^4\,dx$

(c) $\displaystyle\int_0^2 x\,\sqrt{4-x^2}\,dx$

(d) $\displaystyle\int_1^2 \frac{x}{x^2+1}\,dx$

Problem 9.6

Compute the following definite integrals by means of antiderivatives:

(a) $\int_0^2 x \exp\left(-\frac{x^2}{2}\right) dx$

(b) $\int_0^3 (x-1)^2 \, x \, dx$

(c) $\int_0^1 x \exp(x) \, dx$

(d) $\int_0^2 x^2 \exp(x) \, dx$

(e) $\int_1^2 x^2 \ln x \, dx$

Problem 9.7

Compute $\int_{-2}^{2} x^2 f(x)\, dx$ for function

$$f(x) = \begin{cases} 1 + x, & \text{for } -1 \le x < 0, \\ 1 - x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

Problem 9.8

Compute $F(x) = \int_{-2}^{x} f(t)\,dt$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$

Summary

- antiderivate
- Riemann sum and Riemann integral
- indefinite and definite integral
- Fundamental Theorem of Calculus
- integration rules