Chapter 8

Monotone, Convex and Extrema

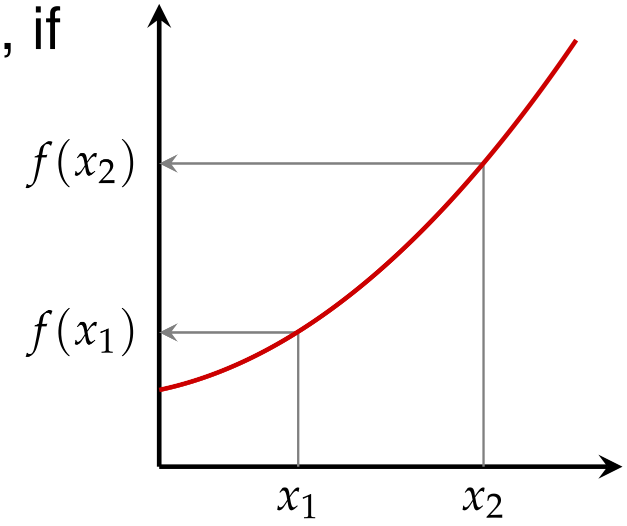
Monotone Functions

Function f is called **monotonically increasing**, if

$$x_1 \leq x_2 \;\Rightarrow\; f(x_1) \leq f(x_2)$$

It is called *strictly monotonically increasing*, if

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)$$

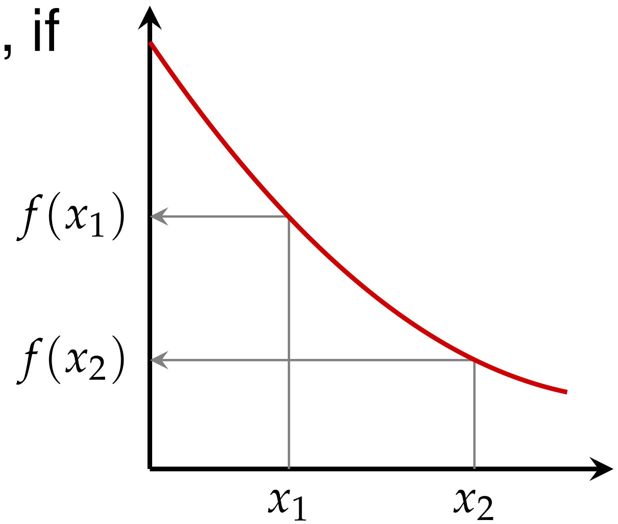


Function f is called **monotonically decreasing**, if

$$x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)$$

It is called *strictly monotonically decreasing*, if

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) > f(x_2)$$



Monotone Functions

For differentiable functions we have

f monotonically increasing	$\Leftrightarrow$	$f'(x) \geq 0$	for all $x \in D_f$
f monotonically decreasing	$\Leftrightarrow$	$f'(x) \leq 0$	for all $x \in D_f$

f strictly monotonically increasing	$\Leftarrow$	$f'(x) > 0$	for all $x \in D_f$
f strictly monotonically decreasing	$\Leftarrow$	$f'(x) < 0$	for all $x \in D_f$

Function $f\colon (0, \infty),\ x \mapsto \ln(x)$ is strictly monotonically increasing, as

$$f'(x) = (\ln(x))' = \frac{1}{x} > 0 \quad \text{for all } x > 0$$

Locally Monotone Functions

A function f can be monotonically increasing in some interval and decreasing in some other interval.

For *continuously* differentiable functions (i.e., when $f'(x)$ is continuous) we can use the following procedure:

1. Compute first derivative $f'(x)$.
2. Determine all roots of $f'(x)$.
3. We thus obtain intervals where $f'(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f'(x_i)$.

Locally Monotone Functions

In which region is function $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ monotonically increasing?

We have to solve inequality $f'(x) \geq 0$:

1. $f'(x) = 6\,x^2 - 24\,x + 18$
2. Roots: $x^2 - 4\,x + 3 = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 3$
3. Obtain 3 intervals: $(-\infty, 1]$, $[1, 3]$, and $[3, \infty)$
4. Sign of $f'(x)$ at appropriate points in each interval:
   $f'(0) = 3 > 0$, $f'(2) = -1 < 0$, and $f'(4) = 3 > 0$.
5. $f'(x)$ cannot change sign in each interval:
   $f'(x) \geq 0$ in $(-\infty, 1]$ and $[3, \infty)$.

Function $f(x)$ is monotonically increasing in $(-\infty, 1]$ and in $[3, \infty)$.

Monotone and Inverse Function

If f is *strictly monotonically increasing*, then

$$x_1 < x_2 \Leftrightarrow f(x_1) < f(x_2)$$

immediately implies

$$x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)$$

That is, f is one-to-one.

So if f is onto and strictly monotonically increasing (or decreasing), then f is invertible.

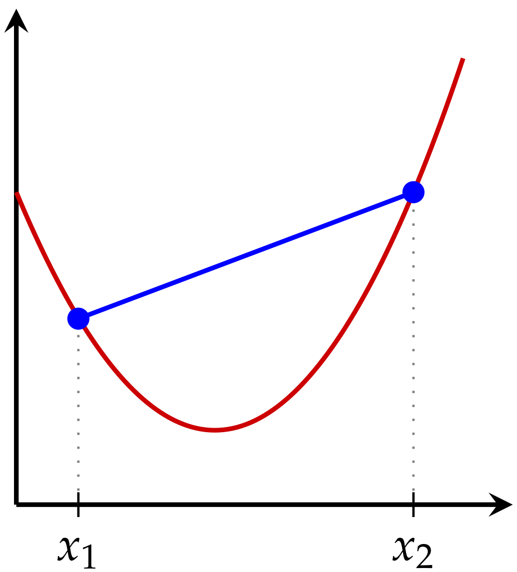
Convex and Concave Functions

Function f is called **convex**, if its domain D_f is an interval and

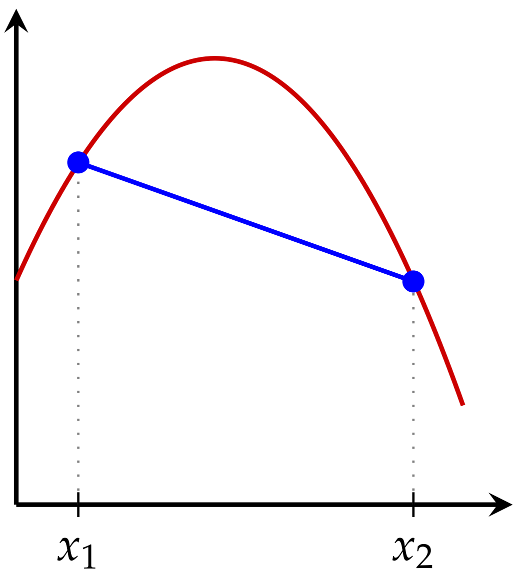
$$f((1-h)\,x_1 + h\,x_2) \le (1-h)\,f(x_1) + h\,f(x_2)$$

for all $x_1, x_2 \in D_f$ and all $h \in [0,1]$. It is called **concave**, if

$$f((1-h)\,x_1 + h\,x_2) \ge (1-h)\,f(x_1) + h\,f(x_2)$$



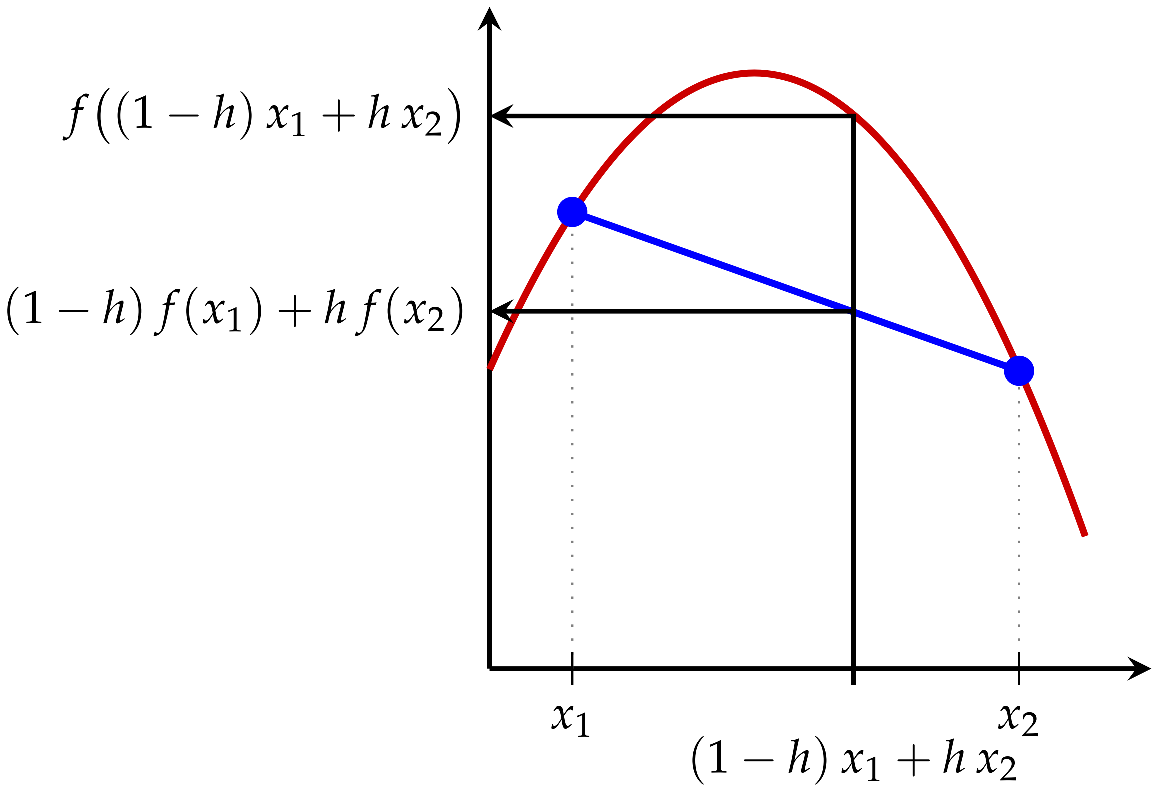
convex



concave

Concave Function

$$f\big((1-h)\,x_1 + h\,x_2\big) \;\geq\; (1-h)\,f(x_1) + h\,f(x_2)$$

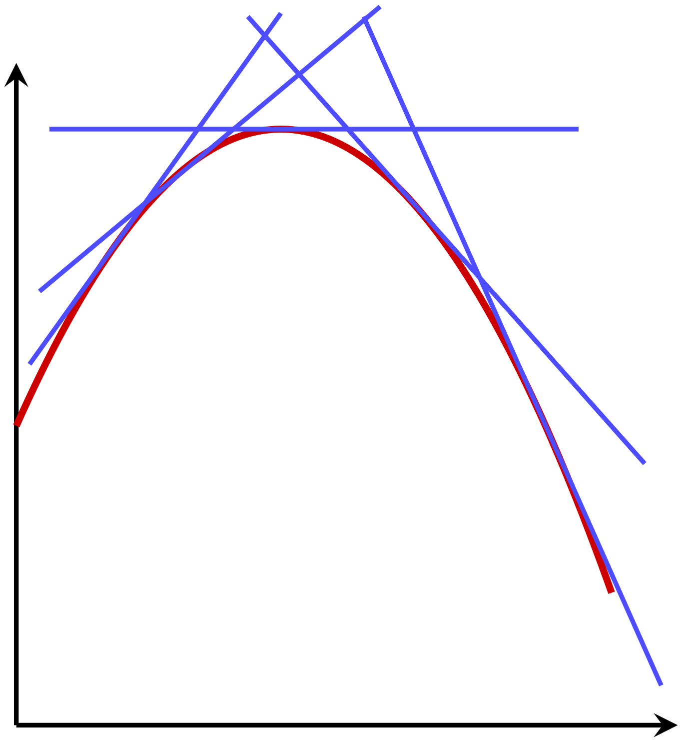


Secant below graph of function

Convex and Concave Functions

For two times differentiable functions we have

f convex	$\Leftrightarrow$	$f''(x) \geq 0$	for all $x \in D_f$
f concave	$\Leftrightarrow$	$f''(x) \leq 0$	for all $x \in D_f$



$f'(x)$ is
monotonically decreasing,

thus $f''(x) \leq 0$

Strictly Convex and Concave Functions

Function f is called **strictly convex**, if its domain D_f is an interval and

$$f((1-h)\,x_1 + h\,x_2) < (1-h)\,f(x_1) + h\,f(x_2)$$

for all $x_1, x_2 \in D_f$, $x_1 \neq x_2$ and all $h \in (0,1)$.

It is called **strictly concave**, if its domain D_f is an interval and

$$f((1-h)\,x_1 + h\,x_2) > (1-h)\,f(x_1) + h\,f(x_2)$$

For two times differentiable functions we have

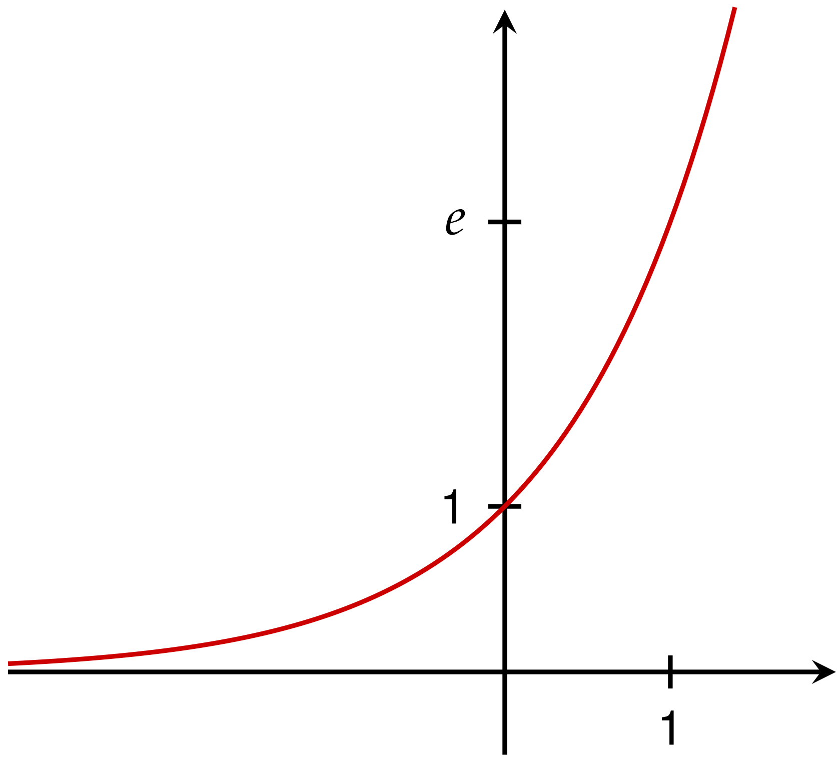
f strictly convex	$\Leftarrow$	$f''(x) > 0$	for all $x \in D_f$
f strictly concave	$\Leftarrow$	$f''(x) < 0$	for all $x \in D_f$

Convex Function

Exponential function:

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x > 0 \quad \text{for all } x \in \mathbb{R}$$

$\exp(x)$ is (strictly) convex.

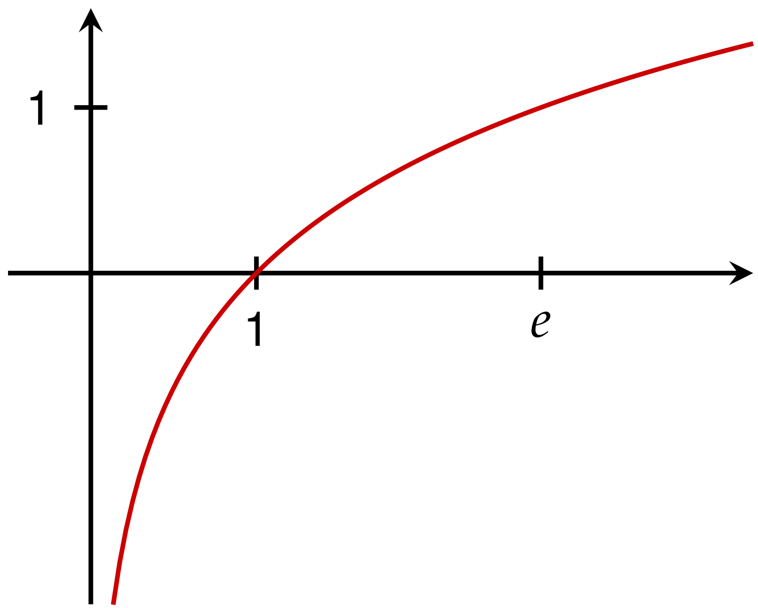


Concave Function

Logarithm function: $(x > 0)$

$$f(x) = \ln(x)$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2} < 0 \quad \text{for all } x > 0$$

$\ln(x)$ is (strictly) concave.



Locally Convex Functions

A function f can be convex in some interval and concave in some other interval.

For two times *continuously* differentiable functions (i.e., when $f''(x)$ is continuous) we can use the following procedure:

1. Compute second derivative $f''(x)$.
2. Determine all roots of $f''(x)$.
3. We thus obtain intervals where $f''(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f''(x_i)$.

Locally Concave Function

In which region is $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ concave?

We have to solve inequality $f''(x) \leq 0$.

1. $f''(x) = 12\,x - 24$
2. Roots: $12\,x - 24 = 0 \quad \Rightarrow \quad x = 2$
3. Obtain 2 intervals: $(-\infty, 2]$ and $[2, \infty)$
4. Sign of $f''(x)$ at appropriate points in each interval:
   $f''(0) = -24 < 0$ and $f''(4) = 24 > 0$.
5. $f''(x)$ cannot change sign in each interval: $f''(x) \leq 0$ in $(-\infty, 2]$

Function $f(x)$ is concave in $(-\infty, 2]$.

Problem 8.1

Determine whether the following functions are concave or convex (or neither).

(a) $\exp(x)$

(b) $\ln(x)$

(c) $\log_{10}(x)$

(d) x^α for $x > 0$ for an $\alpha \in \mathbb{R}$.

Problem 8.2

In which region is function

$$f(x) = x^3 - 3x^2 - 9x + 19$$

monotonically increasing or decreasing?
In which region is it convex or concave?

Problem 8.3

In which region the following functions monotonically increasing or decreasing?
In which region is it convex or concave?

(a) $f(x) = x\, e^{x^2}$

(b) $f(x) = e^{-x^2}$

(c) $f(x) = \dfrac{1}{x^2 + 1}$

Problem 8.4

Function

$$f(x) = b\,x^{1-a}\,, \qquad 0 < a < 1,\, b > 0,\, x \geq 0$$

is an example of a *production function*.
Production functions usually have the following properties:

(1) $f(0) = 0, \quad \lim_{x\to\infty} f(x) = \infty$

(2) $f'(x) > 0, \quad \lim_{x\to\infty} f'(x) = 0$

(3) $f''(x) < 0$

(a) Verify these properties for the given function.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

Problem 8.5

Function

$$f(x) = b\ \ln(ax+1)\ , \qquad a, b > 0,\ x \geq 0$$

is an example of a utility function.
Utility functions have the same properties as production functions.

(a) Verify the properties from Problem 8.4.

(b) Draw (sketch) the graphs of $f(x)$, $f'(x)$, and $f'(x)$.
(Use appropriate values for a and b.)

(c) What is the economic interpretation of these properties?

Problem 8.6

Use the definition of convexity and show that $f(x) = x^2$ is strictly convex.

Hint: Show that inequality $\left(\frac{1}{2}x + \frac{1}{2}y\right)^2 - \left(\frac{1}{2}x^2 + \frac{1}{2}y^2\right) < 0$ holds for all $x \neq y$.

Problem 8.7

Show:

If $f(x)$ is a two times differentiable concave function, then $g(x) = -f(x)$ convex.

Problem 8.8

Show:

If $f(x)$ is a concave function, then $g(x) = -f(x)$ convex.
You may not assume that f is differentiable.

Problem 8.9

Let $f(x)$ and $g(x)$ be two differentiable concave functions.
Show that

$$h(x) = \alpha\, f(x) + \beta\, g(x)\,, \qquad \text{for } \alpha, \beta > 0,$$

is a concave function.

What happens, if $\alpha > 0$ and $\beta < 0$?

Problem 8.10

Sketch the graph of a function $f\colon [0,2] \to \mathbb{R}$ with the properties:

- continuous,
- monotonically decreasing,
- strictly concave,
- $f(0) = 1$ and $f(1) = 0$.

In addition find a particular term for such a function.

Problem 8.11

Suppose we relax the condition *strict concave* into *concave* in Problem 8.10.
Can you find a much simpler example?

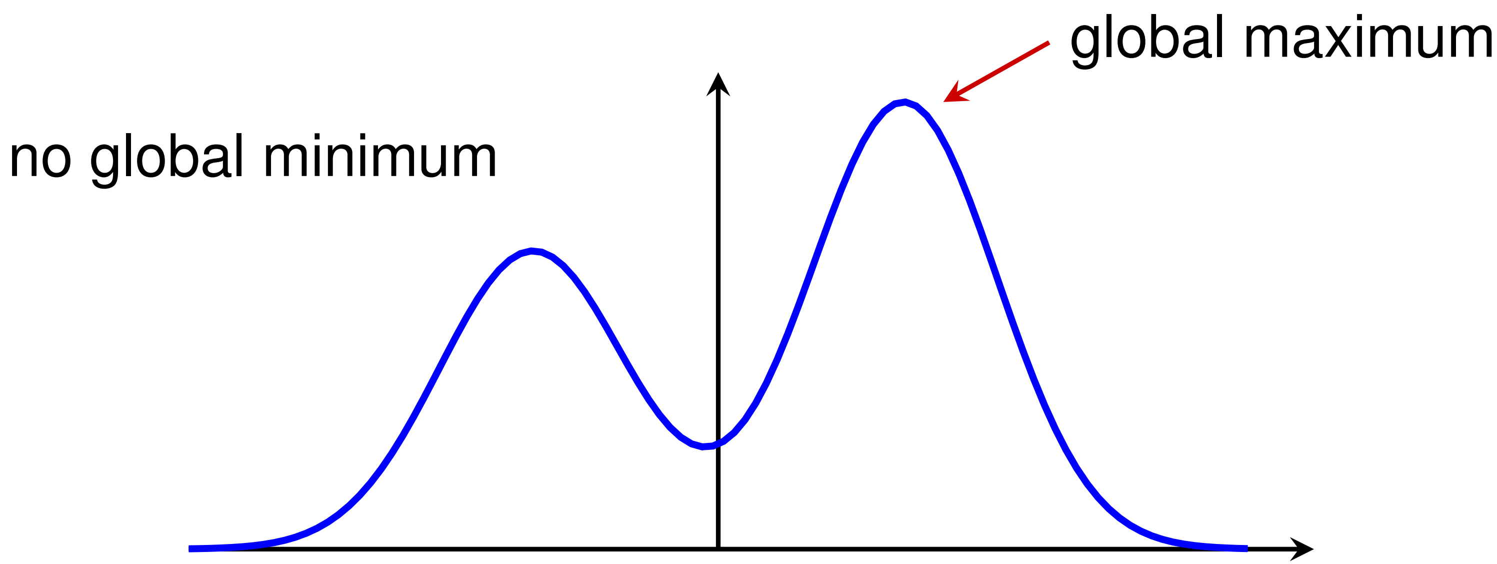
Global Extremum (Optimum)

A point x^* is called **global maximum** (*absolute maximum*) of f, if for all $x \in D_f$,

$$f(x^*) \geq f(x) .$$

A point x^* is called **global minimum** (*absolute minimum*) of f, if for all $x \in D_f$,

$$f(x^*) \leq f(x) .$$



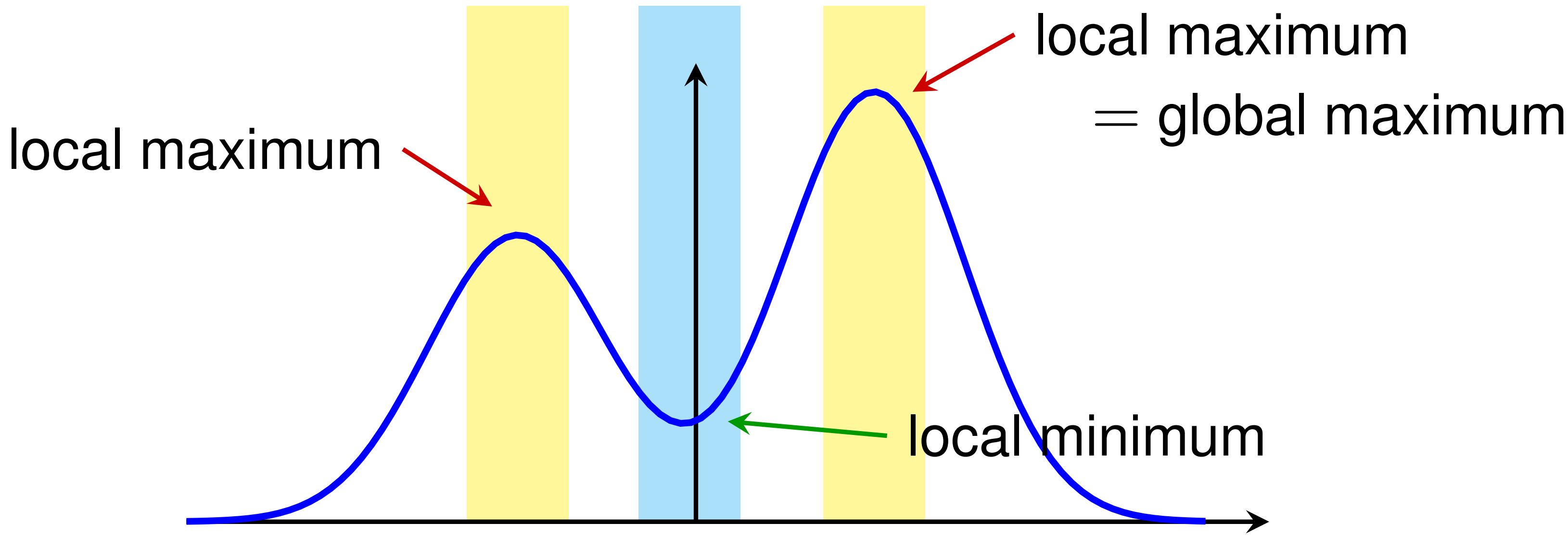
Local Extremum (Optimum)

A point x_0 is called **local maximum** (*relative* maximum) of f, if for all x in some *neighborhood* of x_0,

$$f(x_0) \geq f(x) .$$

A point x_0 is called **local minimum** (*relative minimum*) of f, if for all x in some neighborhood of x_0,

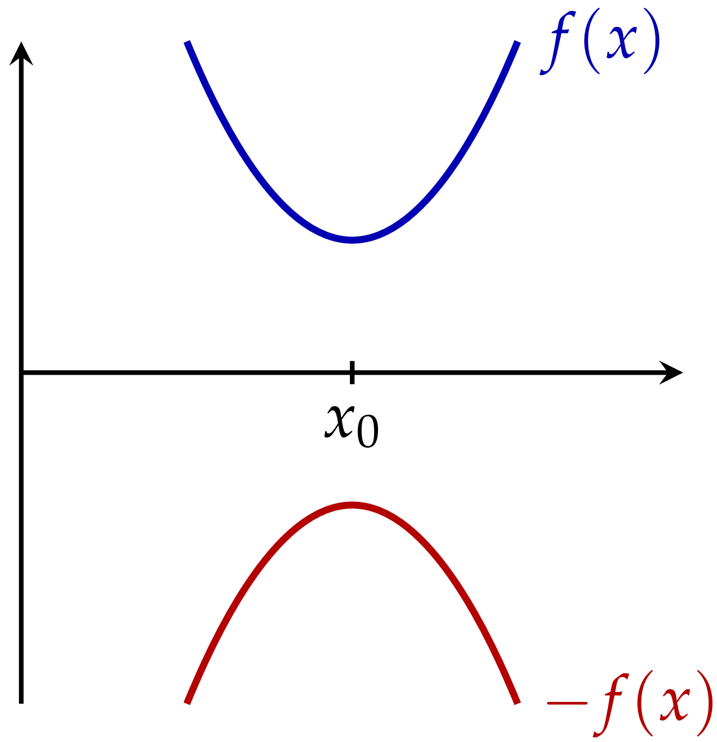
$$f(x_0) \leq f(x) .$$



Minima and Maxima

Notice!
Every minimization problem can be transformed into a maximization problem (and vice versa).

Point x_0 is a minimum of $f(x)$,
if and only if x_0 is
a maximum of $-f(x)$.



Critical Point

At a (local) maximum or minimum the first derivative of the function must vanish (i.e., must be equal 0).

A point x_0 is called a **critical point** (or *stationary point*) of function f, if

$$f'(x_0) = 0$$

Necessary condition for differentiable functions:

Each extremum of f is a critical point of f.

Global Extremum

Sufficient condition:

Let x_0 be a critical point of f.
If f is **concave** then x_0 is a **global maximum** of f.
If f is *convex* then x_0 is a *global minimum* of f.

If f is **strictly** concave (or convex), then the extremum is *unique*.

Global Extremum

Let $f(x) = e^x - 2x$.

Function f is strictly convex:

$$f'(x) = e^x - 2$$
$$f''(x) = e^x > 0 \quad \textbf{for all } x \in \mathbb{R}$$

Critical point:

$$f'(x) = e^x - 2 = 0 \quad \Rightarrow \quad x_0 = \ln 2$$

$x_0 = \ln 2$ is the (unique) global minimum of f.

Local Extremum

A point x_0 is a **local maximum** (or *local minimum*) of f, if

- x_0 is a **critical point** of f,
- f is **locally concave** (and *locally convex*, resp.) around x_0.

Local Extremum

Sufficient condition for two times differentiable functions:

Let x_0 be a critical point of f. Then

- $f''(x_0) < 0 \quad \Rightarrow \quad x_0$ is local maximum
- $f''(x_0) > 0 \quad \Rightarrow \quad x_0$ is local minimum

It is sufficient to evaluate $f''(x)$ at the critical point x_0.
(In opposition to the condition for global extrema.)

Necessary and Sufficient

We want to explain two important concepts using the example of local minima.

Condition "$f'(x_0) = 0$" is **necessary** for a local minimum:

Every local minimum must have this properties.
However, not every point with such a property is a local minimum (e.g. $x_0 = 0$ in $f(x) = x^3$).
Stationary points are *candidates* for local extrema.

Condition "$f'(x_0) = 0$ *and* $f''(x_0) > 0$" is **sufficient** for a local minimum.

If it is satisfied, then x_0 is a local minimum.
However, there are local minima where this condition is not satisfied (e.g. $x_0 = 0$ in $f(x) = x^4$).
If it is *not* satisfied, we cannot draw *any conclusion*.

Procedure for Local Extrema

Sufficient condition
for local extrema of a differentiable function in *one* variable:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all roots x_i of $f'(x_i) = 0$ (critical points).
3. If $f''(x_i) < 0 \quad \Rightarrow \quad x_i$ is a *local maximum*.

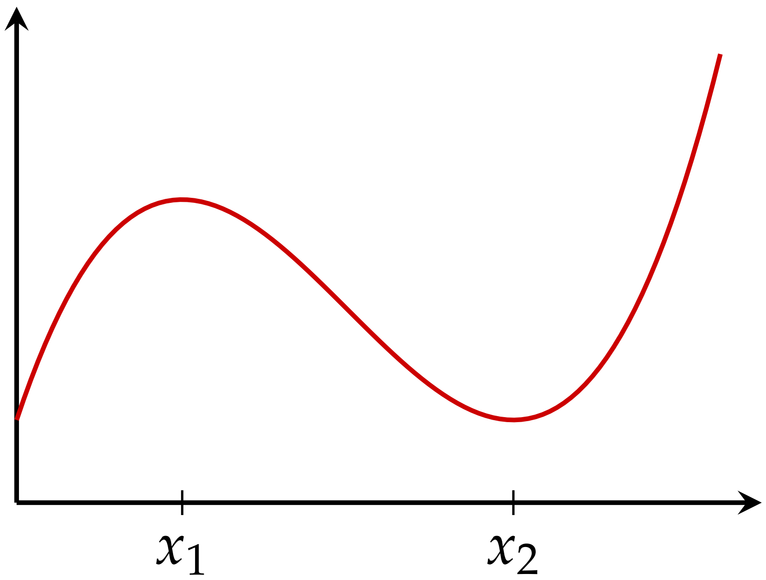
   If $f''(x_i) > 0 \quad \Rightarrow \quad x_i$ is a *local minimum*.

   If $f''(x_i) = 0 \quad \Rightarrow$ *no conclusion possible*!

Local Extrema

Find all local extrema of

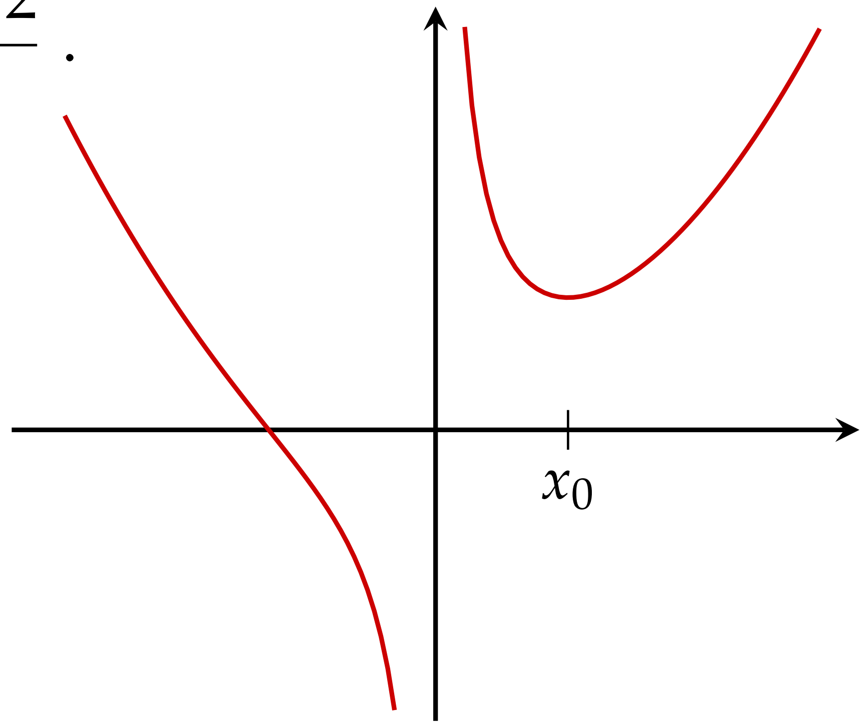
$$f(x) = \frac{1}{12}x^3 - x^2 + 3x + 1$$



1. $f'(x) = \frac{1}{4}x^2 - 2x + 3$,
   $f''(x) = \frac{1}{2}x - 2$.
2. $\frac{1}{4}x^2 - 2x + 3 = 0$
   has roots
   $x_1 = 2$ and $x_2 = 6$.
3. $f''(2) = -1 \Rightarrow$ x_1 is a local maximum.
   $f''(6) = 1 \Rightarrow$ x_2 is a local minimum.

Sources of Errors

Find all global minima of $f(x) = \frac{x^3+2}{3x}$.



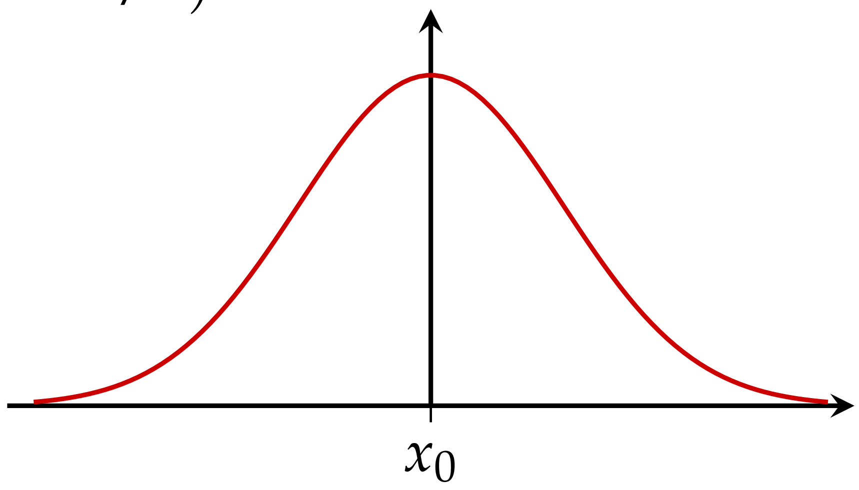
1. $f'(x) = \frac{2(x^3-1)}{3x^2}$,
   $f''(x) = \frac{2x^3+4}{3x^3}$.
2. critical point at $x_0 = 1$.
3. $f''(1) = 2 > 0$
   $\Rightarrow$ global minimum **???**

However, looking *just* at $f''(\mathbf{1})$ is not sufficient as we are looking for *global* minima!

Beware! We have to look at $f''(x)$ *at all* $x \in D_f$.
However, $f''(-1) = -\frac{2}{3} < 0$.
Moreover, domain $D = \mathbb{R} \setminus \{0\}$ is not an interval.
So f is not convex and we cannot apply our theorem.

Sources of Errors

Find all global maxima of $f(x) = \exp(-x^2/2)$.



1. $f'(x) = x\exp(-x^2)$,
   $f''(x) = (x^2 - 1)\exp(-x^2)$.
2. critical point at $x_0 = 0$.
3. However,
   $f''(0) = -1 < 0$ but $f''(2) = 2e^{-2} > 0$.
   So f is not concave and thus there cannot be a global maximum.
   Really ???

Beware! We are checking a *sufficient* condition.
Since an assumption does not hold (f is not concave),
we simply **cannot apply** the theorem.
We *cannot* conclude that f does not have a global maximum.

Global Extrema in $[a, b]$

Extrema of $f(x)$ in **closed** interval $[a, b]$.

Procedure for differentiable functions:

(1) Compute $f'(x)$.

(2) Find all stationary points x_i (i.e., $f'(x_i) = 0$).

(3) Evaluate $f(x)$ for all *candidates*:
- all stationary points x_i,
- boundary points a and b.

(4) Largest of these values is **global maximum**,
smallest of these values is **global minimum**.

It is *not* necessary to compute $f''(x_i)$.

Global Extrema in $[a, b]$

Find all *global* extrema of function

$$f\colon [0{,}5; 8{,}5] \to \mathbb{R},\; x \mapsto \frac{1}{12}\, x^3 - x^2 + 3\,x + 1$$

(1) $f'(x) = \frac{1}{4}\, x^2 - 2\,x + 3$.

(2) $\frac{1}{4}\, x^2 - 2\,x + 3 = 0$ has roots $x_1 = 2$ and $x_2 = 6$.

(3)
$$\begin{aligned} f(0.5) &= 2.260 \\ f(2) &= 3.667 \\ f(6) &= 1.000 \quad \Rightarrow \quad \text{global minimum} \\ f(8.5) &= 5.427 \quad \Rightarrow \quad \text{global maximum} \end{aligned}$$

(4) $x_2 = 6$ is the global minimum and
$b = 8.5$ is the global maximum of f.

Global Extrema in (a, b)

Extrema of $f(x)$ in **open** interval (a, b) (or $(-\infty, \infty)$).

Procedure for differentiable functions:

(1) Compute $f'(x)$.

(2) Find all stationary points x_i (i.e., $f'(x_i) = 0$).

(3) Evaluate $f(x)$ for all *stationary* points x_i.

(4) Determine $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$.

(5) Largest of these values is **global maximum**, smallest of these values is **global minimum**.

(6) A global extremum exists **only if** the largest (smallest) value is obtained in a *stationary point*!

Global Extrema in (a, b)

Compute all *global* extrema of

$$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-x^2}$$

(1) $f'(x) = -2x\, e^{-x^2}$.

(2) $f'(x) = -2x\, e^{-x^2} = 0$ has unique root $x_1 = 0$.

(3) $f(0) = 1 \quad \Rightarrow \quad$ global maximum

$\lim_{x\to-\infty} f(x) = 0 \quad \Rightarrow \quad$ no global minimum

$\lim_{x\to\infty} f(x) = 0$

(4) The function has a global maximum in $x_1 = 0$, but no global minimum.

Existence and Uniqueness

- A function need not have maxima or minima:

$$f : (0,1) \to \mathbb{R},\ x \mapsto x$$

(Points 1 and -1 are not in domain $(0,1)$.)

- (Global) maxima need not be unique:

$$f : \mathbb{R} \to \mathbb{R},\, x \mapsto x^4 - 2\,x^2$$

has two global minima at -1 and 1.

Problem 8.12

Find all local extrema of the following functions.

(a) $f(x) = e^{-x^2}$

(b) $g(x) = \frac{x^2+1}{x}$

(c) $h(x) = (x-3)^6$

Problem 8.13

Find all global extrema of the following functions.

(a) $f\colon (0,\infty) \to \mathbb{R},\ x \mapsto \frac{1}{x} + x$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x} - x$

(c) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-2x} + 2x$

(d) $f\colon (0,\infty) \to \mathbb{R},\ x \mapsto x - \ln(x)$

(e) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-x^2}$

Problem 8.14

Compute all global maxima and minima of the following functions.

(a) $f(x) = \frac{x^3}{12} - \frac{5}{4}x^2 + 4x - \frac{1}{2}$ in interval $[1, 12]$

(b) $f(x) = \frac{2}{3}x^3 - \frac{5}{2}x^2 - 3x + 2$ in interval $[-2, 6]$

(c) $f(x) = x^4 - 2\,x^2$ in interval $[-2, 2]$

Summary

- monotonically increasing and decreasing
- convex and concave
- global and local extrema