Chapter 7

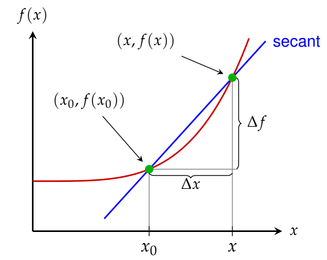
Derivatives

Difference Quotient

Let $f\colon \mathbb{R} \to \mathbb{R}$ be some function. Then the ratio

$$\boxed{\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}}$$

is called **difference quotient**.



Differential Quotient

If the *limit*

$$\boxed{\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

exists, then function f is called **differentiable** at x_0. This limit is then called **differential quotient** or **(first) derivative** of function f at x_0.

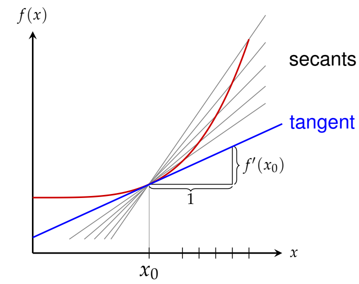
We write

$$\boxed{f'(x_0) \quad \text{or} \quad \left.\frac{df}{dx}\right|_{x=x_0}}$$

Function f is called *differentiable*, if it is differentiable at each point of its domain.

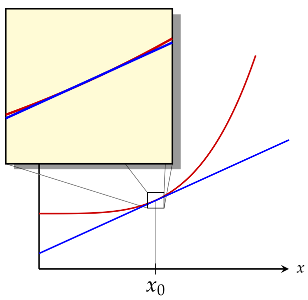
Slope of Tangent

- The differential quotient gives the *slope of the tangent* to the graph of function $f(x)$ at x_0.



Marginal Function

- Instantaneous change of function f.
- "Marginal function" (as in *marginal utility*)



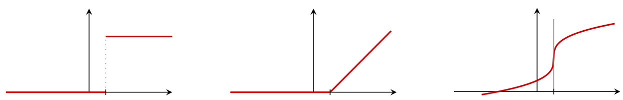
Existence of Differential Quotient

Function f is differentiable at all points, where we can draw the tangent (with finite slope) uniquely to the graph.

Function f is *not* differentiable at all points where this is not possible.

In particular these are

- jump discontinuities
- "kinks" in the graph of the function
- vertical tangents



Computation of the Differential Quotient

We can compute a differential quotient by determining the limit of the difference quotient.

Let $f(x) = x^2$. The we find for the first derivative

$$\begin{aligned}
f'(x_0) &= \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{x_0^2 + 2\,x_0\,h + h^2 - x_0^2}{h} \\
&= \lim_{h \to 0} \frac{2\,x_0\,h + h^2}{h} = \lim_{h \to 0} (2\,x_0 + h) \\
&= 2\,x_0
\end{aligned}$$

Problem 7.1

Draw (sketch) the graphs of the following functions.
At which points are these function differentiable?

(a) $f(x) = 2x + 2$

(b) $f(x) = 3$

(c) $f(x) = |x|$

(d) $f(x) = \sqrt{|x^2 - 1|}$

(e) $f(x) = \begin{cases} -\frac{1}{2}x^2, & \text{for } x \leq -1, \\ x, & \text{for } -1 < x \leq 1, \\ \frac{1}{2}x^2, & \text{for } x > 1. \end{cases}$

(f) $f(x) = \begin{cases} 2 + x, & \text{for } x \leq -1, \\ x^2, & \text{for } x > -1. \end{cases}$

Derivative of a Function

Function

$$\boxed{f' : D \to \mathbb{R},\ x \mapsto f'(x) = \left.\frac{df}{dx}\right|_x}$$

is called the **first derivative** of function f.
Its domain D is the set of all points where the differential quotient (i.e., the limit of the difference quotient) exists.

Derivatives of Elementary Functions

$f(x)$	$f'(x)$
c	0
x^α	$\alpha \cdot x^{\alpha-1}$
e^x	e^x
$\ln(x)$	$\frac{1}{x}$
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$

Computation Rules for Derivatives

- $(c \cdot f(x))' = c \cdot f'(x)$
- $(f(x) + g(x))' = f'(x) + g'(x)$ Summation rule
- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ Product rule
- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ Chain rule
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$ Quotient rule

Computation Rules for Derivatives

$$(3\,x^3 + 2\,x - 4)' = 3 \cdot 3 \cdot x^2 + 2 \cdot 1 - 0 = 9\,x^2 + 2$$

$$(e^x \cdot x^2)' = (e^x)' \cdot x^2 + e^x \cdot (x^2)' = e^x \cdot x^2 + e^x \cdot 2\,x$$

$$((3\,x^2 + 1)^2)' = 2\,(3\,x^2 + 1) \cdot 6\,x$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \tfrac{1}{2} \cdot x^{-\frac{1}{2}} = \tfrac{1}{2\sqrt{x}}$$

$$(a^x)' = \left(e^{\ln(a)\cdot x}\right)' = e^{\ln(a)\cdot x} \cdot \ln(a) = a^x\,\ln(a)$$

$$\left(\frac{1+x^2}{1-x^3}\right)' = \frac{2x \cdot (1-x^3) - (1+x^2) \cdot 3x^2}{(1-x^3)^2}$$

Higher Order Derivatives

We can compute derivatives of the derivative of a function.

Thus we obtain the

- **second derivative** $f''(x)$ of function f,
- **third derivative** $f'''(x)$, etc.,
- ***n*-th derivative** $f^{(n)}(x)$.

Other notations:

- $f''(x) = \frac{d^2 f}{dx^2}(x) = \left(\frac{d}{dx}\right)^2 f(x)$
- $f^{(n)}(x) = \frac{d^n f}{dx^n}(x) = \left(\frac{d}{dx}\right)^n f(x)$

Higher Order Derivatives

The first five derivatives of function

$$f(x) = x^4 + 2x^2 + 5x - 3$$

are

$$\begin{aligned} f'(x) &= (x^4 + 2x^2 + 5x - 3)' = 4x^3 + 4x + 5 \\ f''(x) &= (4x^3 + 4x + 5)' = 12x^2 + 4 \\ f'''(x) &= (12x^2 + 4)' = 24x \\ f^{\text{IV}}(x) &= (24x)' = 24 \\ f^{\text{V}}(x) &= 0 \end{aligned}$$

Problem 7.2

Compute the first and second derivative of the following functions:

(a) $f(x) = 4x^4 + 3x^3 - 2x^2 - 1$

(b) $f(x) = e^{-\frac{x^2}{2}}$

(c) $f(x) = \exp\left(-\frac{x^2}{2}\right)$

(d) $f(x) = \frac{x+1}{x-1}$

Problem 7.3

Compute the first and second derivative of the following functions:

(a) $f(x) = \frac{1}{1+x^2}$

(b) $f(x) = \frac{1}{(1+x)^2}$

(c) $f(x) = x\ln(x) - x + 1$

(d) $f(x) = \ln(|x|)$

Problem 7.4

Compute the first and second derivative of the following functions:

(a) $f(x) = \tan(x) = \dfrac{\sin(x)}{\cos(x)}$

(b) $f(x) = \cosh(x) = \frac{1}{2}(e^x + e^{-x})$

(c) $f(x) = \sinh(x) = \frac{1}{2}(e^x - e^{-x})$

(d) $f(x) = \cos(1 + x^2)$

Problem 7.5

Derive the quotient rule by means of product rule and chain rule.

Marginal Change

We can estimate the derivative $f'(x_0)$ approximately by means of the difference quotient with *small* change Δx:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx \frac{\Delta f}{\Delta x}$$

Vice verse we can estimate the change Δf of f for *small* changes Δx approximately by the first derivative of f:

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

Beware:

- $f'(x_0) \cdot \Delta x$ is a *linear function* in Δx.
- It is the *best possible* approximation of f by a linear function *around* x_0.
- This approximation is useful only for *"small"* values of Δx.

Differential

Approximation

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

becomes exact if Δx (and thus Δf) becomes *infinitesimally small*.
We then write dx and df instead of Δx and Δf, resp.

$$\boxed{df = f'(x_0)\, dx}$$

Symbols df and dx are called the **differentials** of function f and the independent variable x, resp.

Differential

Differential df can be seen as a linear function in dx.
We can use it to compute f approximately around x_0.

$$f(x_0 + dx) \approx f(x_0) + df$$

Let $f(x) = e^x$.

Differential of f at point $x_0 = 1$:
$df = f'(1)\, dx = e^1\, dx$

Approximation of $f(1.1)$ by means of this differential:
$\Delta x = (x_0 + dx) - x_0 = 1.1 - 1 = 0.1$

$f(1.1) \approx f(1) + df = e + e \cdot 0.1 \approx 2.99$

Exact value: $f(1.1) = 3.004166\ldots$

Problem 7.6

Let $f(x) = \dfrac{\ln(x)}{x}$.
Compute $\Delta f = f(3.1) - f(3)$ approximately by means of the differential at point $x_0 = 3$.
Compare your approximation to the exact value.

Elasticity

The first derivative of a function gives *absolute* rate of change of f at x_0.
Hence it depends on the scales used for argument and function values.

However, often *relative* rates of change are more appropriate.

We obtain *scale invariance* and *relative* rate of changes by

$$\frac{\text{change of function value relative to value of function}}{\text{change of argument relative to value of argument}}$$

and thus

$$\lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x) - f(x)}{f(x)}}{\frac{\Delta x}{x}} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \cdot \frac{x}{f(x)} = f'(x) \cdot \frac{x}{f(x)}$$

Elasticity

The expression

$$\boxed{\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)}}$$

is called the **elasticity** of f at point x.

Let $f(x) = 3\, e^{2x}$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{6\, e^{2x}}{3\, e^{2x}} = 2\, x$$

Let $f(x) = \beta\, x^\alpha$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{\beta\, \alpha\, x^{\alpha-1}}{\beta\, x^\alpha} = \alpha$$

Elasticity II

The relative rate of change of f can be expressed as

$$\ln(f(x))' = \frac{f'(x)}{f(x)}$$

What happens if we compute the derivative of $\ln(f(x))$ w.r.t. $\ln(x)$?

Let $\quad v = \ln(x) \quad \Leftrightarrow \quad x = e^v$

Derivation by means of the chain rule yields:

$$\frac{d(\ln(f(x)))}{d(\ln(x))} = \frac{d(\ln(f(e^v)))}{dv} = \frac{f'(e^v)}{f(e^v)}\, e^v = \frac{f'(x)}{f(x)}\, x = \varepsilon_f(x)$$

$$\boxed{\varepsilon_f(x) = \frac{d(\ln(f(x)))}{d(\ln(x))}}$$

Elasticity II

We can use the chain rule *formally* in the following way:

Let
- $u = \ln(y)$,
- $y = f(x)$,
- $x = e^v \quad \Leftrightarrow \quad v = \ln(x)$

Then we find

$$\frac{d(\ln f)}{d(\ln x)} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = \frac{1}{y} \cdot f'(x) \cdot e^v = \frac{f'(x)}{f(x)}\, x$$

Elastic Functions

A Function f is called

- **elastic** in x, if $|\varepsilon_f(x)| > 1$
- **1-elastic** in x, if $|\varepsilon_f(x)| = 1$
- **inelastic** in x, if $|\varepsilon_f(x)| < 1$

For elastic functions we then have:
The value of the function changes *relatively* faster than the value of the argument.

Function $f(x) = 3\, e^{2x}$ is $\qquad [\, \varepsilon_f(x) = 2\,x \,]$
- 1-elastic, for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$;
- inelastic, for $-\frac{1}{2} < x < \frac{1}{2}$;
- elastic, for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.

Source of Errors

Beware!
Function f is elastic if the **absolute value** of the *elasticity* is greater than 1.

Elastic Demand

Let $q(p)$ be an *elastic* demand function, where p is the price.
We have: $p > 0$, $q > 0$, and $q' < 0$ (q is decreasing). Hence

$$\varepsilon_q(p) = p \cdot \frac{q'(p)}{q(p)} < -1$$

What happens to the revenue (= price × selling)?

$$\begin{aligned} u'(p) &= (p \cdot q(p))' = 1 \cdot q(p) + p \cdot q'(p) \\ &= q(p) \cdot (1 + \underbrace{p \cdot \frac{q'(p)}{q(p)}}_{=\varepsilon_q < -1}) \\ &< 0 \end{aligned}$$

In other words, the revenue decreases if we raise prices.

Problem 7.7

Compute the regions where the following functions are elastic, 1-elastic and inelastic, resp.

(a) $g(x) = x^3 - 2\,x^2$

(b) $h(x) = \alpha\, x^\beta, \quad \alpha, \beta \neq 0$

Problem 7.8

Which of the following statements are correct?
Suppose function $y = f(x)$ is elastic in its domain.

(a) If x changes by one unit, then the change of y is greater than one unit.

(b) If x changes by one percent, then the relative change of y is greater than one percent.

(c) The relative rate of change of y is larger than the relative rate of change of x.

(d) The larger x is the larger will be y.

Partial Derivative

We investigate the rate of change of function $f(x_1, \dots, x_n)$, when variable x_i changes and the other variables remain fixed.
Limit

$$\boxed{\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(\dots, x_i + \Delta x_i, \dots) - f(\dots, x_i, \dots)}{\Delta x_i}}$$

is called the (first) **partial derivative** of f w.r.t. x_i.

Other notations for partial derivative $\frac{\partial f}{\partial x_i}$:

- $f_{x_i}(\mathbf{x})$ (derivative w.r.t. variable x_i)
- $f_i(\mathbf{x})$ (derivative w.r.t. the i-th variable)
- $f'_i(\mathbf{x})$ (i-th component of the gradient)

Computation of Partial Derivatives

We obtain partial derivatives $\frac{\partial f}{\partial x_i}$ by applying the rules for *univariate* functions for variable x_i while we treat *all other* variables *as constants*.

First partial derivatives of

$$f(x_1, x_2) = \sin(2\,x_1) \cdot \cos(x_2)$$

$$f_{x_1} = 2 \cdot \cos(2\,x_1) \cdot \underbrace{\cos(x_2)}_{\text{treated as constant}}$$

$$f_{x_2} = \underbrace{\sin(2\,x_1)}_{\text{treated as constant}} \cdot (-\sin(x_2))$$

Higher Order Partial Derivatives

We can compute partial derivatives of partial derivatives analogously to their univariate counterparts and obtain **higher order partial derivatives**:

$$\boxed{\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) \quad \text{and} \quad \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x})}$$

Other notations for partial derivative $\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x})$:

- $f_{x_i x_k}(\mathbf{x})$ (derivative w.r.t. variables x_i and x_k)
- $f_{ik}(\mathbf{x})$ (derivative w.r.t. the i-th and k-th variable)
- $f''_{ik}(\mathbf{x})$ (component of the Hessian matrix with index ik)

Higher Order Partial Derivatives

If all second order partial derivatives exists and are *continuous*, then the order of differentiation does not matter (Schwarz's theorem):

$$\boxed{\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_k}(\mathbf{x})}$$

Remark: Practically all differentiable functions in economic models have this property.

Higher Order Partial Derivatives

Compute the first and second order partial derivatives of

$$f(x, y) = x^2 + 3\,x\,y$$

First order partial derivatives:

$$f_x = 2\,x + 3\,y \qquad f_y = 0 + 3\,x$$

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & f_{xy} &= 3 \\ f_{yx} &= 3 & f_{yy} &= 0 \end{aligned}$$

Problem 7.9

Compute the first and second order partial derivatives of the following functions at point $(1, 1)$:

(a) $f(x, y) = x + y$

(b) $f(x, y) = x\,y$

(c) $f(x, y) = x^2 + y^2$

(d) $f(x, y) = x^2\,y^2$

(e) $f(x, y) = x^\alpha\,y^\beta, \quad \alpha, \beta > 0$

Problem 7.10

Compute the first and second order partial derivatives of

$$f(x, y) = \exp(x^2 + y^2)$$

at point $(0, 0)$.

Problem 7.11

Compute the first and second order partial derivatives of the following functions at point $(1, 1)$:

(a) $f(x, y) = \sqrt{x^2 + y^2}$

(b) $f(x, y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x, y) = (x^p + y^p)^{\frac{1}{p}}$

Gradient

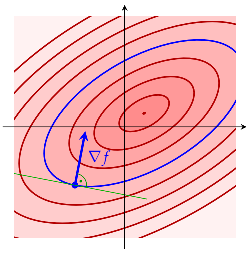
We collect all *first order partial derivatives* into a (row) vector which is called the **gradient** at point $\mathbf{x}$.

$$\boxed{\nabla f(\mathbf{x}) = (f_{x_1}(\mathbf{x}), \ldots, f_{x_n}(\mathbf{x}))}$$

- read: "gradient of f" or "nabla f".
- Other notation: $f'(\mathbf{x})$
- Alternatively the gradient can also be a column vector.
- The gradient is the analog of the first derivative of univariate functions.

Properties of the Gradient

- The gradient of f always points in the direction of *steepest ascent*.
- Its length is equal to the slope at this point.
- The gradient is *normal* (i.e. in right angle) to the corresponding *contour line* (level set).



Gradient

Compute the gradient of

$$f(x,y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (3,2)$.

$$f_x = 2\,x + 3\,y$$
$$f_y = 0 + 3\,x$$

$$\nabla f(\mathbf{x}) = (2x + 3y,\, 3x)$$
$$\nabla f(3,2) = (12, 9)$$

Problem 7.12

Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$

(b) $f(x,y) = x\,y$

(c) $f(x,y) = x^2 + y^2$

(d) $f(x,y) = x^2\,y^2$

(e) $f(x,y) = x^\alpha\,y^\beta, \quad \alpha,\beta > 0$

Problem 7.13

Compute the gradients of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

Hessian Matrix

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be two times differentiable. Then matrix

$$\boxed{\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \dots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \dots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \dots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}}$$

is called the **Hessian matrix** of f at $\mathbf{x}$.

- The Hessian matrix is symmetric, i.e., $f_{x_ix_k}(\mathbf{x}) = f_{x_kx_i}(\mathbf{x})$.
- Other notation: $f''(\mathbf{x})$
- The Hessian matrix is the analog of the second derivative of univariate functions.

Gradient

Compute the Hessian matrix of

$$f(x,y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (1,2)$.

Second order partial derivatives:

$$f_{xx} = 2 \qquad f_{xy} = 3$$
$$f_{yx} = 3 \qquad f_{yy} = 0$$

Hessian matrix:

$$\mathbf{H}_f(x,y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} = \mathbf{H}_f(1,2)$$

Problem 7.14

Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = x + y$

(b) $f(x,y) = x\,y$

(c) $f(x,y) = x^2 + y^2$

(d) $f(x,y) = x^2\,y^2$

(e) $f(x,y) = x^\alpha\,y^\beta, \quad \alpha,\beta > 0$

Problem 7.15

Compute the Hessian matrix of the following functions at point $(1,1)$:

(a) $f(x,y) = \sqrt{x^2 + y^2}$

(b) $f(x,y) = (x^3 + y^3)^{\frac{1}{3}}$

(c) $f(x,y) = (x^p + y^p)^{\frac{1}{p}}$

Jacobian Matrix

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$

The $m \times n$ matrix

$$\boxed{D\mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}}$$

is called the **Jacobian matrix** of $\mathbf{f}$ at point $\mathbf{x}_0$.

It is the generalization of *derivatives* (and gradients) for vector-valued functions.

Jacobian Matrix

- $f(\mathbf{x}) = f(x_1, x_2) = \exp(-x_1^2 - x_2^2)$
  $D f(\mathbf{x}) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}\right) = \nabla f(\mathbf{x})$
  $= (-2\,x_1 \exp(-x_1^2 - x_2^2), -2\,x_2 \exp(-x_1^2 - x_2^2))$
- $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1, x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \end{pmatrix}$
  $D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2\,x_1 & 2\,x_2 \\ 2\,x_1 & -2\,x_2 \end{pmatrix}$
- $\mathbf{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$
  $D\mathbf{s}(t) = \begin{pmatrix} \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$

Chain Rule

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g}\colon \mathbb{R}^m \to \mathbb{R}^k$. Then

$$\boxed{(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x})}$$

$$\mathbf{f}(x,y) = \begin{pmatrix} e^x \\ e^y \end{pmatrix} \qquad \mathbf{g}(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$$

$$\mathbf{f}'(x,y) = \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \qquad \mathbf{g}'(x,y) = \begin{pmatrix} 2\,x & 2\,y \\ 2\,x & -2\,y \end{pmatrix}$$

$$(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 2\,e^x & 2\,e^y \\ 2\,e^x & -2\,e^y \end{pmatrix} \cdot \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix}$$
$$= \begin{pmatrix} 2\,e^{2x} & 2\,e^{2y} \\ 2\,e^{2x} & -2\,e^{2y} \end{pmatrix}$$

Example – Indirect Dependency

Let $f(x_1, x_2, t)$ where $x_1(t)$ and $x_2(t)$ also depend on t.
What is the total derivative of f w.r.t. t?

Chain rule:
Let $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \\ t \end{pmatrix}$

$$\begin{aligned}
\frac{df}{dt} &= (f \circ \mathbf{x})'(t) = f'(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\
&= \nabla f(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} = (f_{x_1}(\mathbf{x}(t)), f_{x_2}(\mathbf{x}(t)), f_t(\mathbf{x}(t))) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} \\
&= f_{x_1}(\mathbf{x}(t)) \cdot x_1'(t) + f_{x_2}(\mathbf{x}(t)) \cdot x_2'(t) + f_t(\mathbf{x}(t)) \\
&= f_{x_1}(x_1, x_2, t) \cdot x_1'(t) + f_{x_2}(x_1, x_2, t) \cdot x_2'(t) + f_t(x_1, x_2, t)
\end{aligned}$$

Problem 7.16

Let
$$f(x,y) = x^2 + y^2 \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} t \\ t^2 \end{pmatrix}.$$

Compute the derivative of the composite functions

(a) $h = f \circ \mathbf{g}$, and
(b) $\mathbf{p} = \mathbf{g} \circ f$

by means of the chain rule.

Problem 7.17

Let $\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^3 - x_2 \\ x_1 - x_2^3 \end{pmatrix}$ and $\mathbf{g}(\mathbf{x}) = \begin{pmatrix} x_2^2 \\ x_1 \end{pmatrix}$.

Compute the derivatives of the composite functions

(a) $\mathbf{g} \circ \mathbf{f}$, and
(b) $\mathbf{f} \circ \mathbf{g}$

by means of the chain rule.

Problem 7.18

Let $Q(K, L, t)$ be a production function, where $L = L(t)$ and $K = K(t)$ also depend on time t. Compute the total derivative $\frac{dQ}{dt}$ by means of the chain rule.

Summary

- difference quotient and differential quotient
- differential quotient and derivative
- derivatives of elementary functions
- differentiation rules
- higher order derivatives
- total differential
- elasticity
- partial derivatives
- gradient and Hessian matrix
- Jacobian matrix and chain rule