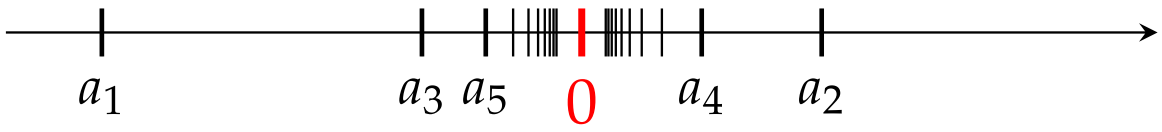
Chapter 6

Limits

Limit of a Sequence

Consider the following sequence of numbers

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \frac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right)$$



The terms of this sequence *tend* to 0 with increasing n.
We say that sequence (a_n) **converges** to 0.

We write

$$\boxed{(a_n) \to 0 \quad \text{or} \quad \lim_{n\to\infty} a_n = 0}$$

(read: “limit of a_n for n tends to ∞”)

Limit of a Sequence / Definition

Definition:
A number $a \in \mathbb{R}$ is a **limit** of sequence (a_n), if there *exists an* N *for every interval* $(a - \varepsilon, a + \varepsilon)$ such that $a_n \in (a - \varepsilon, a + \varepsilon)$ for all $n \geq N$; i.e., all terms following a_N are contained in this interval.

Equivalent Definition: A sequence (a_n) converges to **limit** $a \in \mathbb{R}$ if there *exists an* N *for every* $\varepsilon > 0$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

[Mathematicians like to use ε for a very small positive number.]

A sequence that has a *limit* is called **convergent**.
It **converges** to its limit.

It can be shown that a limit of a sequence is *unique*ly defined *(if it exists)*.

A sequence *without* a limit is called **divergent**.

Limit of a Sequence / Example

Sequence

$$\left(a_n\right)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\right)$$

has limit $a = 0$.

For example, if we set $\varepsilon = 0.3$, then all terms following a_4 are contained in interval $(a - \varepsilon, a + \varepsilon)$.

If we set $\varepsilon = \frac{1}{1\,000\,000}$, then all terms starting with the 1 000 001-st term are contained in the interval.

Thus

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0\,.$$

Limit of a Sequence / Example

Sequence $(a_n)_{n=1}^{\infty} = \left(\frac{1}{2^n}\right)_{n=1}^{\infty} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ converges to 0:

$$\lim_{n\to\infty} a_n = 0$$

Sequence $(b_n)_{n=1}^{\infty} = \left(\frac{n-1}{n+1}\right)_{n=1}^{\infty} = \left(0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots\right)$ is convergent:

$$\lim_{n\to\infty} b_n = 1$$

Sequence $(c_n)_{n=1}^{\infty} = \left((-1)^n\right)_{n=1}^{\infty} = \left(-1, 1, -1, 1, -1, 1, \dots\right)$ is divergent.

Sequence $(d_n)_{n=1}^{\infty} = \left(2^n\right)_{n=1}^{\infty} = \left(2, 4, 8, 16, 32, \dots\right)$ is divergent, but tends to ∞. By abuse of notation we write:

$$\lim_{n\to\infty} d_n = \infty$$

Limits of Important Sequences

$$\lim_{n\to\infty} n^a = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a = 0 \\ \infty & \text{for } a > 0 \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{for } |q| < 1 \\ 1 & \text{for } q = 1 \\ \infty & \text{for } q > 1 \\ \nexists & \text{for } q \leq -1 \end{cases}$$

$$\lim_{n\to\infty} \frac{n^a}{q^n} = \begin{cases} 0 & \text{for } |q| > 1 \\ \infty & \text{for } 0 < q < 1 \\ \nexists & \text{for } -1 < q < 0 \end{cases} \qquad (|q| \notin \{0,1\})$$

Rules for Limits

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, resp., and let $(c_n)_{n=1}^{\infty}$ be a bounded sequence.
Then

(1)	$\lim_{n\to\infty} (k \cdot a_n + d) = k \cdot a + d$	
(2)	$\lim_{n\to\infty} (a_n + b_n) = a + b$	
(3)	$\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$	
(4)	$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$	for $b \neq 0$
(5)	$\lim_{n\to\infty} (a_n \cdot c_n) = 0$	provided $a = 0$
(6)	$\lim_{n\to\infty} a_n^k = a^k$	

Rules for Limits

$$\lim_{n\to\infty}\left(2+\frac{3}{n^2}\right) = 2+3\underbrace{\lim_{n\to\infty} n^{-2}}_{=0} = 2+3\cdot 0 = 2$$

$$\lim_{n\to\infty}(2^{-n}\cdot n^{-1}) = \lim_{n\to\infty}\frac{n^{-1}}{2^n} = 0$$

$$\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2-\frac{3}{n^2}} = \frac{\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim\limits_{n\to\infty}\left(2-\frac{3}{n^2}\right)} = \frac{1}{2}$$

$$\lim_{n\to\infty}\underbrace{\sin(n)}_{\text{bounded}}\cdot\underbrace{\frac{1}{n^2}}_{\to 0} = 0$$

Rules for Limits / Rational Terms

Important!
When we apply these rules we have to take care that we never obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0 \cdot \infty$.

These expressions are **not defined**!

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \frac{\lim\limits_{n\to\infty} 3n^2+1}{\lim\limits_{n\to\infty} n^2-1} = \frac{\infty}{\infty} \quad \text{(not defined)}$$

Trick: Reduce the fraction by the *largest power* in its **denominator**.

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{3+n^{-2}}{1-n^{-2}} = \frac{\lim\limits_{n\to\infty} 3+n^{-2}}{\lim\limits_{n\to\infty} 1-n^{-2}} = \frac{3}{1} = 3$$

Euler's Number

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182818284590\ldots}$$

This limit is very important in many applications including finance (continuous compounding).

$$\begin{aligned}\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{1}{n/x}\right)^n \\ &= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} \qquad \left(m=\frac{n}{x}\right) \\ &= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = e^x\end{aligned}$$

Problem 6.1

Compute the following limits:

(a) $\lim_{n\to\infty} \left(7 + \left(\frac{1}{2}\right)^n\right)$

(b) $\lim_{n\to\infty} \left(\frac{2n^3 - 6n^2 + 3n - 1}{7n^3 - 16}\right)$

(c) $\lim_{n\to\infty} \left(n^2 - (-1)^n n^3\right)$

(d) $\lim_{n\to\infty} \left(\frac{n^2 + 1}{n + 1}\right)$

(e) $\lim_{n\to\infty} \left(\frac{n \bmod 10}{(-2)^n}\right)$

$a \bmod b$ is the remainder after integer division, e.g., $17 \bmod 5 = 2$ and $12 \bmod 4 = 0$.

Problem 6.2

Compute the following limits:

(a) $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}$

(b) $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{n}$

(c) $\lim_{n\to\infty}\left(1+\frac{1}{nx}\right)^{n}$

Limit of a Function

What happens with the value of a function f, if the argument x tends to some value x_0 (which need not belong to the domain of f)?

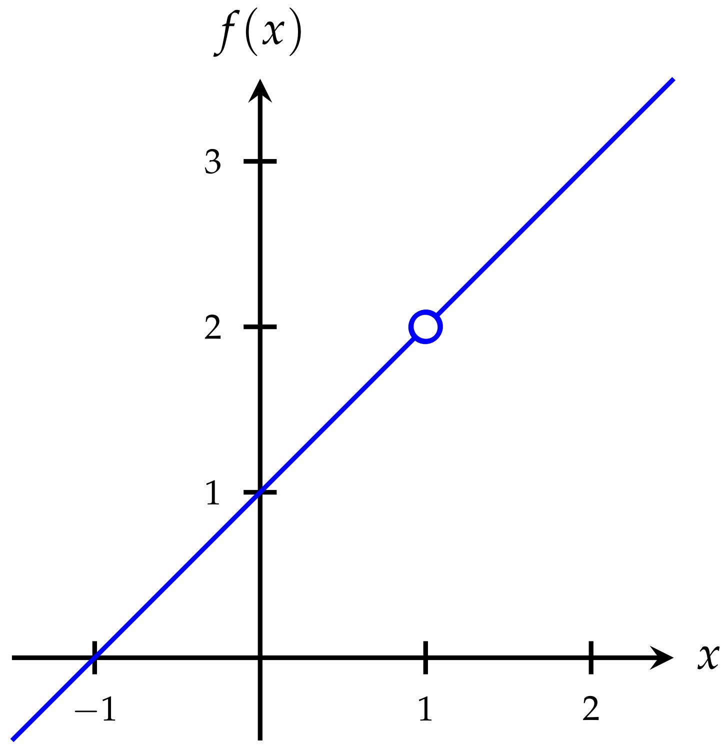
Function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

is not defined in $x = 1$.

By factorizing and reducing we get function

$$g(x) = x + 1 = \begin{cases} f(x), & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$



Limit of a Function

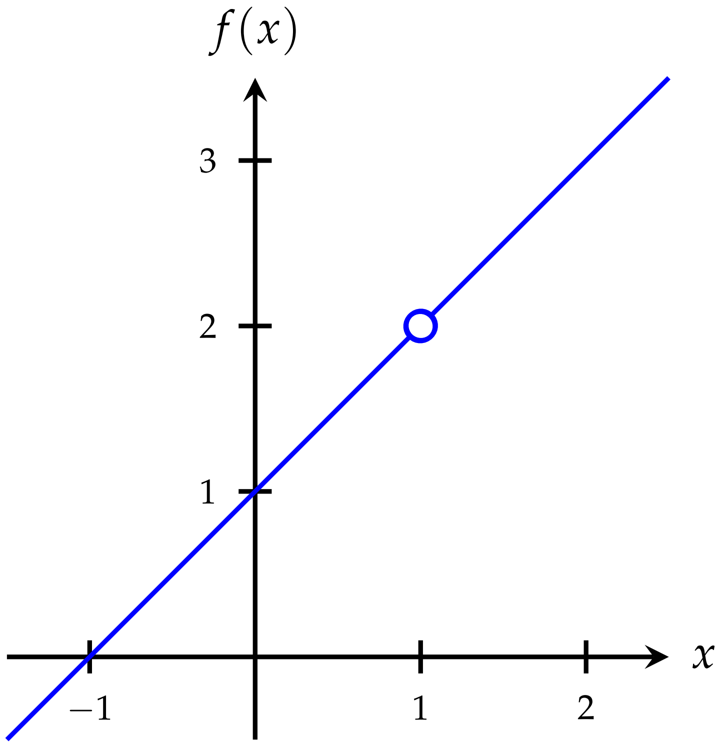
Suppose we approach argument $x_0 = 1$.

Then the value of function $f(x) = \dfrac{x^2 - 1}{x - 1}$ tends to 2.

We say:
$f(x)$ **converges** to 2 when x tends to 1

and write:

$$\boxed{\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2}$$



Limit of a Function

Formal **definition**:
If sequence $\big(f(x_n)\big)_{n=1}^{\infty}$ of function values converges to number y_0 for every convergent sequence $(x_n)_{n=1}^{\infty} \to x_0$ of arguments, then y_0 is called the **limit** of f as x approaches x_0.

We write

$$\boxed{\lim_{x\to x_0} f(x) = y_0 \quad \text{or} \quad f(x) \to y_0 \text{ for } x \to x_0}$$

x_0 need not belong to the domain of f.
y_0 need not belong to the codomain of f.

Rules for Limits

Rules for limits of functions are analogous to rules for sequences.

Let $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$.

$$
\begin{array}{lll}
(1) & \lim_{x \to x_0} (c \cdot f(x) + d) = c \cdot a + d & \\
(2) & \lim_{x \to x_0} (f(x) + g(x)) = a + b & \\
(3) & \lim_{x \to x_0} (f(x) \cdot g(x)) = a \cdot b & \\
(4) & \lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b} & \text{for } b \neq 0 \\
(5) & \lim_{x \to x_0} (f(x))^k = a^k & \text{for } k \in \mathbb{N}
\end{array}
$$

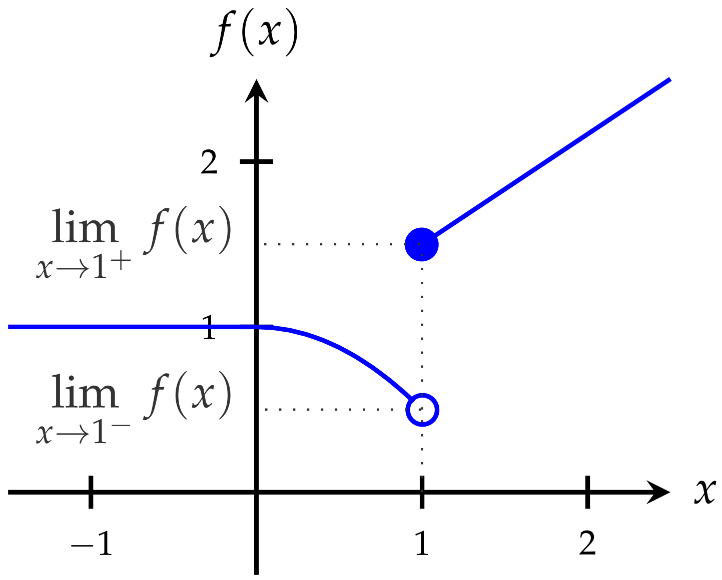
How to Find Limits?

The following recipe is suitable for “simple” functions:

1. Draw the graph of the function.
2. Mark x_0 on the x-axis.
3. Follow the graph with your pencil until we reach x_0 starting from *right* of x_0.
4. The y-coordinate of your pencil in this point is then the so called **right-handed limit** of f as x approaches x_0 (from above):
   $\lim_{x \to x_0^+} f(x)$. (Other notations: $\lim_{x \downarrow x_0} f(x)$ or $\lim_{x \searrow x_0} f(x)$)
5. Analogously we get the **left-handed limit** of f as x approaches x_0 (from below): $\lim_{x \to x_0^-} f(x)$.
6. If both limits *coincide*, then the limit exists and we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$$

How to Find Limits?



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

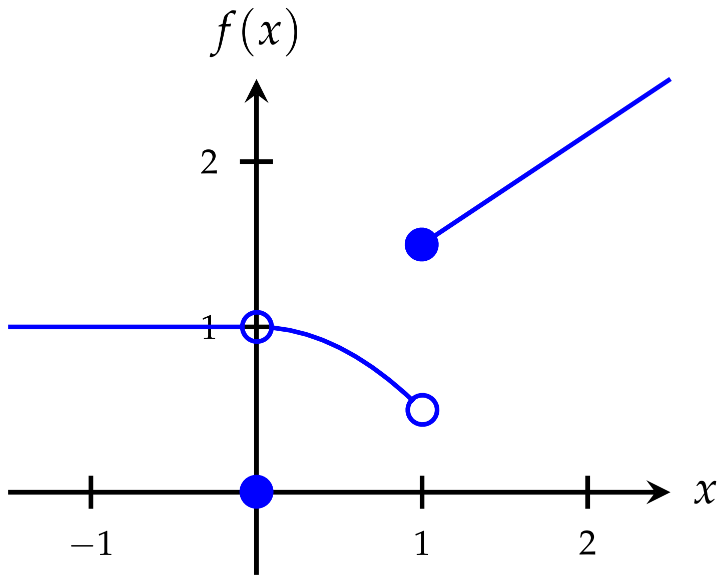
$0.5 = \lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x) = 1.5$

i.e., the limit of f at $x_0 = 1$ does not exist.

The limits at other points, however, do exist,

e.g. $\lim\limits_{x\to 0} f(x) = 1.$

How to Find Limits?



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

The only difference is to above is the function value at $x_0 = 0$.
Nevertheless, the limit does exist:

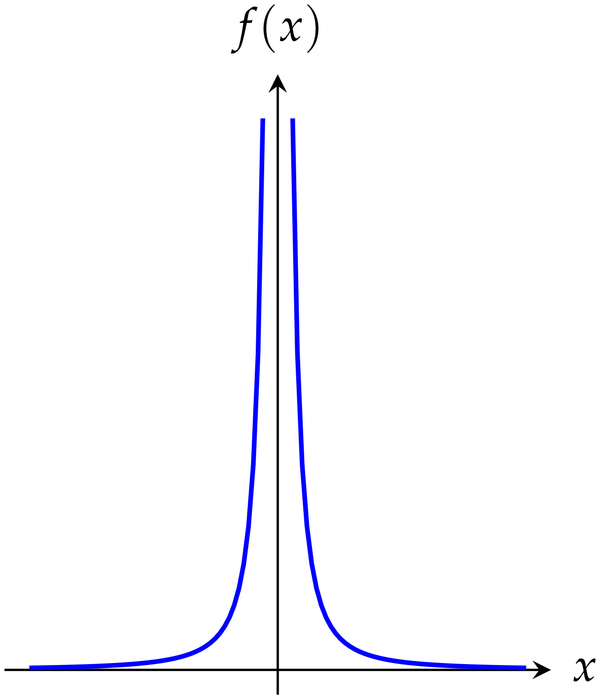
$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x) \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 1\,.$$

Unbounded Function

It may happen that $f(x)$ tends to ∞ (or $-\infty$) if x tends to x_0.

We then write (by abuse of notation):

$$\boxed{\lim_{x \to x_0} f(x) = \infty}$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Limit as $x \to \infty$

By abuse of language we can define the *limit* analogously for $x_0 = \infty$ and $x_0 = -\infty$, resp.

Limit

$$\lim_{x\to\infty} f(x)$$

exists, if $f(x)$ converges whenever x tends to infinity.

$$\lim_{x\to\infty} \frac{1}{x^2} = 0 \qquad \text{and} \qquad \lim_{x\to-\infty} \frac{1}{x^2} = 0$$

Problem 6.3

Draw the graph of function

$$f(x) = \begin{cases} -\frac{x^2}{2}, & \text{for } x \le -2, \\ x+1, & \text{for } -2 < x < 2, \\ \frac{x^2}{2}, & \text{for } x \ge 2. \end{cases}$$

and determine $\lim_{x \to x_0^+} f(x)$, $\lim_{x \to x_0^-} f(x)$, and $\lim_{x \to x_0} f(x)$ for $x_0 = -2$, 0 and 2:

$$\lim_{x \to -2^+} f(x) \qquad \lim_{x \to -2^-} f(x) \qquad \lim_{x \to -2} f(x)$$

$$\lim_{x \to 0^+} f(x) \qquad \lim_{x \to 0^-} f(x) \qquad \lim_{x \to 0} f(x)$$

$$\lim_{x \to 2^+} f(x) \qquad \lim_{x \to 2^-} f(x) \qquad \lim_{x \to 2} f(x)$$

Problem 6.4

Determine the following left-handed and right-handed limits:

(a) $\lim_{x \to 0^-} f(x)$

$\lim_{x \to 0^+} f(x)$

$$\text{for } f(x) = \begin{cases} 1, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

(b) $\lim_{x \to 0^-} \frac{1}{x}$

$\lim_{x \to 0^+} \frac{1}{x}$

(c) $\lim_{x \to 1^-} x$

$\lim_{x \to 1^+} x$

Problem 6.5

Determine the following limits:

(a) $\lim_{x\to\infty} \frac{1}{x+1}$

(b) $\lim_{x\to 0} x^2$

(c) $\lim_{x\to\infty} \ln(x)$

(d) $\lim_{x\to 0} \ln|x|$

(e) $\lim_{x\to\infty} \frac{x+1}{x-1}$

Problem 6.6

Determine

(a) $\lim_{x \to 1^+} \frac{x^{3/2} - 1}{x^3 - 1}$

(b) $\lim_{x \to -2^-} \frac{\sqrt{|x^2 - 4|^2}}{x + 2}$

(c) $\lim_{x \to 0^-} \lfloor x \rfloor$

(d) $\lim_{x \to 1^+} \frac{x - 1}{\sqrt{x - 1}}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Problem 6.7

Determine

(a) $\lim_{x\to 2^+} \frac{2x^2-3x-2}{|x-2|}$

(b) $\lim_{x\to 2^-} \frac{2x^2-3x-2}{|x-2|}$

(c) $\lim_{x\to -2^+} \frac{|x+2|^{3/2}}{2+x}$

(d) $\lim_{x\to 1^-} \frac{x+1}{x^2-1}$

(e) $\lim_{x\to -7^+} \frac{2|x+7|}{x^2+4x-21}$

Problem 6.8

Compute

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

for

(a) $f(x) = x$

(b) $f(x) = x^2$

(c) $f(x) = x^3$

(d) $f(x) = x^n$, for $n \in \mathbb{N}$.

L'Hôpital's Rule

Suppose we want to compute

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

and find

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \qquad \text{(or } = \pm\infty\text{)}$$

However, expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are not defined.

(You must not reduce the fraction by 0 or ∞!)

L'Hôpital's Rule

If $\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$ (or $= \infty$ or $= -\infty$), then

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

Assumption: f and g are differentiable in x_0.

This formula is called **l'Hôpital's rule** (also written as *l'Hospital's rule*).

L'Hôpital's Rule

$$\lim_{x\to 2} \frac{x^3 - 7\,x + 6}{x^2 - x - 2} = \lim_{x\to 2} \frac{3\,x^2 - 7}{2\,x - 1} = \frac{5}{3}$$

$$\lim_{x\to\infty} \frac{\ln\, x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2\,x} = \lim_{x\to\infty} \frac{1}{2\,x^2} = 0$$

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{1 - (1+x)^{-1}}{2\,x} = \lim_{x\to 0} \frac{(1+x)^{-2}}{2} = \frac{1}{2}$$

L'Hôpital's Rule

L'Hôpital's rule can be applied iteratively:

$$\lim_{x \to 0} \frac{e^x - x - 1}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Problem 6.9

Compute the following limits:

(a) $$\lim_{x \to 4} \frac{x^2 - 2\,x - 8}{x^3 - 2\,x^2 - 11\,x + 12}$$

(b) $$\lim_{x \to -1} \frac{x^2 - 2\,x - 8}{x^3 - 2\,x^2 - 11\,x + 12}$$

(c) $$\lim_{x \to 2} \frac{x^3 - 5\,x^2 + 8\,x - 4}{x^3 - 3\,x^2 + 4}$$

(d) $$\lim_{x \to 0} \frac{1 - \cos(x)}{x^2}$$

(e) $$\lim_{x \to 0^+} x\,\ln(x)$$

(f) $$\lim_{x \to \infty} x\,\ln(x)$$

Problem 6.10

If we apply l'Hôpital's rule on the following limit we obtain

$$\lim_{x\to 1} \frac{x^3 + x^2 - x - 1}{x^2 - 1} = \lim_{x\to 1} \frac{3\,x^2 + 2\,x - 1}{2\,x} = \lim_{x\to 1} \frac{6\,x + 2}{2} = 4\,.$$

However, the correct value for the limit is 2.

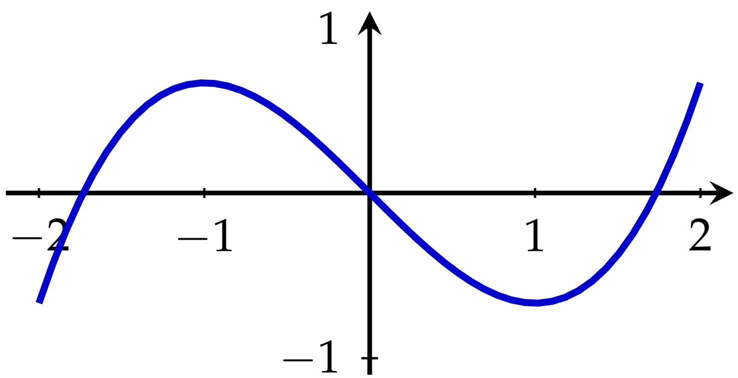
Why does l'Hôpital's rule not work for this problem?

How do you get the correct value?

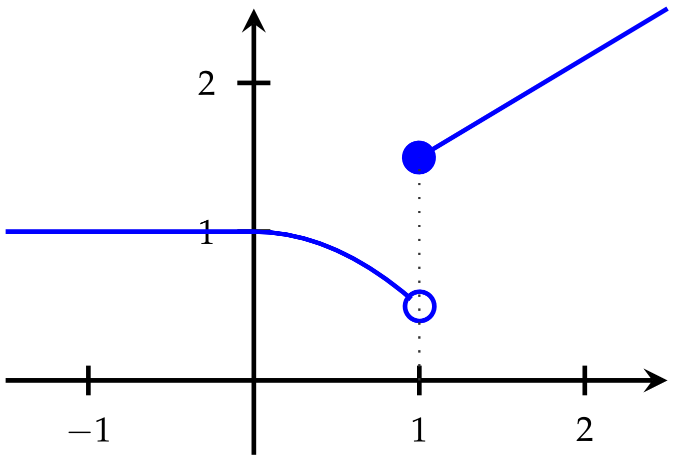
Continuous Functions

We may observe that we can draw the graph of a function *without removing the pencil from paper*. We call such functions **continuous**.

For some other functions we *have to remove* the pencil. At such points the function has a **jump discontinuity**.



continuous



jump discontinuity at $x = 1$

Continuous Functions

Formal **Definition:**

Function $f\colon D \to \mathbb{R}$ is called **continuous** *at* $x_0 \in D$, if

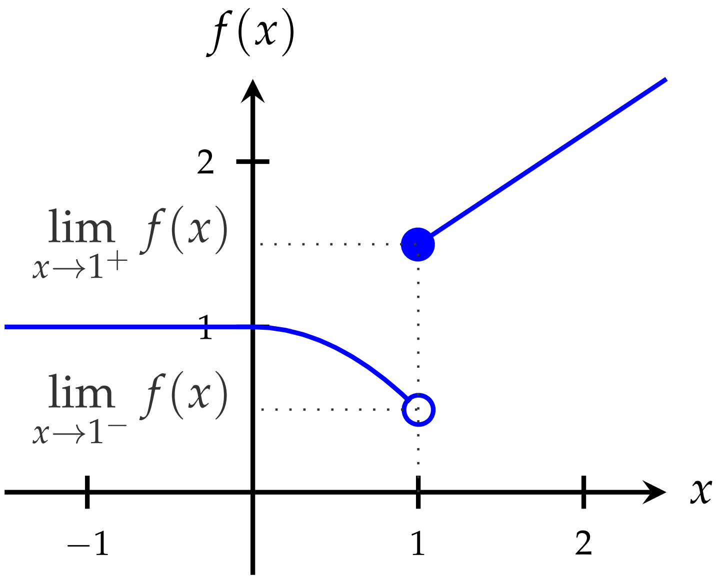
1. $\lim\limits_{x \to x_0} f(x)$ exists, and

2. $\boxed{\lim\limits_{x \to x_0} f(x) = f(x_0)}$.

The function is called **continuous** if it is continuous *at all* points of its domain.

Note that continuity is a *local* property of a function.

Discontinuous Function



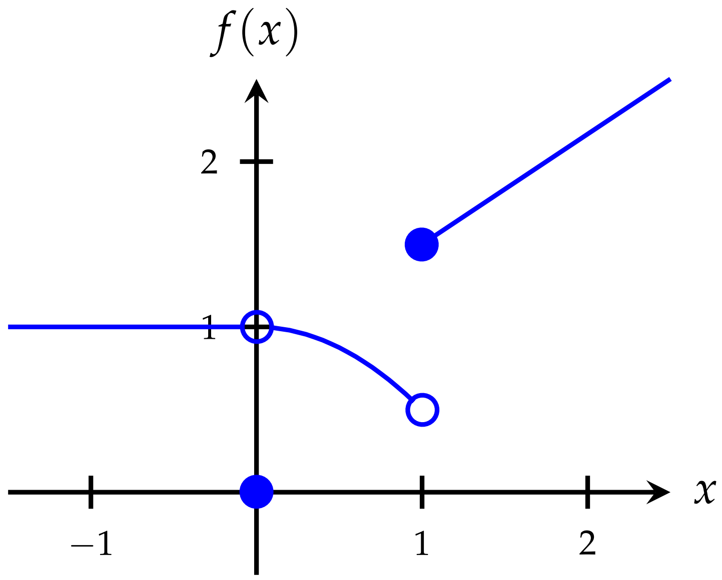
$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Not continuous in $x = 1$ as $\lim_{x \to 1} f(x)$ does not exist.

So f is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{1\}$.
For example at $x = 0$, $\lim_{x \to 0} f(x)$ does exist and $\lim_{x \to 0} f(x) = 1 = f(0)$.

Discontinuous Function



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Not continuous in all $x = 0$, either.

$\lim_{x \to 0} f(x) = 1$ does exist but $\lim_{x \to 0} f(x) \neq f(0)$.

So f is not a continuous function.

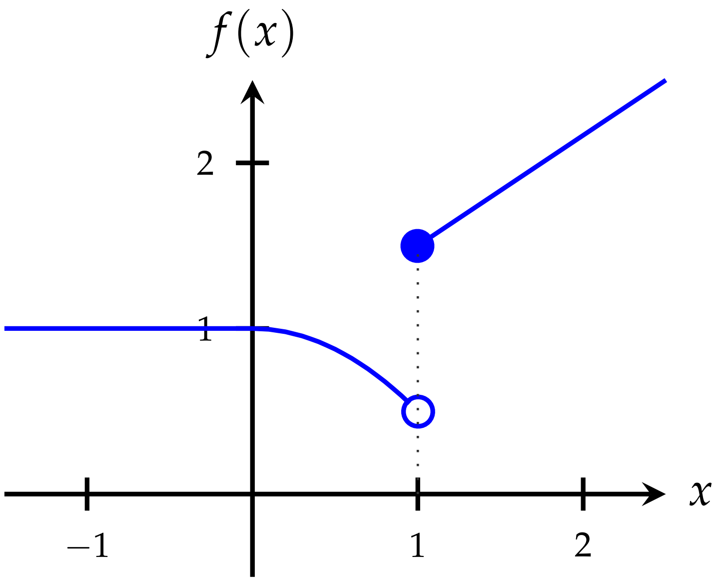
However, it is still continuous in all $x \in \mathbb{R} \setminus \{0, 1\}$.

Recipe for "Nice" Functions

(1) Draw the graph of the given function.

(2) At all points of the *domain*, where we *have to remove* the pencil from paper the function is *not continuous*.

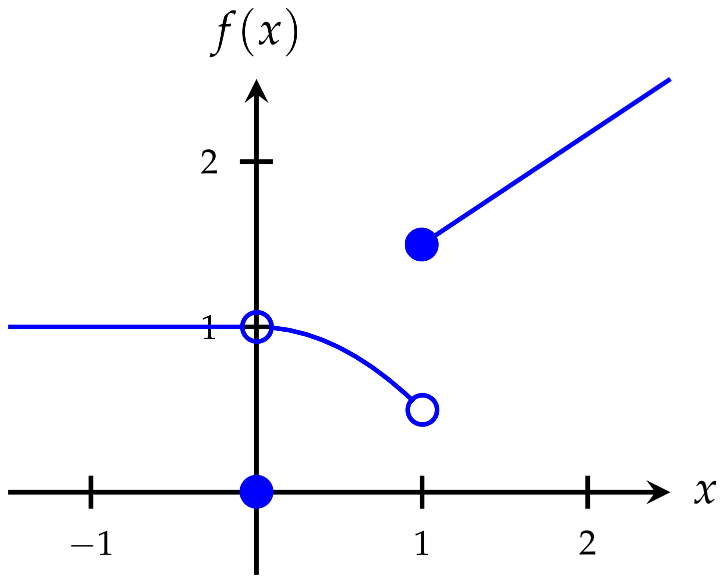
(3) At all other points of the domain (where we need not remove the pencil) the function is *continuous*.



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1, \\ \frac{x}{2} + 1, & \text{for } x \ge 1. \end{cases}$$

f is continuous
except at point $x = 1$.

Discontinuous Function



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Function f is continuous *except* at points $x = 0$ and $x = 1$.

Problem 6.11

Draw the graph of function

$$f(x) = \begin{cases} -\frac{x^2}{2}, & \text{for } x \leq -2, \\ x+1, & \text{for } -2 < x < 2, \\ \frac{x^2}{2}, & \text{for } x \geq 2. \end{cases}$$

and compute $\lim\limits_{x\to x_0^+} f(x)$, $\lim\limits_{x\to x_0^-} f(x)$, and $\lim\limits_{x\to x_0} f(x)$
for $x_0 = -2$, 0, and 2.

Is function f continuous at these points?

Problem 6.12

Determine the left and right-handed limits of function

$$f(x) = \begin{cases} x^2 + 1, & \text{for } x > 0, \\ 0, & \text{for } x = 0, \\ -x^2 - 1, & \text{for } x < 0. \end{cases}$$

at $x_0 = 0$.

Is function f continuous at this point?

Is function f differentiable at this point?

Problem 6.13

Is function

$$f(x) = \begin{cases} x + 1, & \text{for } x \leq 1, \\ \frac{x}{2} + \frac{3}{2}, & \text{for } x > 1, \end{cases}$$

continuous at $x_0 = 1$?

Is it differentiable at $x_0 = 1$?

Compute the limit of f at $x_0 = 1$.

Problem 6.14

Sketch the graphs of the following functions.
Which of these are continuous (on its domain)?

(a) $D = \mathbb{R}$, $f(x) = x$

(b) $D = \mathbb{R}$, $f(x) = 3x + 1$

(c) $D = \mathbb{R}$, $f(x) = e^{-x} - 1$

(d) $D = \mathbb{R}$, $f(x) = |x|$

(e) $D = \mathbb{R}^+$, $f(x) = \ln(x)$

(f) $D = \mathbb{R}, f(x) = \lfloor x \rfloor$

(g) $D = \mathbb{R}$, $f(x) = \begin{cases} 1, & \text{for } x \leq 0, \\ x + 1, & \text{for } 0 < x \leq 2, \\ x^2, & \text{for } x > 2. \end{cases}$

Remark: $\lfloor x \rfloor$ is the largest integer less than or equal to x.

Problem 6.15

Sketch the graph of

$$f(x) = \frac{1}{x} \,.$$

Is it continuous?

Problem 6.16

Determine a value for h, such that function

$$f(x) = \begin{cases} x^2 + 2hx, & \text{for } x \leq 2, \\ 3x - h, & \text{for } x > 2, \end{cases}$$

is continuous.

Limits of Continuous Functions

If function f is known to be *continuous*, then its limit $\lim_{x\to x_0} f(x)$ exists for all $x_0 \in D_f$ and we obviously find

$$\lim_{x\to x_0} f(x) = f(x_0)\ .$$

Polynomials are always continuous. Hence

$$\lim_{x\to 2} 3x^2 - 4x + 5 = 3\cdot 2^2 - 4\cdot 2 + 5 = 9\ .$$

Summary

- limit of a sequence
- limit of a function
- convergent and divergent
- Euler’s number
- rules for limits
- l’Hôpital’s rule
- continuous functions