Chapter 5

Real Functions

Real Function

Real functions are maps where both *domain* and *codomain* are (unions of) intervals in $\mathbb{R}$.

Often only function terms are given but neither domain nor codomain. Then domain and codomain are implicitly given as following:

- *Domain* of the function is the largest *sensible* subset of the domain of the function terms (i.e., where the terms are defined).
- *Codomain* is the image (range) of the function

$$f(D) = \{y \mid y = f(x) \text{ for an } x \in D_f\} .$$

Implicit Domain

Production function $f(x) = \sqrt{x}$ is an abbreviation for

$$f\colon [0,\infty) \to [0,\infty),\ x \mapsto f(x) = \sqrt{x}$$

(There are no negative amounts of goods.
Moreover, $\sqrt{x}$ is not real for $x < 0$.)

Its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is an abbreviation for

$$f'\colon (0,\infty) \to (0,\infty),\ x \mapsto f'(x) = \frac{1}{2\sqrt{x}}$$

(Note the open interval $(0,\infty)$; $\frac{1}{2\sqrt{x}}$ is not defined for $x = 0$.)

Problem 5.1

Give the largest possible domain of the following functions?

(a) $h(x) = \frac{x-1}{x-2}$

(b) $D(p) = \frac{2p+3}{p-1}$

(c) $f(x) = \sqrt{x-2}$

(d) $g(t) = \frac{1}{\sqrt{2t-3}}$

(e) $f(x) = 2 - \sqrt{9-x^2}$

(f) $f(x) = 1 - x^3$

(g) $f(x) = 2 - |x|$

Problem 5.2

Give the largest possible domain of the following functions?

(a) $f(x) = \frac{|x-3|}{x-3}$

(b) $f(x) = \ln(1+x)$

(c) $f(x) = \ln(1+x^2)$

(d) $f(x) = \ln(1-x^2)$

(e) $f(x) = \exp(-x^2)$

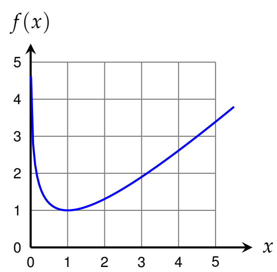
(f) $f(x) = (e^x - 1)/x$

Graph of a Function

Each tuple $(x, f(x))$ corresponds to a point in the xy-plane.
The set of all these points forms a curve called the **graph** of function f.

$$\mathcal{G}_f = \{(x, y) \mid x \in D_f,\ y = f(x)\}$$

Graphs can be used to visualize functions.
They allow to detect many properties of the given function.



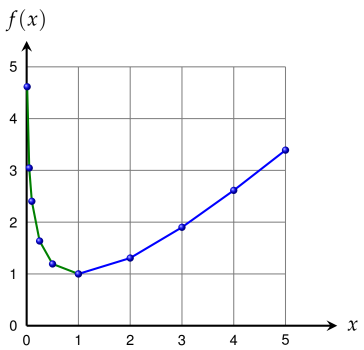
$f(x) = x - \ln(x)$

How to Draw a Graph

1. Get an idea about the possible shape of the graph. One should be able to sketch graphs of elementary functions by heart.
2. Find an appropriate range for the x-axis.
   (It should show a characteristic detail of the graph.)
3. Create a table of function values and draw the corresponding points into the xy-plane.

   If known, use characteristic points like local extrema or inflection points.
4. Check if the curve can be constructed from the drawn points.
   If not add adapted points to your table of function values.
5. Fit the curve of the graph through given points in a proper way.

How to Draw a Graph



Graph of function

$$f(x) = x - \ln x$$

Table of values:

x	$f(x)$
0	ERROR
1	1
2	1.307
3	1.901
4	2.614
5	3.391
0.5	1.193
0.25	1.636
0.1	2.403
0.05	3.046

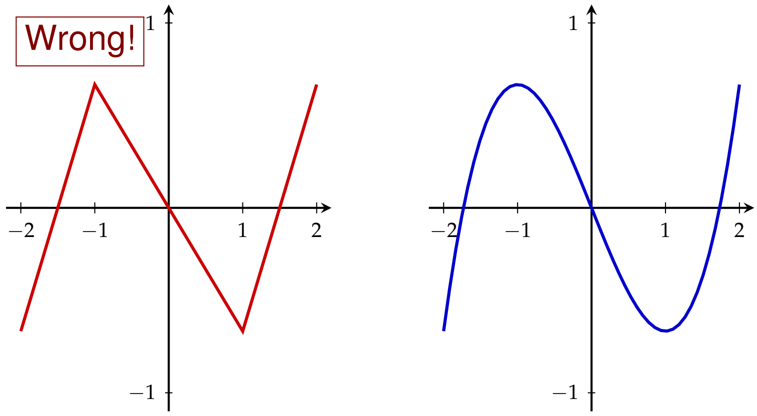
Sources of Errors

Most frequent errors when drawing function graphs:

- **Table of values is too small:**
  It is not possible to construct the curve from the computed function values.
- **Important points are ignored:**
  Ideally extrema and inflection points should be known and used.
- **Range for x and y-axes not suitable:**
  The graph is tiny or important details vanish in the “noise” of handwritten lines (or pixel size in case of a computer program).

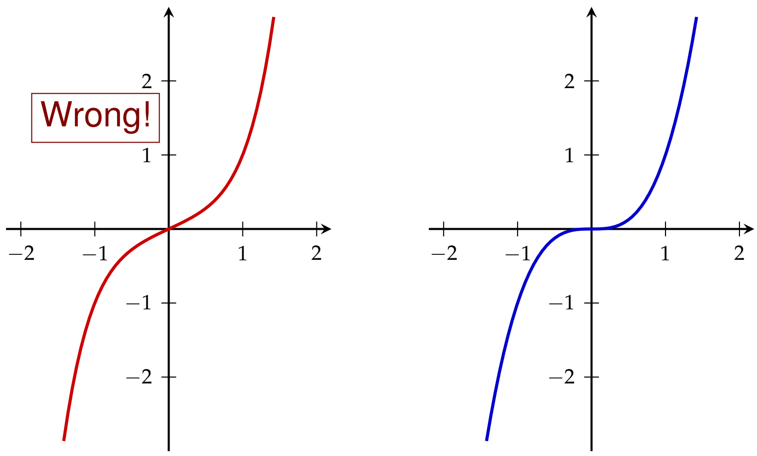
Sources of Errors

Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:



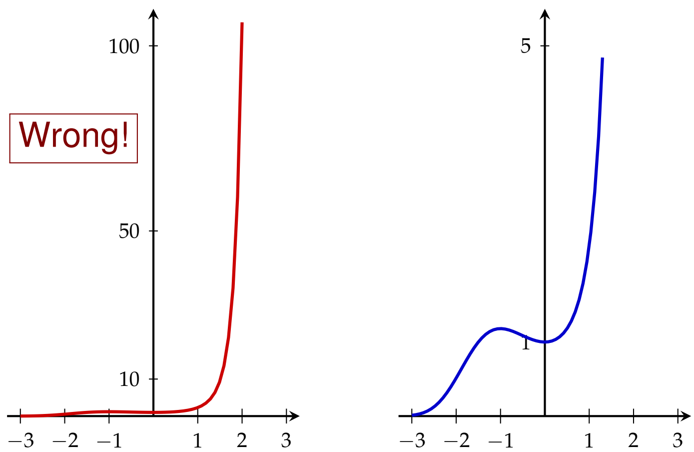
Sources of Errors

Graph of $f(x) = x^3$ has slope 0 in $x = 0$:



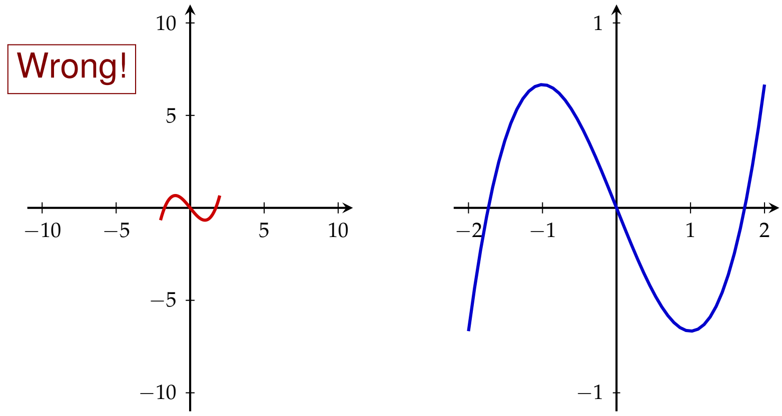
Sources of Errors

Function $f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$ has a local maximum in $x = -1$:



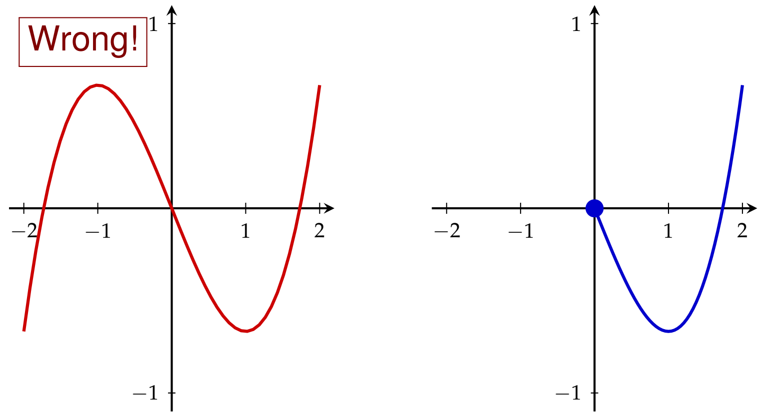
Sources of Errors

Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:



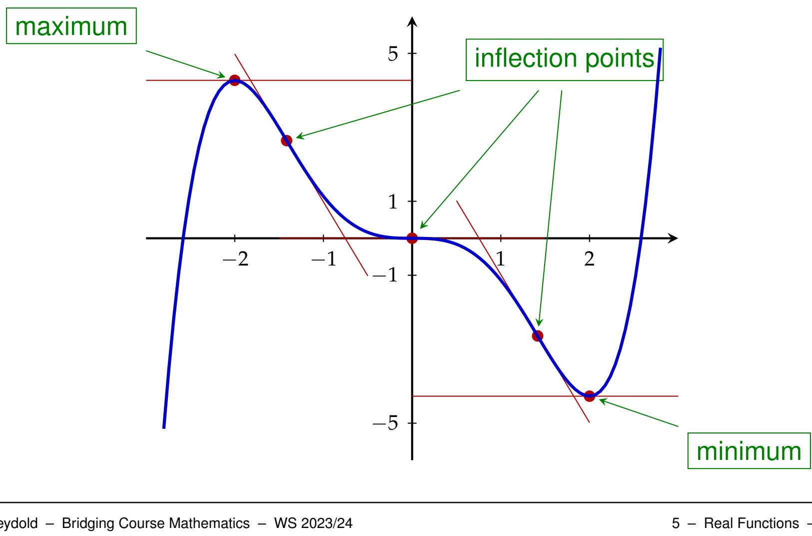
Sources of Errors

Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[0, 2]$: (not in $[-2, 2]$!)



Extrema and Inflection Points

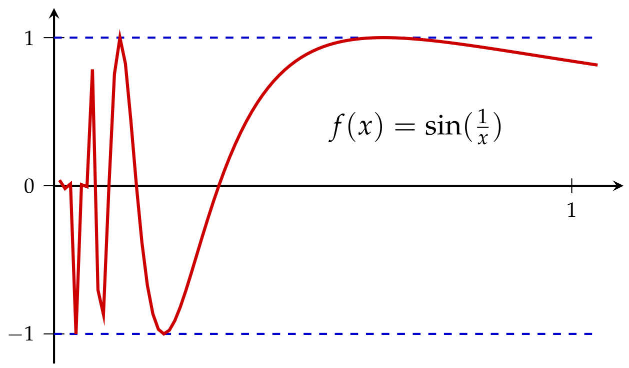
Graph of function $f(x) = \frac{1}{15}(3x^5 - 20x^3)$:



Sources of Errors

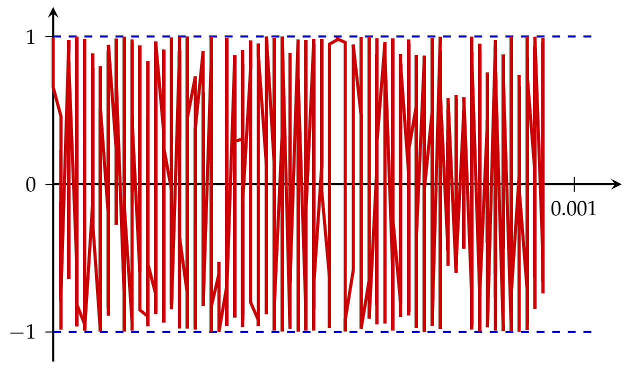
It is *important* that one already has an *idea of the shape* of the function graph **before** drawing the curve.

Even a graph drawn by means of a computer program can differ significantly from the true curve.



Sources of Errors

$$f(x) = \sin(\tfrac{1}{x})$$



Sketch of a Function Graph

Often a **sketch** of the graph is sufficient. Then the exact function values are not so important. Axes may not have scales.

However, it is important that the sketch clearly shows all characteristic details of the graph (like extrema or important function values).

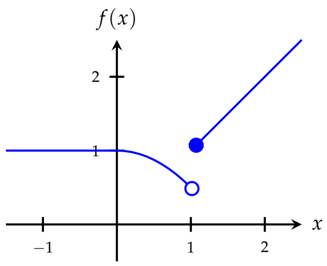
Sketches can also be drawn like a caricature:
They stress prominent parts and properties of the function.

Piece-wise Defined Functions

The function term can be defined differently in subintervals of the domain.

At the boundary points of these subintervals we have to mark which points belong to the graph and which do not:

- • (belongs) and ○ (does not belong).



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1, \\ x, & \text{for } x \geq 1. \end{cases}$$

Problem 5.3

Draw the graph of function

$$f(x) = -x^4 + 2x^2$$

in interval $[-2, 2]$.

Problem 5.4

Draw the graph of function

$$f(x) = e^{-x^4+2x^2}$$

in interval $[-2, 2]$.

Problem 5.5

Draw the graph of function

$$f(x) = \frac{x-1}{|x-1|}$$

in interval $[-2, 2]$.

Problem 5.6

Draw the graph of function

$$f(x) = \sqrt{|1-x^2|}$$

in interval $[-2, 2]$.

Bijectivity

Recall that each argument has exactly one image and that the number of preimages of an element in the codomain can vary.
Thus we can characterize maps by their possible number of preimages.

- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Also recall that a function has an *inverse* if and only if it is one-to-one and onto (i.e., *bijective*).

A Simple Horizontal Test

How can we determine whether a real function is one-to-one or onto?
I.e., how many preimage may a $y \in W_f$ have?

(1) Draw the graph of the given function.

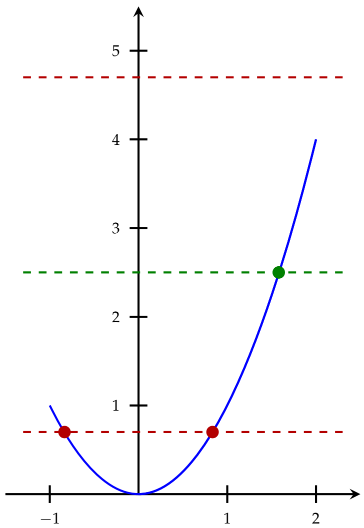
(2) Mark some $y \in W$ on the y-axis and draw a line parallel to the x-axis (*horizontal*) through this point.

(3) The number of intersection points of horizontal line and graph coincides with the number of preimages of y.

(4) Repeat Steps (2) and (3) for a *representative* set of y-values.

(5) Interpretation: If all horizontal lines intersect the graph in
- **(a)** *at most one* point, then f is **one-to-one**;
- **(b)** *at least one* point, then f is **onto**;
- **(c)** *exactly one* point, then f is **bijective**.

Example



$f\colon [-1,2] \to \mathbb{R},\ x \mapsto x^2$
- is not one-to-one;
- is not onto.

$f\colon [0,2] \to \mathbb{R},\ x \mapsto x^2$
- is one-to-one;
- is not onto.

$f\colon [0,2] \to [0,4],\ x \mapsto x^2$
- is one-to-one and onto.

Beware! *Domain* and *codomain* are part of the function!

Problem 5.7

Draw the graphs of the following functions and determine whether these functions are one-to-one or onto (or both).

(a) $f\colon [-2,2] \to \mathbb{R},\ x \mapsto 2\,x + 1$

(b) $f\colon \mathbb{R} \setminus \{0\} \to \mathbb{R},\ x \mapsto \frac{1}{x}$

(c) $f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^3$

(d) $f\colon [2,6] \to \mathbb{R},\ x \mapsto (x-4)^2 - 1$

(e) $f\colon [2,6] \to [-1,3],\ x \mapsto (x-4)^2 - 1$

(f) $f\colon [4,6] \to [-1,3],\ x \mapsto (x-4)^2 - 1$

Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.

$$\boxed{g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))}$$

is called **composite function**.
(read: "g composed with f", "g circle f", or "g after f")

Let $g\colon \mathbb{R} \to [0,\infty),\ x \mapsto g(x) = x^2,$
$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = 3x - 2.$

Then $(g \circ f)\colon \mathbb{R} \to [0,\infty),$
$x \mapsto (g \circ f)(x) = g(f(x)) = g(3x-2) = (3x-2)^2$

and $(f \circ g)\colon \mathbb{R} \to \mathbb{R},$
$x \mapsto (f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 2$

Problem 5.8

Let $f(x) = x^2 + 2x - 1$ and $g(x) = 1 + |x|^{\frac{3}{2}}$.

Compute

(a) $(f \circ g)(4)$

(b) $(f \circ g)(-9)$

(c) $(g \circ f)(0)$

(d) $(g \circ f)(-1)$

Problem 5.9

Determine $f \circ g$ and $g \circ f$.
What are the domains of f, g, $f \circ g$ and $g \circ f$?

(a) $f(x) = x^2, \quad g(x) = 1 + x$

(b) $f(x) = \sqrt{x} + 1, \quad g(x) = x^2$

(c) $f(x) = \frac{1}{x+1}, \quad g(x) = \sqrt{x} + 1$

(d) $f(x) = 2 + \sqrt{x}, \quad g(x) = (x-2)^2$

(e) $f(x) = x^2 + 2, \quad g(x) = x - 3$

(f) $f(x) = \frac{1}{1+x^2}, \quad g(x) = \frac{1}{x}$

(g) $f(x) = \ln(x), \quad g(x) = \exp(x^2)$

(h) $f(x) = \ln(x-1), \quad g(x) = x^3 + 1$

Inverse Function

If $f\colon D_f \to W_f$ is a **bijection**, then there exists a so called **inverse function**

$$\boxed{f^{-1}\colon W_f \to D_f,\; y \mapsto x = f^{-1}(y)}$$

with the property

$$\boxed{f^{-1} \circ f = \mathrm{id} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}}$$

We get the function term of the inverse by *interchanging* the roles of *argument* x and *image* y.

Example

We get the term for the inverse function by expressing x as function of y

We need the inverse function of

$$y = f(x) = 2x - 1$$

By rearranging we obtain

$$y = 2x - 1 \quad \Leftrightarrow \quad y + 1 = 2x \quad \Leftrightarrow \quad \frac{1}{2}(y+1) = x$$

Thus the term of the inverse function is $f^{-1}(y) = \frac{1}{2}(y+1)$.

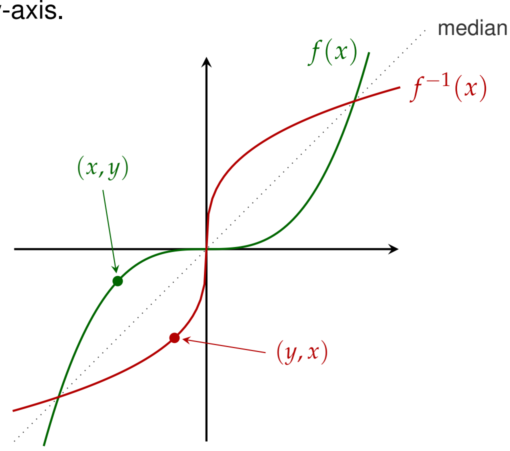
Arguments are usually denoted by x. So we write

$$f^{-1}(x) = \frac{1}{2}(x+1)\ .$$

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

Geometric Interpretation

Interchanging of x and y corresponds to reflection across the median between x and y-axis.



(Graph of function $f(x) = x^3$ and its inverse.)

Problem 5.10

Find the inverse function of

$$f(x) = \ln(1 + x)$$

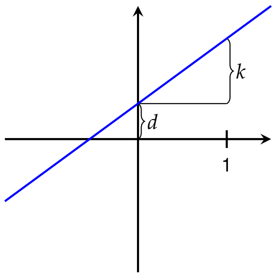
Draw the graphs of f and f^{-1}.

Linear Function and Absolute Value

- **Linear function**

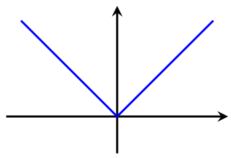
$$\boxed{f(x) = k\,x + d}$$

k ... **slope**
d ... **intercept**



- **Absolute value** (or *modulus*)

$$\boxed{f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}}$$



Problem 5.11

Draw the graph of function

$$f(x) = 2x + 1$$

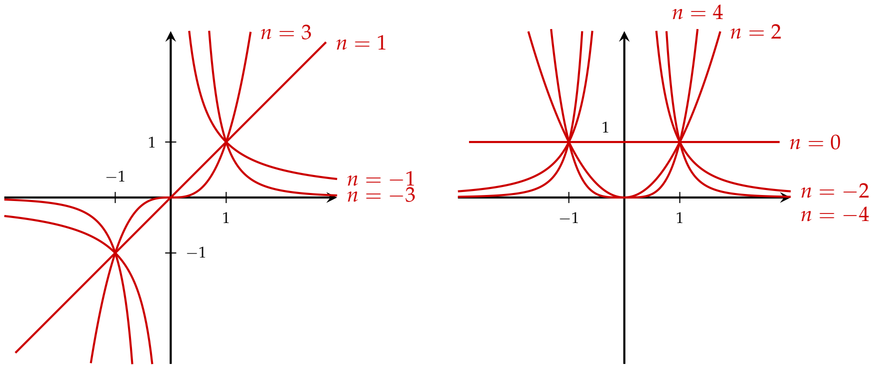
in interval $[-2, 2]$.

Hint: Two points and a ruler are sufficient.

Power Function

Power function with integer exponents:

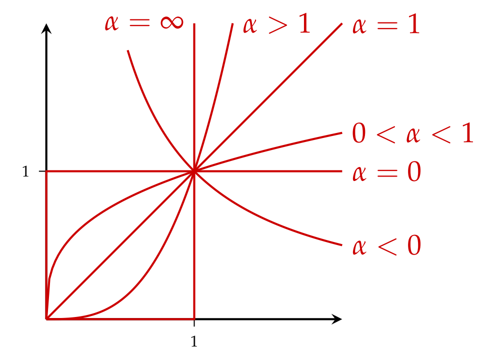
$$\boxed{f\colon x \mapsto x^n, \quad n \in \mathbb{Z}} \qquad D = \begin{cases} \mathbb{R} & \text{for } n \geq 0 \\ \mathbb{R} \setminus \{0\} & \text{for } n < 0 \end{cases}$$



Power Function

Power function with *real* exponents:

$$\boxed{f\colon x \mapsto x^\alpha \qquad \alpha \in \mathbb{R}} \qquad D = \begin{cases} [0, \infty) & \text{for } \alpha \geq 0 \\ (0, \infty) & \text{for } \alpha < 0 \end{cases}$$



Problem 5.12

Draw (sketch) the graph of power function

$$f(x) = x^n$$

in interval $[0, 2]$ for

$$n = -4, -2, -1, -\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{2}, 1, 2, 3, 4\,.$$

Polynomial and Rational Functions

- **Polynomial** of degree n:

$$\boxed{f(x) = \sum_{k=0}^{n} a_k\, x^k}$$

$a_i \in \mathbb{R}$, for $i = 1, \dots, n$, $\quad a_n \neq 0$.

- **Rational Function:**

$$\boxed{D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}}$$

$p(x)$ and $q(x)$ are polynomials
$D = \mathbb{R} \setminus \{\text{roots of } q\}$

Problem 5.13

Draw (sketch) the graphs of the following functions in interval $[-2, 2]$:

(a) $f(x) = \dfrac{x}{x^2+1}$

(b) $f(x) = \dfrac{x}{x^2-1}$

(c) $f(x) = \dfrac{x^2}{x^2+1}$

(d) $f(x) = \dfrac{x^2}{x^2-1}$

Exponential Function

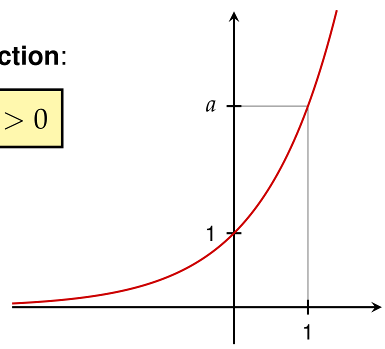
- **Exponential function**:

$$\boxed{\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \exp(x) = e^x}$$

$e = 2{,}7182818\dots$ Euler's number

- **Generalized exponential function**:

$$\boxed{\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto a^x \qquad a > 0}$$



Problem 5.14

Draw (sketch) the graph of the following functions:

(a) $f(x) = e^x$

(b) $f(x) = 3^x$

(c) $f(x) = e^{-x}$

(d) $f(x) = e^{x^2}$

(e) $f(x) = e^{-x^2}$

(f) $f(x) = e^{-1/x^2}$

(g) $\cosh(x) = (e^x + e^{-x})/2$

(h) $\sinh(x) = (e^x - e^{-x})/2$

Logarithm Function

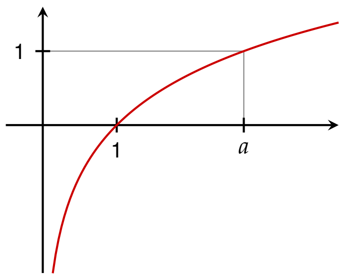
- **Logarithm**:

  Inverse of *exponential function*.

  $$\boxed{\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log(x) = \ln(x)}$$

- **Generalized Logarithm** to basis a:

  $$\boxed{\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log_a(x)}$$



Problem 5.15

Draw (sketch) the graph of the following functions:

(a) $f(x) = \ln(x)$

(b) $f(x) = \ln(x+1)$

(c) $f(x) = \ln\left(\frac{1}{x}\right)$

(d) $f(x) = \log_{10}(x)$

(e) $f(x) = \log_{10}(10x)$

(f) $f(x) = (\ln(x))^2$

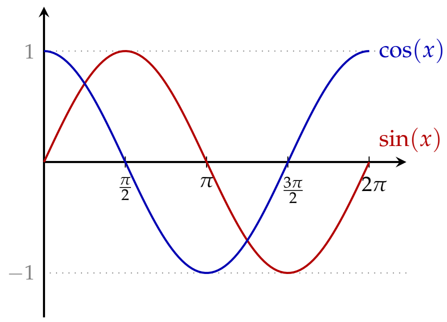
Trigonometric Functions

- **Sine**:

$$\mathbb{R} \to [-1,1],\ x \mapsto \sin(x)$$

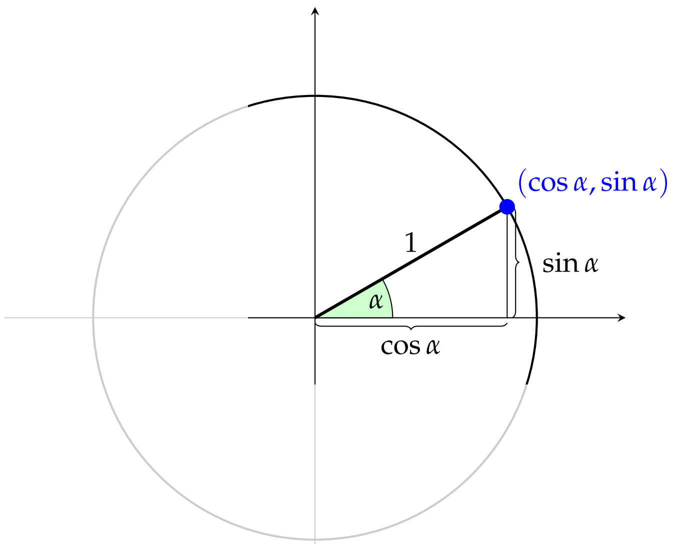
- **Cosine**:

$$\mathbb{R} \to [-1,1],\ x \mapsto \cos(x)$$



Beware!
These functions use **radian** for their arguments, i.e., angles are measured by means of the length of arcs on the unit circle and not by degrees. A right angle then corresponds to $x = \pi/2$.

Sine and Cosine



Sine and Cosine

Important formulas:

Periodic: For all $k \in \mathbb{Z}$,

$$\boxed{\begin{aligned}\sin(x + 2k\pi) &= \sin(x)\\ \cos(x + 2k\pi) &= \cos(x)\end{aligned}}$$

Relation between sin and cos:

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$

Problem 5.16

Assign the following functions to the graphs $\boxed{1}$ – $\boxed{18}$:

(a) $f(x) = x^2$

(b) $f(x) = \frac{x}{x+1}$

(c) $f(x) = \frac{1}{x+1}$

(d) $f(x) = \sqrt{x}$

(e) $f(x) = x^3 - 3x^2 + 2x$

(f) $f(x) = \sqrt{|2x - x^2|}$

(g) $f(x) = -x^2 - 2x$

(h) $f(x) = (x^3 - 3x^2 + 2x)\operatorname{sgn}(1 - x) + 1$

(sgn$(x) = 1$ if $x \geq 0$ and -1 otherwise.)

Problem 5.16 / 2

(i) $f(x) = e^x$

(j) $f(x) = e^{x/2}$

(k) $f(x) = e^{2x}$

(l) $f(x) = 2^x$

(m) $f(x) = \ln(x)$

(n) $f(x) = \log_{10}(x)$

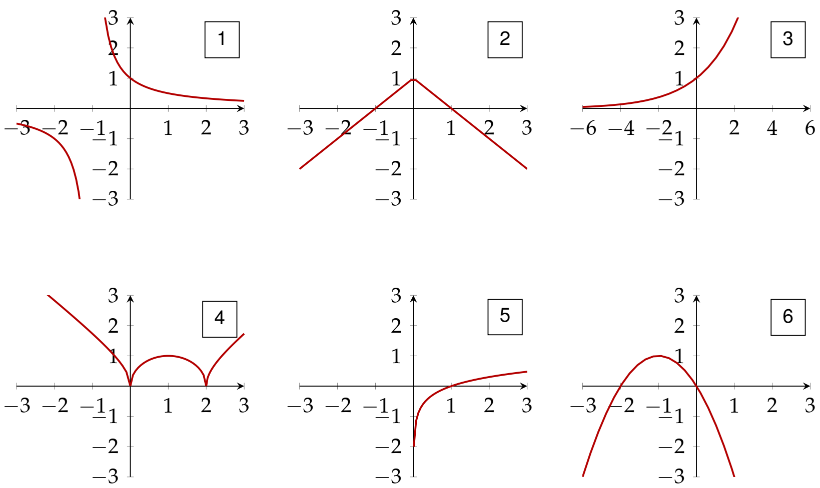
(o) $f(x) = \log_2(x)$

(p) $f(x) = \sqrt{4 - x^2}$

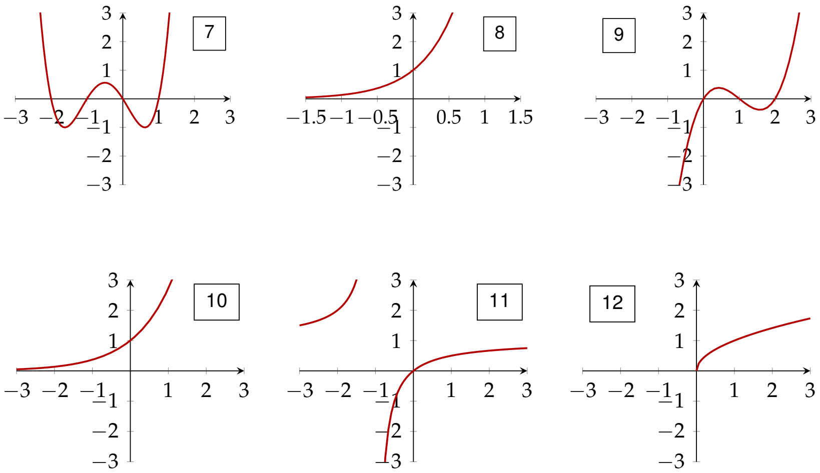
(q) $f(x) = 1 - |x|$

(r) $f(x) = \prod_{k=-1}^{2}(x + k)$

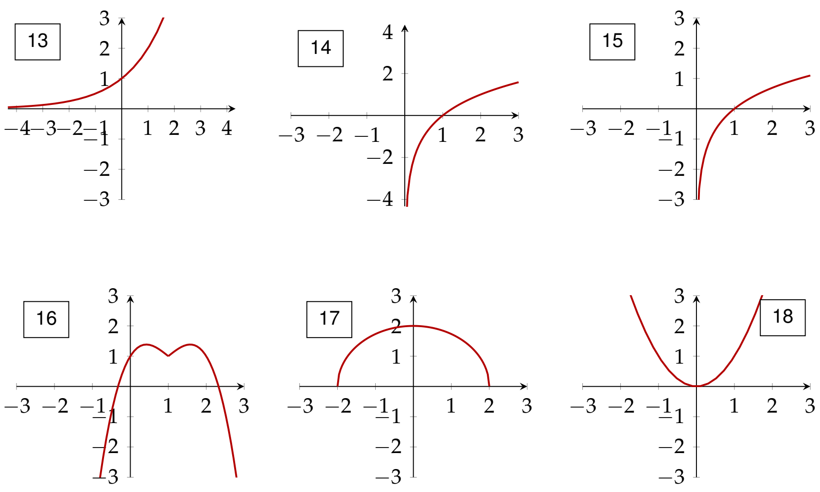
Problem 5.16 / 3



Problem 5.16 / 4



Problem 5.16 / 5



Multivariate Function

A **function of several variables** (or **multivariate function**) is a function with more than one argument which evaluates to a real number.

$$\boxed{f\colon \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)}$$

Arguments x_i are the **variables** of function f.

$$f(x,y) = \exp(-x^2 - 2y^2)$$

is a bivariate function in variables x and y.

$$p(x_1, x_2, x_3) = x_1^2 + x_1 x_2 - x_2^2 + 5x_1 x_3 - 2x_2 x_3$$

is a function in the three variables x_1, x_2, and x_3.

Graphs of Bivariate Functions

Bivariate functions (i.e., of *two* variables) can be visualized by its graph:

$$\boxed{\mathcal{G}_f = \{(x, y, z) \mid z = f(x, y) \text{ for } x, y \in \mathbb{R}\}}$$

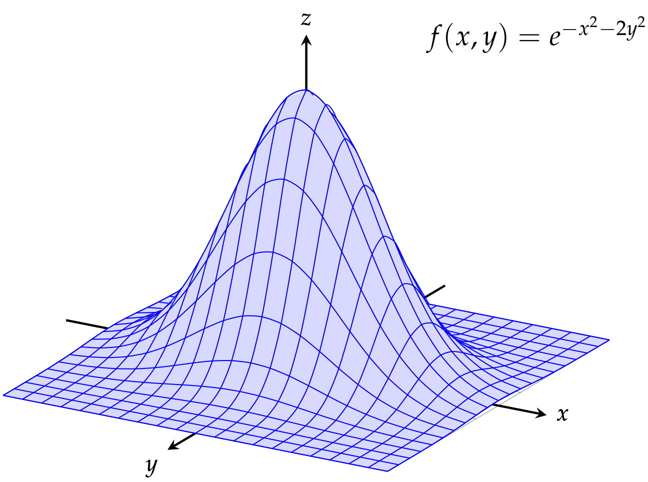
It can be seen as the two-dimensional *surface* of a three-dimensional landscape.

The notion of *graph* exists analogously for functions of three or more variables.

$$\mathcal{G}_f = \{(\mathbf{x}, y) \mid y = f(\mathbf{x}) \text{ for an } \mathbf{x} \in \mathbb{R}^n\}$$

However, it can hardly be used to visualize such functions.

Graphs of Bivariate Functions



Contour Lines of Bivariate Functions

Let $c \in \mathbb{R}$ be fixed. Then the set of all points (x, y) in the real plane with $f(x, y) = c$ is called **contour line** of function f.

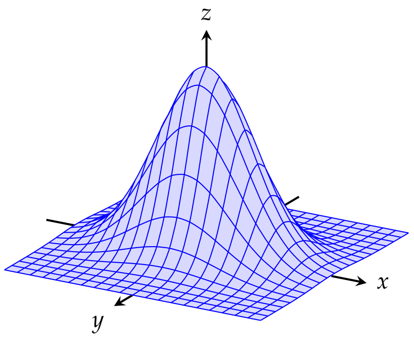
Function f is constant on each of its contour lines.

Other names:

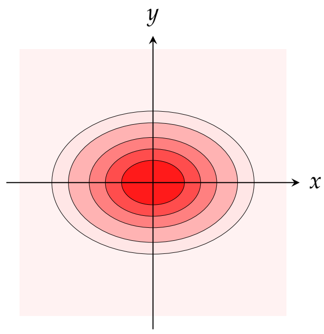
- *Indifference curve*
- *Isoquant*
- *Level set* (is a generalization of a contour line for functions of any number of variables.)

A collection of contour lines can be seen as a kind of “hiking map” for the “landscape” of the function.

Contour Lines of Bivariate Functions



graph



contour lines

$$f(x, y) = e^{-x^2-2y^2}$$

Problem 5.17

In a simplistic model we are given utility function U of a household w.r.t. two complementary goods (e.g. left and right shoes):

$$U(x_1, x_2) = \sqrt{\min\{x_1, x_2\}}, \quad x_1, x_2 \geq 0.$$

(a) Sketch the graph of U.

(b) Sketch the contour lines for $U = U_0 = 1$ and $U = U_1 = 2$.

Indifference Curves

Indifference curves are determined by an equation

$$F(x, y) = 0$$

We can (try to) draw such curves by expressing one of the variables as function of the other one
(i.e., solve the equation w.r.t. one of the two variables).

So we may get an univariate function. The graph of this function coincides with the indifference curve.

We then draw the graph of this univariate function by the method described above.

Cobb-Douglas-Function

We want to draw indifference curve

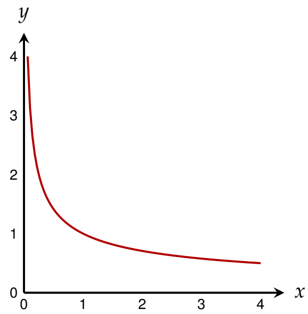
$$x^{\frac{1}{3}} y^{\frac{2}{3}} = 1\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$x = \frac{1}{y^2}$$

Alternatively we can express y by x:

$$y = \frac{1}{\sqrt{x}}$$



CES-Function

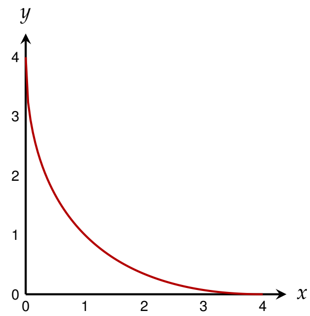
We want to draw indifference curve

$$\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 = 4\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$y = \left(2 - x^{\frac{1}{2}}\right)^2$$

(Take care about the domain of this curve!)



Problem 5.18

Draw the following indifference curves:

(a) $x + y^2 - 1 = 0$

(b) $x^2 + y^2 - 1 = 0$

(c) $x^2 - y^2 - 1 = 0$

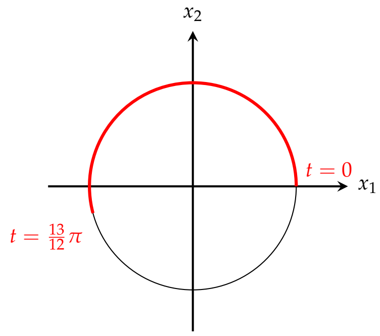
Paths

A function

$$\boxed{s\colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto s(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}}$$

is called a **path** in $\mathbb{R}^n$.
Variable t is often interpreted as *time*.

$$[0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$



Problem 5.19

Sketch the graphs of the following paths:

(a) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$

(b) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \end{pmatrix}$

(c) $s\colon [0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} t\ \cos(2\pi t) \\ t\ \sin(2\pi t) \end{pmatrix}$

Vector-valued Function

Generalized vector-valued function:

$$\boxed{\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}}$$

- Univariate functions:
  $\mathbb{R} \to \mathbb{R},\ x \mapsto y = x^2$
- Multivariate functions:
  $\mathbb{R}^2 \to \mathbb{R},\ \mathbf{x} \mapsto y = x_1^2 + x_2^2$
- Paths:
  $[0,1) \to \mathbb{R}^n,\ s \mapsto (s, s^2)^t$
- Linear maps:
  $\mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x}$ $\qquad$ $\mathbf{A} \dots\ m \times n$-Matrix

Summary

- real functions
- implicit domain
- graph of a function
- sources of errors
- piece-wise defined functions
- one-to-one and onto
- function composition
- inverse function
- elementary functions
- multivariate functions
- paths
- vector-valued functions