

## Chapter 4

# Sequences and Series

# Sequences

A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$a: \mathbb{N} \rightarrow \mathbb{R}, n \mapsto a_n$$

Sequences are denoted by  $(a_n)_{n=1}^{\infty}$  or just  $(a_n)$  for short.

An alternative notation used in literature is  $\langle a_n \rangle_{n=1}^{\infty}$ .

# Sequences

Sequences can be defined

- ▶ by **enumerating** of its terms,
- ▶ by a **formula**, or
- ▶ by **recursion**.

Each term is determined by its predecessor(s).

Enumeration:  $(a_n) = (1, 3, 5, 7, 9, \dots)$

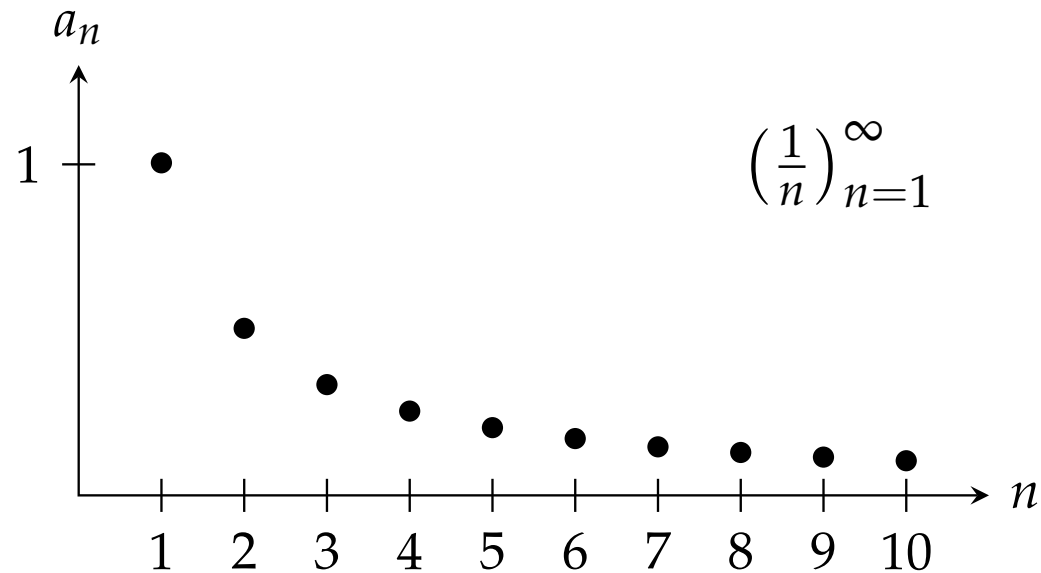
Formula:  $(a_n) = (2n - 1)$

Recursion:  $a_1 = 1, a_{n+1} = a_n + 2$

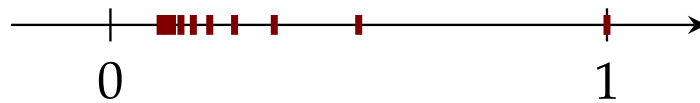
# Graphical Representation

A sequence  $(a_n)$  can be represented

(1) by drawing tuples  $(n, a_n)$  in the plane, or



(2) by drawing points on the number line.



# Properties

**Properties** of a sequence  $(a_n)$ :

Property	Definition
monotonically increasing	$a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$
monotonically decreasing	$a_{n+1} \leq a_n$
alternating	$a_{n+1} \cdot a_n < 0$ i.e. the sign changes
bounded	$ a_n  \leq M$ for some $M \in \mathbb{R}$

Sequence  $(\frac{1}{n})$  is

- ▶ monotonically decreasing
- ▶ bounded, as for all  $n \in \mathbb{N}$ ,  $|a_n| = |1/n| \leq M = 1$   
(we could also choose  $M = 1000$ )
- ▶ but *not* alternating.

# Problem 4.1

Draw the first 10 elements of the following sequences.

Which of these sequences are monotone, alternating, or bounded?

**(a)**  $(n^2)_{n=1}^{\infty}$

**(b)**  $(n^{-2})_{n=1}^{\infty}$

**(c)**  $(\sin(\pi/n))_{n=1}^{\infty}$

**(d)**  $a_1 = 1, a_{n+1} = 2a_n$

**(e)**  $a_1 = 1, a_{n+1} = -\frac{1}{2}a_n$

# Series

The sum of the first  $n$  terms of sequence  $(a_i)_{i=1}^{\infty}$

$$s_n = \sum_{i=1}^n a_i$$

is called the  $n$ -th **partial sum** of the *sequence*.

The sequence  $(s_n)$  of all partial sums is called the **series** of the sequence.

The series of sequence  $(a_i) = (2i - 1)$  is

$$(s_n) = \left( \sum_{i=1}^n (2i - 1) \right) = (1, 4, 9, 16, 25, \dots) = (n^2) .$$

## Problem 4.2

Compute the first 5 partial sums of the following sequences:

(a)  $2n$

(b)  $\frac{1}{2+n}$

(c)  $2^{n/10}$



# Arithmetic Sequence

Formula and recursion:

$$a_n = a_1 + (n - 1) \cdot d$$

$$a_{n+1} = a_n + d$$

*Differences* of consecutive terms are constant:

$$a_{n+1} - a_n = d$$

Each term is the *arithmetic mean* of its neighboring terms:

$$a_n = \frac{1}{2}(a_{n+1} + a_{n-1})$$

**Arithmetic series:**

$$s_n = \frac{n}{2}(a_1 + a_n)$$

# Geometric Sequence

Formula and recursion:

$$a_n = a_1 \cdot q^{n-1}$$

$$a_{n+1} = a_n \cdot q$$

*Ratios* of consecutive terms are constant:

$$\frac{a_{n+1}}{a_n} = q$$

Each term is the *geometric mean* of its neighboring terms:

$$a_n = \sqrt{a_{n+1} \cdot a_{n-1}}$$

**Geometric series:**

$$s_n = a_1 \cdot \frac{q^n - 1}{q - 1} \quad \text{for } q \neq 1$$

# Sources of Errors

Indices of sequences may also start with 0 (instead of 1).

**Beware!**

Formulæ then are slightly changed.

*Arithmetic* sequence:

$$a_n = a_0 + n \cdot d \quad \text{and} \quad s_n = \frac{n+1}{2} (a_0 + a_n)$$

*Geometric* sequence:

$$a_n = a_0 \cdot q^n \quad \text{and} \quad s_n = a_0 \cdot \frac{q^{n+1} - 1}{q - 1} \quad (\text{for } q \neq 1)$$

## Problem 4.3

We are given a geometric sequence  $(a_n)$  with  $a_1 = 2$  and relative change 0.1, i.e., each term of the sequence is increased by 10% compared to its predecessor.

Give formula and term  $a_7$ .

## Problem 4.4

Compute the first 10 partial sums of the arithmetic series for

**(a)**  $a_1 = 0$  and  $d = 1$ ,

**(b)**  $a_1 = 1$  and  $d = 2$ .

## Problem 4.5

Compute  $\sum_{n=1}^N a_n$  for

**(a)**  $N = 7$  and  $a_n = 3^{n-2}$

**(b)**  $N = 7$  and  $a_n = 2(-1/4)^n$

# Applications of Geometric Sequences

See your favorite book /course on finance and accounting.

# Summary

- ▶ sequence
- ▶ formula and recursion
- ▶ series and partial sums
- ▶ arithmetic and geometric sequence