Chapter 4

Sequences and Series

Sequences

A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$a\colon \mathbb{N} \to \mathbb{R},\ n \mapsto a_n$$

Sequences are denoted by $(a_n)_{n=1}^{\infty}$ or just (a_n) for short.

An alternative notation used in literature is $\langle a_n \rangle_{n=1}^{\infty}$.

Sequences

Sequences can be defined

- by **enumerating** of its terms,
- by a **formula**, or
- by **recursion**.
  Each term is determined by its predecessor(s).

Enumeration: $(a_n) = (1, 3, 5, 7, 9, \ldots)$

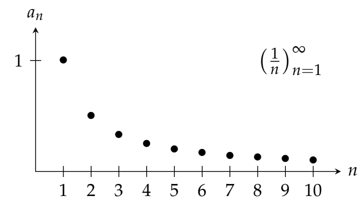
Formula: $(a_n) = (2\,n - 1)$

Recursion: $a_1 = 1,\ \ a_{n+1} = a_n + 2$

Graphical Representation

A sequence (a_n) can by represented

(1) by drawing tuples (n, a_n) in the plane, or



(2) by drawing points on the number line.

Properties

Properties of a sequence (a_n):

Property	Definition	
monotonically increasing	$a_{n+1} \geq a_n$	for all $n \in \mathbb{N}$
monotonically decreasing	$a_{n+1} \leq a_n$	
alternating	$a_{n+1} \cdot a_n < 0$	i.e. the sign changes
bounded	$\lvert a_n \rvert \leq M$	for some $M \in \mathbb{R}$

Sequence $\left(\frac{1}{n}\right)$ is

- monotonically decreasing
- bounded, as for all $n \in \mathbb{N}$, $\lvert a_n \rvert = \lvert 1/n \rvert \leq M = 1$
  (we could also choose $M = 1000$)
- but *not* alternating.

Problem 4.1

Draw the first 10 elements of the following sequences.
Which of these sequences are monotone, alternating, or bounded?

(a) $(n^2)_{n=1}^{\infty}$

(b) $(n^{-2})_{n=1}^{\infty}$

(c) $\left(\sin(\pi/n)\right)_{n=1}^{\infty}$

(d) $a_1 = 1,\ \ a_{n+1} = 2a_n$

(e) $a_1 = 1,\ \ a_{n+1} = -\frac{1}{2}a_n$

Series

The sum of the first n terms of sequence $(a_i)_{i=1}^{\infty}$

$$s_n = \sum_{i=1}^{n} a_i$$

is called the n-th **partial sum** of the *sequence*.

The sequence (s_n) of all partial sums is called the **series** of the sequence.

The series of sequence $(a_i) = (2\,i - 1)$ is

$$(s_n) = \left(\sum_{i=1}^{n} (2\,i - 1)\right) = (1, 4, 9, 16, 25, \ldots) = (n^2)\ .$$

Problem 4.2

Compute the first 5 partial sums of the following sequences:

(a) $2n$

(b) $\dfrac{1}{2+n}$

(c) $2^{n/10}$

Arithmetic Sequence

Formula and recursion:

$$a_n = a_1 + (n-1)\cdot d \qquad a_{n+1} = a_n + d$$

Differences of consecutive terms are constant:

$$a_{n+1} - a_n = d$$

Each term is the *arithmetic mean* of its neighboring terms:

$$a_n = \tfrac{1}{2}(a_{n+1} + a_{n-1})$$

Arithmetic series:

$$s_n = \tfrac{n}{2}(a_1 + a_n)$$

Geometric Sequence

Formula and recursion:

$$a_n = a_1 \cdot q^{n-1} \qquad a_{n+1} = a_n \cdot q$$

Ratios of consecutive terms are constant:

$$\frac{a_{n+1}}{a_n} = q$$

Each term is the *geometric mean* of its neighboring terms:

$$a_n = \sqrt{a_{n+1}\cdot a_{n-1}}$$

Geometric series:

$$s_n = a_1 \cdot \frac{q^n - 1}{q-1} \qquad \text{for } q \neq 1$$

Sources of Errors

Indices of sequences may also start with 0 (instead of 1).

Beware!
Formulæ then are slightly changed.

Arithmetic sequence:

$$a_n = a_0 + n\cdot d \qquad \text{and} \qquad s_n = \tfrac{n+1}{2}(a_0 + a_n)$$

Geometric sequence:

$$a_n = a_0 \cdot q^n \qquad \text{and} \qquad s_n = a_0 \cdot \frac{q^{n+1}-1}{q-1} \quad (\text{for } q \neq 1)$$

Problem 4.3

We are given a geometric sequence (a_n) with $a_1 = 2$ and relative change 0.1, i.e., each term of the sequence is increased by 10% compared to its predecessor.
Give formula and term a_7.

Problem 4.4

Compute the first 10 partial sums of the arithmetic series for

(a) $a_1 = 0$ and $d = 1$,

(b) $a_1 = 1$ and $d = 2$.

Problem 4.5

Compute $\sum_{n=1}^{N} a_n$ for

(a) $N = 7$ and $a_n = 3^{n-2}$

(b) $N = 7$ and $a_n = 2(-1/4)^n$

Applications of Geometric Sequences

See your favorite book /course on finance and accounting.

Summary

- sequence
- formula and recursion
- series and partial sums
- arithmetic and geometric sequence