Chapter 3

Equations and Inequalities

Equation

We get an **equation** by **equating** two terms.

$$\boxed{\text{l.h.s.} = \text{r.h.s.}}$$

- **Domain**:
  Intersection of domains of all involved terms restricted to a feasible region (e.g., non-negative numbers).

- **Solution set**:
  Set of all objects from the domain that solve the equation(s).

Transform into Equivalent Equation

Idea:
The equation is transformed into an *equivalent* but *simpler* equation, i.e., one with the *same solution* set.

- *Add* or *subtract* a number or term on both sides of the equation.
- *Multiply* or *divide* by a **non-zero** number or term on both sides of the equation.
- Take the *logarithm* or *antilogarithm* on both sides.

A useful strategy is to *isolate* the unknown quantity on one side of the equation.

Sources of Errors

Beware!
These operations may change the *domain* of the equation.
This may or may not alter the solution set.

In particular this happens if a rational term is reduced or expanded by a factor that contains the unknown.

Important!
Verify that both sides are *strictly positive* before taking the *logarithm*.

Beware!
Any term that contains the unknown may *vanish* (become 0).

- Multiplication may result in an *additional* but *invalid* “solution”.
- Division may *eliminate* a *valid* solution.

Non-equivalent Domains

Equation

$$\frac{(x-1)(x+1)}{x-1} = 1$$

can be transformed into the seemingly equivalent equation

$$x + 1 = 1$$

by reducing the rational term by factor $(x-1)$.

However, the latter has domain $\mathbb{R}$
while the given equation has domain $\mathbb{R} \setminus \{1\}$.

Fortunately, the solution set $L = \{0\}$ remains unchanged by this transformation.

Multiplication

By multiplication of

$$\frac{x^2 + x - 2}{x - 1} = 1$$

by $(x - 1)$ we get

$$x^2 + x - 2 = x - 1$$

with solution set $L = \{-1, 1\}$.

However, $x = 1$ is not in the domain of $\frac{x^2 + x - 2}{x - 1}$ and thus not a valid solution of our equation.

Division

If we divide both sides of equation

$$(x-1)(x-2)=0 \qquad \text{(solution set } L=\{1,2\}\text{)}$$

by $(x-1)$ we obtain equation

$$x-2=0 \qquad \text{(solution set } L=\{2\}\text{)}$$

Thus solution $x=1$ has been “lost” by this division.

Division

Important!
We need a *case-by-case* analysis when we divide by some term that contains an unknown:

Case 1: Division is *allowed* (the divisor is **non-zero**).

Case 2: Division is *forbidden* (the divisor is **zero**).

Find all solutions of $(x-1)(x-2)=0$:

Case $x-1 \neq 0$:
By division we get equation $x-2=0$ with solution $x_1=2$.

Case $x-1=0$:
This implies solution $x_2=1$.

We shortly will discuss a better method for finding roots.

Division

System of two equations in two unknowns

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

Addition and division in the first equation yields:

$$x\,y = x \;\rightsquigarrow\; y = \frac{x}{x} = 1$$

Substituting into the second equation then gives $x = \pm 1$.
Seemingly solution set: $L = \{(-1, 1), (1, 1)\}$.

However: Division is only allowed if $x \neq 0$.

$x = 0$ also satisfies the first equation (for every y).

Correct solution set: $L = \{(-1, 1), (1, 1), (0, \sqrt{2}), (0, -\sqrt{2})\}$.

Factorization

Factorizing a term can be a suitable method for finding roots (points where a term vanishes).

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

The first equation $\quad x\,y - x = x \cdot (y-1) = 0 \quad$ implies

$$x = 0 \quad \textit{or} \quad y - 1 = 0 \quad \text{(or both).}$$

Case $x = 0$: $\quad y = \pm\sqrt{2}$

Case $y - 1 = 0$: $\quad y = 1$ and $x = \pm 1$.

Solution set $L = \{(-1,1), (1,1), (0,\sqrt{2}), (0,-\sqrt{2})\}$.

Verification

A (seemingly correct) solution can be easily *verified* by **substituting** it into the given equation.

If unsure, verify the correctness of a solution.

Hint for your exams:
If a (homework or exam) problem asks for verification of a given solution, then simply substitute into the equation.
There is no need to solve the equation from scratch.

Linear Equation

Linear equations only contain *linear terms* and can (almost) always be solved.

Express annuity R from the formula for the present value

$$B_n = R \cdot \frac{q^n - 1}{q^n(q-1)} \; .$$

As R is to the power 1 only we have a linear equation which can be solved by dividing by (non-zero) constant $\frac{q^n-1}{q^n(q-1)}$:

$$R = B_n \cdot \frac{q^n(q-1)}{q^n - 1}$$

Equation with Absolute Value

An equation with *absolute value* can be seen as an abbreviation for a *system* of two (or more) equations:

$$|x| = 1 \quad \Leftrightarrow \quad x = 1 \text{ or } -x = 1$$

Find all solutions of $|2x - 3| = |x + 1|$.

Union of the respective solutions of the two equations

$$(2x - 3) = (x + 1) \qquad \Rightarrow \; x = 4$$
$$-(2x - 3) = (x + 1) \qquad \Rightarrow \; x = \frac{2}{3}$$

(Equations $-(2x - 3) = -(x + 1)$ and $(2x - 3) = -(x + 1)$ are equivalent to the above ones.)

We thus find solution set: $L = \{\frac{2}{3}, 4\}$.

Problem 3.1

Solve the following equations:

(a) $|x(x-2)| = 1$

(b) $|x+1| = \dfrac{1}{|x-1|}$

(c) $\left|\dfrac{x^2-1}{x+1}\right| = 2$

Equation with Exponents

Equations where the unknown is an exponent can (sometimes) be solved by taking the logarithm:

- Isolate the term with the unknown on one side of the equation.
- Take the **logarithm** on both sides.

Solve equation $2^x = 32$.
By taking the logarithm we obtain

$$\begin{aligned} & 2^x = 32 \\ \Leftrightarrow \quad & \ln(2^x) = \ln(32) \\ \Leftrightarrow \quad & x \ln(2) = \ln(32) \\ \Leftrightarrow \quad & x = \frac{\ln(32)}{\ln(2)} = 5 \end{aligned}$$

Solution set: $L = \{5\}$.

Equation with Exponents

Compute the term n of a loan over K monetary units and accumulation factor q from formula

$$X = K \cdot q^n \frac{q-1}{q^n - 1}$$

for installment X.

$$\begin{aligned}
X &= K \cdot q^n \tfrac{q-1}{q^n-1} && \big|\ \cdot (q^n - 1) \\
X\,(q^n - 1) &= K\, q^n\,(q-1) && \big|\ -K\, q^n\,(q-1) \\
q^n\,(X - K(q-1)) - X &= 0 && \big|\ +X \\
q^n\,(X - K(q-1)) &= X && \big|\ : (X - K(q-1)) \\
q^n &= \tfrac{X}{X-K(q-1)} && \big|\ \ln \\
n\,\ln(q) &= \ln(X) - \ln(X - K(q-1)) && \big|\ : \ln(q) \\
n &= \tfrac{\ln(X) - \ln(X-K(q-1))}{\ln(q)}
\end{aligned}$$

Equation with logarithms

Equations which contain (just) the logarithm of the unknown can (sometimes) be solved by taking the antilogarithm.

We get solution of $\ln(x+1) = 0$ by:

$$
\begin{aligned}
& \ln(x+1) = 0 \\
\Leftrightarrow\quad & e^{\ln(x+1)} = e^0 \\
\Leftrightarrow\quad & x+1 = 1 \\
\Leftrightarrow\quad & x = 0
\end{aligned}
$$

Solution set: $L = \{0\}$.

Problem 3.2

Solve the following equations:

(a) $2^x = 3^{x-1}$

(b) $3^{2-x} = 4^{\frac{x}{2}}$

(c) $2^x 5^{2x} = 10^{x+2}$

(d) $2 \cdot 10^{x-2} = 0.1^{3x}$

(e) $\frac{1}{2^{x+1}} = 0.2^x 10^4$

(f) $(3^x)^2 = 4 \cdot 5^{3x}$

Problem 3.3

Function

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

is called the *hyperbolic cosine*.
Find all solutions of

$$\cosh(x) = a$$

Hint: Use auxiliary variable $y = e^x$. Then the equation simplifies to $(y + \frac{1}{y})/2 = a$.

Problem 3.4

Solve the following equation:

$$\ln\left(x^2\left(x-\frac{7}{4}\right)+\left(\frac{x}{4}+1\right)^2\right)=0$$

Equation with Powers

An Equation that contains **only one** power of the unknown which in addition has *integer* degree can be solved by **calculating roots**.

Important!

- Take care that the equation may not have a (unique) solution (in $\mathbb{R}$) if the power has *even* degree.
- If its degree is *odd*, then the solution always exists and is unique (in $\mathbb{R}$).

The solution set of $x^2 = 4$ is $L = \{-2, 2\}$.

Equation $x^2 = -4$ does not have a (real) solution, $L = \emptyset$.

The solution set of $x^3 = -8$ is $L = \{-2\}$.

Equation with Roots

We can solve an **equation with roots** by squaring or taking a power of both sides.

We get the solution of $\sqrt[3]{x-1} = 2$ by taking the third power:

$$\sqrt[3]{x-1} = 2 \quad \Leftrightarrow \quad x - 1 = 2^3 \quad \Leftrightarrow \quad x = 9$$

Square Root

Beware!
Squaring an equation with square roots may create *additional* but *invalid* solutions (cf. multiplying with possible negative terms).

Squaring "non-equality" $-3 \neq 3$ yields equality $(-3)^2 = 3^2$.

Beware!
The domain of an equation with roots often is just a subset of $\mathbb{R}$.

For roots with even root degree the *radicand* must not be negative.

Important!

Always *verify* solutions of equations with roots!

Square Root

Solve equation $\sqrt{x-1} = 1 - \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 - \sqrt{x-4} && |^2 \\
x - 1 &= 1 - 2 \cdot \sqrt{x-4} + (x-4) && | - x + 3 \quad | : 2 \\
1 &= -\sqrt{x-4} && |^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

However, substitution gives $\sqrt{5-1} = 1 - \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 0$, which is **false**. Thus we get solution set $L = \emptyset$.

Square Root

Solve equation $\sqrt{x-1} = 1 + \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 + \sqrt{x-4} && |^2 \\
x - 1 &= 1 + 2 \cdot \sqrt{x-4} + (x-4) && | - x + 3 \quad | : 2 \\
1 &= \sqrt{x-4} && |^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

Now, verification yields $\sqrt{5-1} = 1 + \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 2$,
which is a **true** statement. Thus we get non-empty solution $L = \{5\}$.

Problem 3.5

Solve the following equations:

(a) $\sqrt{x+3} = x + 1$

(b) $\sqrt{x-2} = \sqrt{x+1} - 1$

Quadratic Equation

A **quadratic equation** is one of the form

$$a\,x^2 + b\,x + c = 0 \qquad \text{Solution: } \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or in standard form

$$x^2 + p\,x + q = 0 \qquad \text{Solution: } \quad x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Roots of Polynomials

Quadratic equations are a special case of **algebraic equations** (*polynomial equations*)

$$P_n(x) = 0$$

where $P_n(x)$ is a polynomial of degree n.

There exist closed form solutions for algebraic equations of degree 3 (*cubic equations*) and 4, resp. However, these are rather tedious.

For polynomials of degree 5 or higher no general formula does exist.

Roots of Polynomials

A polynomial equation can be solved by reducing its degree recursively.

1. Search for a root x_1 of $P_n(x)$
   (e.g. by trial and error, by means of Vieta's formulas,
   or by means of Newton's method)
2. We obtain linear factor $(x - x_1)$ of $P_n(X)$.
3. By division $P_n(x) : (x - x_1)$
   we get a polynomial $P_{n-1}(x)$ of degree $n - 1$.
4. If $n - 1 = 2$, solve the resulting quadratic equation.

   Otherwise goto Step 1.

Roots of Polynomials

Find all solutions of

$$x^3 - 6x^2 + 11x - 6 = 0.$$

By educated guess we find solution $x_1 = 1$.

Division by the linear factor $(x - 1)$ yields

$$(x^3 - 6x^2 + 11x - 6) : (x - 1) = x^2 - 5x + 6$$

Quadratic equation $x^2 - 5x + 6 = 0$ has solutions $x_2 = 2$ and $x_3 = 3$.

The solution set is thus $L = \{1, 2, 3\}$.

Roots of Products

A *product* of two (or more) terms $f(x) \cdot g(x)$ is zero if and only if *at least one* factor is zero:

$$f(x) = 0 \quad \textit{or} \quad g(x) = 0 \quad \text{(or both).}$$

Equation $x^2 \cdot (x-1) \cdot e^x = 0$ is satisfied if

- $x^2 = 0 \quad (\Rightarrow x = 0)$, *or*
- $x - 1 = 0 \quad (\Rightarrow x = 1)$, *or*
- $e^x = 0$ (no solution).

Thus we have solution set $L = \{0, 1\}$.

Roots of Products

Important!

If a polynomial is already factorized one should resist to expand this expression.

The roots of polynomial

$$(x - 1) \cdot (x + 2) \cdot (x - 3) = 0$$

are obviously 1, -2 and 3.

Roots of the expanded expression

$$x^3 - 2x^2 - 5x + 6 = 0$$

are hard to find.

Problem 3.6

Compute all roots and decompose into linear factors:

(a) $f(x) = x^2 + 4x + 3$

(b) $f(x) = 3x^2 - 9x + 2$

(c) $f(x) = x^3 - x$

(d) $f(x) = x^3 - 2x^2 + x$

(e) $f(x) = (x^2 - 1)(x - 1)^2$

Problem 3.7

Solve with respect to x and with respect to y:

(a) $xy + x - y = 0$

(b) $3xy + 2x - 4y = 1$

(c) $x^2 - y^2 + x + y = 0$

(d) $x^2y + xy^2 - x - y = 0$

(e) $x^2 + y^2 + 2xy = 4$

(f) $9x^2 + y^2 + 6xy = 25$

(g) $4x^2 + 9y^2 = 36$

(h) $4x^2 - 9y^2 = 36$

(i) $\sqrt{x} + \sqrt{y} = 1$

Problem 3.8

Solve with respect to x and with respect to y:

(a) $xy^2 + yx^2 = 6$

(b) $xy^2 + (x^2 - 1)y - x = 0$

(c) $\frac{x}{x+y} = \frac{y}{x-y}$

(d) $\frac{y}{y+x} = \frac{y-x}{y+x^2}$

(e) $\frac{1}{y-1} = \frac{y+x}{2y+1}$

(f) $\frac{yx}{y+x} = \frac{1}{y}$

(g) $(y + 2x)^2 = \frac{1}{1+x} + 4x^2$

(h) $y^2 - 3xy + \left(2x^2 + x - 1\right) = 0$

(i) $\frac{y}{x+2y} = \frac{2x}{x+y}$

Problem 3.9

Find constants a, b and c such that the following equations hold for all x in the corresponding domains:

(a) $$\frac{x}{1+x} - \frac{2}{2-x} = -\frac{2a + bx + cx^2}{2 + x - x^2}$$

(b) $$\frac{x^2+2x}{x+2} - \frac{x^2+3}{x+3} = \frac{a(x-b)}{x+c}$$

Inequalities

We get an **inequality** by comparing two terms by means of one of the "inequality" symbols

- $\leq$ (less than or equal to),
- $<$ (less than),
- $>$ (greater than),
- $\geq$ (greater than or equal to):

$$\text{l.h.s.} \leq \text{r.h.s.}$$

The inequality is called **strict** if equality does not hold.

Solution set of an inequality is the set of all numbers in its domain that satisfy the inequality.
Usually this is an (open or closed) interval or union of intervals.

Transform into Equivalent Inequality

Idea:
The inequality is transformed into an *equivalent* but *simpler* inequality.
Ideally we try to isolate the unknown on one side of the inequality.

Beware!
If we *multiply* an inequality by some **negative** number, then the *direction* of the inequality symbol is *reverted*.
Thus we need a case analysis:

- Case: term is **greater** than zero:
  Direction of inequality symbol is *not revert*.
- Case: term is **less** than zero:
  Direction of inequality symbol *is revert*.
- Case: term is **equal** to zero:
  Multiplication or division is *forbidden*!

Transform into Equivalent Inequality

Find all solutions of $$\frac{2\,x-1}{x-2} \leq 1.$$

We multiply inequality by $(x-2)$.

- Case $x-2>0 \Leftrightarrow x>2$:
  We find $2\,x-1 \leq x-2 \Leftrightarrow x \leq -1$,
  a contradiction to our assumption $x>2$.

- Case $x-2<0 \Leftrightarrow x<2$:
  The inequality symbol is reverted, and
  we find $2\,x-1 \geq x-2 \Leftrightarrow x \geq -1$.
  Hence $x<2$ and $x \geq -1$.

- Case $x-2=0 \Leftrightarrow x=2$:
  not in domain of inequality.

Solution set is the interval $L=[-1,2)$.

Sources of Errors

Inequalities with polynomials *cannot* be directly solved by transformations.

Important!
One **must not** simply replace the equality sign "$=$" in the formula for quadratic equations by an inequality symbols.

We want to find all solutions of

$$x^2 - 3\,x + 2 \leq 0$$

Invalid approach: $x_{1,2} \leq \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2}$

and thus $x \leq 1$ (and $x \leq 2$) which would imply "solution" set $L = (-\infty, 1]$.

However, $0 \in L$ but violates the inequality as $2 \not\leq 0$.

Inequalities with Polynomials

1. Move all terms on the l.h.s. and obtain an expression of the form
   $T(x) \leq 0$ (and $T(x) < 0$, resp.).

2. Compute all roots $x_1 < \ldots < x_k$ of $T(x)$,
   i.e., solve equation $T(x) = 0$ as we would with any polynomial as described above.

3. These roots decompose the *domain* into intervals I_j.
   These are open if the inequality is *strict* (with $<$ or $>$), and closed otherwise.

   In each of these intervals the inequality now holds
   either in *all* **or** in *none* of its points.

4. Select some point $z_j \in I_j$ which is not on the boundary.
   If z_j satisfies the corresponding **strict** inequality,
   then I_j belongs to the solution set, else none of its points.

Inequalities with Polynomials

Find all solutions of

$$x^2 - 3\,x + 2 \leq 0\ .$$

The solutions of $x^2 - 3\,x + 2 = 0$ are $x_1 = 1$ and $x_2 = 2$.

We obtain three intervals and check by means of three points (0, $\frac{3}{2}$, and 3) whether the inequality is satisfied in each of these:

$(-\infty, 1]$	not satisfied:	$0^2 - 3 \cdot 0 + 2 = 2 \not\leq 0$
$[1, 2]$	satisfied:	$\left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 2 = -\frac{1}{4} < 0$
$[2, \infty)$	not satisfied:	$3^2 - 3 \cdot 3 + 2 = 2 \not\leq 0$

Solution set is $L = [1, 2]$.

Continuous Terms

The above principle also works for inequalities where all terms are **continuous**.

If there is any point where $T(x)$ is *not continuous*, then we also have to use this point for decomposing the domain into intervals.

Furthermore, we have to take care when the domain of the inequality is a union of two or more disjoint intervals.

Continuous Terms

Find all solutions of inequality

$$\frac{x^2+x-3}{x-2} \geq 1 \, .$$

Its domain is the union of two intervals: $(-\infty, 2) \cup (2, \infty)$.

We find for the solutions of the equation $\dfrac{x^2+x-3}{x-2} = 1$:

$$\frac{x^2+x-3}{x-2} = 1 \;\Leftrightarrow\; x^2+x-3 = x-2 \;\Leftrightarrow\; x^2-1 = 0$$

and thus $x_1 = -1$, $x_2 = 1$.
So we get four intervals:

$$(-\infty, -1],\ [-1, 1],\ [1, 2) \text{ and } (2, \infty).$$

Continuous Terms

We check by means of four points whether the inequality holds in these intervals:

$(-\infty, -1]$	not satisfied:	$\frac{(-2)^2+(-2)-3}{(-2)-2} = \frac{1}{4} \not\geq 1$
$[-1, 1]$	satisfied:	$\frac{0^2-0-3}{0-2} = \frac{3}{2} > 1$
$[1, 2)$	not satisfied:	$\frac{1.5^2+1.5-3}{1.5-2} = -\frac{3}{2} \not\geq 1$
$(2, \infty)$	satisfied:	$\frac{3^2+3-3}{3-2} = 9 > 1$

Solution set is $L = [-1, 1] \cup (2, \infty)$.

Inequalities with Absolute Values

Inequalities with *absolute values* can be solved by the above procedure.

However, we also can see such an inequality as a system of two (or more) inequalities:

$$|x| < 1 \quad \Leftrightarrow \quad x < 1 \text{ and } x > -1$$

$$|x| > 1 \quad \Leftrightarrow \quad x > 1 \text{ or } x < -1$$

Problem 3.10

Solve the following inequalities:

(a) $x^3 - 2x^2 - 3x \geq 0$

(b) $x^3 - 2x^2 - 3x > 0$

(c) $x^2 - 2x + 1 \leq 0$

(d) $x^2 - 2x + 1 \geq 0$

(e) $x^2 - 2x + 6 \leq 1$

Problem 3.11

Solve the following inequalities:

(a) $7 \leq |12x + 1|$

(b) $\dfrac{x+4}{x+2} < 2$

(c) $\dfrac{3(4-x)}{x-5} \leq 2$

(d) $25 < (-2x+3)^2 \leq 50$

(e) $42 \leq |12x + 6| < 72$

(f) $5 \leq \dfrac{(x+4)^2}{|x+4|} \leq 10$

Summary

- equations and inequalities
- domain and solution set
- transformation into equivalent problem
- possible errors with multiplication and division
- equations with powers and roots
- equations with polynomials and absolute values
- roots of polynomials
- equations with exponents and logarithms
- method for solving inequalities