Chapter 3

# Equations and Inequalities

# Equation

We get an **equation** by **equating** two terms.

$$\boxed{\text{l.h.s.} = \text{r.h.s.}}$$

- **Domain**:
  Intersection of domains of all involved terms restricted to a feasible region (e.g., non-negative numbers).

- **Solution set**:
  Set of all objects from the domain that solve the equation(s).

# Transform into Equivalent Equation

**Idea:**
The equation is transformed into an *equivalent* but *simpler* equation, i.e., one with the *same solution* set.

- *Add* or *subtract* a number or term on both sides of the equation.
- *Multiply* or *divide* by a **non-zero** number or term on both sides of the equation.
- Take the *logarithm* or *antilogarithm* on both sides.

A useful strategy is to *isolate* the unknown quantity on one side of the equation.

# Sources of Errors

**Beware!**
These operations may change the *domain* of the equation.
This may or may not alter the solution set.

In particular this happens if a rational term is reduced or expanded by a factor that contains the unknown.

**Important!**
Verify that both sides are *strictly positive* before taking the *logarithm*.

**Beware!**
Any term that contains the unknown may *vanish* (become $0$).

- Multiplication may result in an *additional* but *invalid* “solution”.
- Division may *eliminate* a *valid* solution.

# Non-equivalent Domains

Equation

$$\frac{(x-1)(x+1)}{x-1} = 1$$

can be transformed into the seemingly equivalent equation

$$x + 1 = 1$$

by reducing the rational term by factor $(x-1)$.

However, the latter has domain $\mathbb{R}$
while the given equation has domain $\mathbb{R} \setminus \{1\}$.

Fortunately, the solution set $L = \{0\}$ remains unchanged by this transformation.

# Multiplication

By multiplication of

$$\frac{x^2 + x - 2}{x - 1} = 1$$

by $(x - 1)$ we get

$$x^2 + x - 2 = x - 1$$

with solution set $L = \{-1, 1\}$.

However, $x = 1$ is not in the domain of $\dfrac{x^2 + x - 2}{x - 1}$ and thus not a valid solution of our equation.

# Division

If we divide both sides of equation

$$(x-1)(x-2)=0 \qquad \text{(solution set } L=\{1,2\}\text{)}$$

by $(x-1)$ we obtain equation

$$x-2=0 \qquad \text{(solution set } L=\{2\}\text{)}$$

Thus solution $x=1$ has been “lost” by this division.

# Division

**Important!**
We need a *case-by-case* analysis when we divide by some term that contains an unknown:

**Case 1:** Division is *allowed* (the divisor is **non-zero**).

**Case 2:** Division is *forbidden* (the divisor is **zero**).

Find all solutions of $(x-1)(x-2)=0$:

Case $x-1 \neq 0$:
By division we get equation $x-2=0$ with solution $x_1=2$.

Case $x-1=0$:
This implies solution $x_2=1$.

We shortly will discuss a better method for finding roots.

# Division

System of two equations in two unknowns

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

Addition and division in the first equation yields:

$$x\,y = x \;\rightsquigarrow\; y = \frac{x}{x} = 1$$

Substituting into the second equation then gives $x = \pm 1$.
Seemingly solution set: $L = \{(-1,1),(1,1)\}$.

*However:* Division is only allowed if $x \neq 0$.

$x = 0$ also satisfies the first equation (for every $y$).

Correct solution set: $L = \{(-1,1),(1,1),(0,\sqrt{2}),(0,-\sqrt{2})\}$.

# Factorization

Factorizing a term can be a suitable method for finding roots (points where a term vanishes).

$$\begin{cases} x\,y - x = 0 \\ x^2 + y^2 = 2 \end{cases}$$

The first equation $\quad x\,y - x = x \cdot (y-1) = 0 \quad$ implies

$$x = 0 \quad \textit{or} \quad y - 1 = 0 \quad \text{(or both).}$$

Case $x = 0$: $\quad y = \pm\sqrt{2}$

Case $y - 1 = 0$: $\quad y = 1$ and $x = \pm 1$.

Solution set $L = \{(-1,1), (1,1), (0,\sqrt{2}), (0,-\sqrt{2})\}$.

# Verification

A (seemingly correct) solution can be easily *verified* by **substituting** it into the given equation.

**If unsure, verify the correctness of a solution.**

**Hint** for your exams:
If a (homework or exam) problem asks for verification of a given solution, then simply substitute into the equation.
There is no need to solve the equation from scratch.

# Linear Equation

**Linear equations** only contain *linear terms* and can (almost) always be solved.

Express annuity $R$ from the formula for the present value

$$B_n = R \cdot \frac{q^n - 1}{q^n(q-1)} \ .$$

As $R$ is to the power $1$ only we have a linear equation which can be solved by dividing by (non-zero) constant $\frac{q^n-1}{q^n(q-1)}$:

$$R = B_n \cdot \frac{q^n(q-1)}{q^n - 1}$$

# Equation with Absolute Value

An equation with *absolute value* can be seen as an abbreviation for a *system* of two (or more) equations:

$$\boxed{|x| = 1 \quad \Leftrightarrow \quad x = 1 \text{ or } -x = 1}$$

Find all solutions of $\quad |2x - 3| = |x + 1|$.

Union of the respective solutions of the two equations

$$\begin{aligned} (2x - 3) &= (x + 1) \qquad &\Rightarrow\ x = 4 \\ -(2x - 3) &= (x + 1) \qquad &\Rightarrow\ x = \tfrac{2}{3} \end{aligned}$$

(Equations $-(2x - 3) = -(x + 1)$ and $(2x - 3) = -(x + 1)$ are equivalent to the above ones.)

We thus find solution set: $L = \left\{\frac{2}{3}, 4\right\}$.

# Problem 3.1

Solve the following equations:

**(a)** $|x(x-2)| = 1$

**(b)** $|x+1| = \dfrac{1}{|x-1|}$

**(c)** $\left|\dfrac{x^2-1}{x+1}\right| = 2$

# Equation with Exponents

Equations where the unknown is an exponent can (sometimes) be solved by taking the logarithm:

- Isolate the term with the unknown on one side of the equation.
- Take the **logarithm** on both sides.

Solve equation $2^x = 32$.
By taking the logarithm we obtain

$$\begin{aligned} & 2^x = 32 \\ \Leftrightarrow \quad & \ln(2^x) = \ln(32) \\ \Leftrightarrow \quad & x \ln(2) = \ln(32) \\ \Leftrightarrow \quad & x = \frac{\ln(32)}{\ln(2)} = 5 \end{aligned}$$

Solution set: $L = \{5\}$.

# Equation with Exponents

Compute the term $n$ of a loan over $K$ monetary units and accumulation factor $q$ from formula

$$X = K \cdot q^n \frac{q-1}{q^n - 1}$$

for installment $X$.

$$\begin{aligned}
X &= K \cdot q^n \tfrac{q-1}{q^n-1} && \Big| \cdot (q^n - 1) \\
X\,(q^n - 1) &= K\,q^n\,(q-1) && \Big| -K\,q^n\,(q-1) \\
q^n\,(X - K(q-1)) - X &= 0 && \Big| +X \\
q^n\,(X - K(q-1)) &= X && \Big| : (X - K(q-1)) \\
q^n &= \tfrac{X}{X-K(q-1)} && \Big| \ln \\
n\,\ln(q) &= \ln(X) - \ln(X - K(q-1)) && \Big| : \ln(q) \\
n &= \tfrac{\ln(X)-\ln(X-K(q-1))}{\ln(q)}
\end{aligned}$$

# Equation with logarithms

Equations which contain (just) the logarithm of the unknown can (sometimes) be solved by taking the antilogarithm.

We get solution of $\ln(x+1) = 0$ by:

$$\begin{aligned} & \ln(x+1) = 0 \\ \Leftrightarrow \quad & e^{\ln(x+1)} = e^0 \\ \Leftrightarrow \quad & x+1 = 1 \\ \Leftrightarrow \quad & x = 0 \end{aligned}$$

Solution set: $L = \{0\}$.

# Problem 3.2

Solve the following equations:

**(a)** $2^x = 3^{x-1}$

**(b)** $3^{2-x} = 4^{\frac{x}{2}}$

**(c)** $2^x 5^{2x} = 10^{x+2}$

**(d)** $2 \cdot 10^{x-2} = 0.1^{3x}$

**(e)** $\frac{1}{2^{x+1}} = 0.2^x 10^4$

**(f)** $(3^x)^2 = 4 \cdot 5^{3x}$

# Problem 3.3

Function

$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

is called the *hyperbolic cosine*.
Find all solutions of

$$\cosh(x) = a$$

Hint: Use auxiliary variable $y = e^x$. Then the equation simplifies to $(y + \frac{1}{y})/2 = a$.

# Problem 3.4

Solve the following equation:

$$\ln\left(x^2\left(x-\frac{7}{4}\right)+\left(\frac{x}{4}+1\right)^2\right)=0$$

# Equation with Powers

An Equation that contains **only one** power of the unknown which in addition has *integer* degree can be solved by **calculating roots**.

**Important!**

- Take care that the equation may not have a (unique) solution (in $\mathbb{R}$) if the power has *even* degree.
- If its degree is *odd*, then the solution always exists and is unique (in $\mathbb{R}$).

The solution set of $x^2 = 4$ is $L = \{-2, 2\}$.

Equation $x^2 = -4$ does not have a (real) solution, $L = \emptyset$.

The solution set of $x^3 = -8$ is $L = \{-2\}$.

# Equation with Roots

We can solve an **equation with roots** by squaring or taking a power of both sides.

We get the solution of $\sqrt[3]{x-1} = 2$ by taking the third power:

$$\sqrt[3]{x-1} = 2 \quad \Leftrightarrow \quad x - 1 = 2^3 \quad \Leftrightarrow \quad x = 9$$

# Square Root

**Beware!**
*Squaring* an equation with square roots may create *additional* but *invalid* solutions (cf. multiplying with possible negative terms).

Squaring "non-equality" $-3 \neq 3$ yields equality $(-3)^2 = 3^2$.

**Beware!**
The domain of an equation with roots often is just a subset of $\mathbb{R}$.

For roots with even root degree the *radicand* must not be negative.

## Important!

**Always** *verify* solutions of equations with roots!

# Square Root

Solve equation $\sqrt{x-1} = 1 - \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 - \sqrt{x-4} && |^2 \\
x - 1 &= 1 - 2 \cdot \sqrt{x-4} + (x-4) && | - x + 3 \quad | : 2 \\
1 &= -\sqrt{x-4} && |^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

However, substitution gives $\sqrt{5-1} = 1 - \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 0$, which is **false**. Thus we get solution set $L = \emptyset$.

# Square Root

Solve equation $\sqrt{x-1} = 1 + \sqrt{x-4}$.

Domain is $D = \{x | x \geq 4\}$.

Squaring yields

$$\begin{aligned}
\sqrt{x-1} &= 1 + \sqrt{x-4} && |^2 \\
x - 1 &= 1 + 2 \cdot \sqrt{x-4} + (x-4) && | -x+3 \quad | : 2 \\
1 &= \sqrt{x-4} && |^2 \\
1 &= x - 4 \\
x &= 5
\end{aligned}$$

Now, verification yields $\sqrt{5-1} = 1 + \sqrt{5-4} \quad \Leftrightarrow \quad 2 = 2$,
which is a **true** statement. Thus we get non-empty solution $L = \{5\}$.

# Problem 3.5

Solve the following equations:

**(a)** $\sqrt{x+3} = x+1$

**(b)** $\sqrt{x-2} = \sqrt{x+1} - 1$

# Quadratic Equation

A **quadratic equation** is one of the form

$$a\,x^2 + b\,x + c = 0 \qquad \text{Solution:} \quad x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

or in standard form

$$x^2 + p\,x + q = 0 \qquad \text{Solution:} \quad x_{1,2} = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

# Roots of Polynomials

Quadratic equations are a special case of **algebraic equations** (*polynomial equations*)

$$P_n(x) = 0$$

where $P_n(x)$ is a polynomial of degree $n$.

There exist closed form solutions for algebraic equations of degree $3$ (*cubic equations*) and $4$, resp. However, these are rather tedious.

For polynomials of degree $5$ or higher no general formula does exist.

# Roots of Polynomials

A polynomial equation can be solved by reducing its degree recursively.

1. Search for a root $x_1$ of $P_n(x)$
   (e.g. by trial and error, by means of Vieta's formulas,
   or by means of Newton's method)

2. We obtain linear factor $(x - x_1)$ of $P_n(X)$.

3. By division $P_n(x) : (x - x_1)$
   we get a polynomial $P_{n-1}(x)$ of degree $n - 1$.

4. If $n - 1 = 2$, solve the resulting quadratic equation.

   Otherwise goto Step 1.

# Roots of Polynomials

Find all solutions of

$$x^3 - 6x^2 + 11x - 6 = 0.$$

By educated guess we find solution $x_1 = 1$.

Division by the linear factor $(x - 1)$ yields

$$(x^3 - 6x^2 + 11x - 6) : (x - 1) = x^2 - 5x + 6$$

Quadratic equation $x^2 - 5x + 6 = 0$ has solutions $x_2 = 2$ and $x_3 = 3$.

The solution set is thus $L = \{1, 2, 3\}$.

# Roots of Products

A *product* of two (or more) terms $f(x) \cdot g(x)$ is zero if and only if *at least one* factor is zero:

$$f(x) = 0 \quad \textit{or} \quad g(x) = 0 \quad \text{(or both).}$$

Equation $x^2 \cdot (x-1) \cdot e^x = 0$ is satisfied if

- $x^2 = 0 \quad (\Rightarrow x = 0), \quad$ *or*
- $x - 1 = 0 \quad (\Rightarrow x = 1), \quad$ *or*
- $e^x = 0 \quad$ (no solution).

Thus we have solution set $L = \{0, 1\}$.

# Roots of Products

**Important!**

If a polynomial is already factorized one should resist to expand this expression.

The roots of polynomial

$$(x-1)\cdot(x+2)\cdot(x-3)=0$$

are obviously $1$, $-2$ and $3$.

Roots of the expanded expression

$$x^3-2x^2-5x+6=0$$

are hard to find.

# Problem 3.6

Compute all roots and decompose into linear factors:

**(a)** $f(x) = x^2 + 4x + 3$

**(b)** $f(x) = 3x^2 - 9x + 2$

**(c)** $f(x) = x^3 - x$

**(d)** $f(x) = x^3 - 2x^2 + x$

**(e)** $f(x) = (x^2 - 1)(x - 1)^2$

# Problem 3.7

Solve with respect to $x$ and with respect to $y$:

**(a)** $xy + x - y = 0$

**(b)** $3xy + 2x - 4y = 1$

**(c)** $x^2 - y^2 + x + y = 0$

**(d)** $x^2y + xy^2 - x - y = 0$

**(e)** $x^2 + y^2 + 2xy = 4$

**(f)** $9x^2 + y^2 + 6xy = 25$

**(g)** $4x^2 + 9y^2 = 36$

**(h)** $4x^2 - 9y^2 = 36$

**(i)** $\sqrt{x} + \sqrt{y} = 1$

# Problem 3.8

Solve with respect to $x$ and with respect to $y$:

**(a)** $xy^2 + yx^2 = 6$

**(b)** $xy^2 + (x^2 - 1)y - x = 0$

**(c)** $\frac{x}{x+y} = \frac{y}{x-y}$

**(d)** $\frac{y}{y+x} = \frac{y-x}{y+x^2}$

**(e)** $\frac{1}{y-1} = \frac{y+x}{2y+1}$

**(f)** $\frac{yx}{y+x} = \frac{1}{y}$

**(g)** $(y + 2x)^2 = \frac{1}{1+x} + 4x^2$

**(h)** $y^2 - 3xy + (2x^2 + x - 1) = 0$

**(i)** $\frac{y}{x+2y} = \frac{2x}{x+y}$

# Problem 3.9

Find constants $a$, $b$ and $c$ such that the following equations hold for all $x$ in the corresponding domains:

**(a)** $$\frac{x}{1+x} - \frac{2}{2-x} = -\frac{2a+bx+cx^2}{2+x-x^2}$$

**(b)** $$\frac{x^2+2x}{x+2} - \frac{x^2+3}{x+3} = \frac{a(x-b)}{x+c}$$

# Inequalities

We get an **inequality** by comparing two terms by means of one of the "inequality" symbols

$\leq$ (less than or equal to),

$<$ (less than),

$>$ (greater than),

$\geq$ (greater than or equal to):

$$\boxed{\text{l.h.s.} \leq \text{r.h.s.}}$$

The inequality is called **strict** if equality does not hold.

**Solution set** of an inequality is the set of all numbers in its domain that satisfy the inequality.
Usually this is an (open or closed) interval or union of intervals.

# Transform into Equivalent Inequality

**Idea:**
The inequality is transformed into an *equivalent* but *simpler* inequality.
Ideally we try to isolate the unknown on one side of the inequality.

**Beware!**
If we *multiply* an inequality by some **negative** number, then the *direction* of the inequality symbol is *reverted*.
Thus we need a case analysis:

- Case: term is **greater** than zero:
  Direction of inequality symbol is *not revert*.
- Case: term is **less** than zero:
  Direction of inequality symbol *is revert*.
- Case: term is **equal** to zero:
  Multiplication or division is *forbidden*!

# Transform into Equivalent Inequality

Find all solutions of $\dfrac{2\,x-1}{x-2} \leq 1$.

We multiply inequality by $(x-2)$.

- Case $x-2>0 \Leftrightarrow x>2$:
  We find $2\,x-1 \leq x-2 \Leftrightarrow x \leq -1$,
  a contradiction to our assumption $x>2$.
- Case $x-2<0 \Leftrightarrow x<2$:
  The inequality symbol is reverted, and
  we find $2\,x-1 \geq x-2 \Leftrightarrow x \geq -1$.
  Hence $x<2$ and $x \geq -1$.
- Case $x-2=0 \Leftrightarrow x=2$:
  not in domain of inequality.

Solution set is the interval $L=[-1,2)$.

# Sources of Errors

Inequalities with polynomials *cannot* be directly solved by transformations.

**Important!**
One **must not** simply replace the equality sign "$=$" in the formula for quadratic equations by an inequality symbols.

We want to find all solutions of

$$x^2 - 3\,x + 2 \leq 0$$

**Invalid approach:** $x_{1,2} \leq \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} \pm \frac{1}{2}$

and thus $x \leq 1$ (and $x \leq 2$) which would imply "solution" set $L = (-\infty, 1]$.

However, $0 \in L$ but violates the inequality as $2 \not\leq 0$.

# Inequalities with Polynomials

1. Move all terms on the l.h.s. and obtain an expression of the form $T(x) \leq 0$ (and $T(x) < 0$, resp.).

2. Compute all roots $x_1 < \ldots < x_k$ of $T(x)$,
   i.e., solve equation $T(x) = 0$ as we would with any polynomial as described above.

3. These roots decompose the *domain* into intervals $I_j$.
   These are open if the inequality is *strict* (with $<$ or $>$), and closed otherwise.

   In each of these intervals the inequality now holds
   **either** in *all* **or** in *none* of its points.

4. Select some point $z_j \in I_j$ which is not on the boundary.
   If $z_j$ satisfies the corresponding **strict** inequality,
   then $I_j$ belongs to the solution set, else none of its points.

# Inequalities with Polynomials

Find all solutions of

$$x^2 - 3\,x + 2 \leq 0\,.$$

The solutions of $x^2 - 3\,x + 2 = 0$ are $x_1 = 1$ and $x_2 = 2$.

We obtain three intervals and check by means of three points ($0$, $\frac{3}{2}$, and $3$) whether the inequality is satisfied in each of these:

| | | |
|---|---|---|
| $(-\infty, 1]$ | not satisfied: | $0^2 - 3 \cdot 0 + 2 = 2 \not\leq 0$ |
| $[1, 2]$ | satisfied: | $\left(\frac{3}{2}\right)^2 - 3 \cdot \frac{3}{2} + 2 = -\frac{1}{4} < 0$ |
| $[2, \infty)$ | not satisfied: | $3^2 - 3 \cdot 3 + 2 = 2 \not\leq 0$ |

Solution set is $L = [1, 2]$.

# Continuous Terms

The above principle also works for inequalities where all terms are **continuous**.

If there is any point where $T(x)$ is *not continuous*, then we also have to use this point for decomposing the domain into intervals.

Furthermore, we have to take care when the domain of the inequality is a union of two or more disjoint intervals.

# Continuous Terms

Find all solutions of inequality

$$\frac{x^2 + x - 3}{x - 2} \geq 1\,.$$

Its domain is the union of two intervals: $(-\infty, 2) \cup (2, \infty)$.

We find for the solutions of the equation $\dfrac{x^2 + x - 3}{x - 2} = 1$:

$$\frac{x^2 + x - 3}{x - 2} = 1 \;\Leftrightarrow\; x^2 + x - 3 = x - 2 \;\Leftrightarrow\; x^2 - 1 = 0$$

and thus $x_1 = -1$, $x_2 = 1$.
So we get four intervals:

$$(-\infty, -1],\ [-1, 1],\ [1, 2) \text{ and } (2, \infty).$$

# Continuous Terms

We check by means of four points whether the inequality holds in these intervals:

| | | |
|---|---|---|
| $(-\infty, -1]$ | not satisfied: | $\frac{(-2)^2+(-2)-3}{(-2)-2} = \frac{1}{4} \not> 1$ |
| $[-1, 1]$ | satisfied: | $\frac{0^2-0-3}{0-2} = \frac{3}{2} > 1$ |
| $[1, 2)$ | not satisfied: | $\frac{1.5^2+1.5-3}{1.5-2} = -\frac{3}{2} \not> 1$ |
| $(2, \infty)$ | satisfied: | $\frac{3^2+3-3}{3-2} = 9 > 1$ |

Solution set is $L = [-1, 1] \cup (2, \infty)$.

# Inequalities with Absolute Values

Inequalities with *absolute values* can be solved by the above procedure.

However, we also can see such an inequality as a system of two (or more) inequalities:

$$|x| < 1 \qquad \Leftrightarrow \qquad x < 1 \text{ and } x > -1$$

$$|x| > 1 \qquad \Leftrightarrow \qquad x > 1 \text{ or } x < -1$$

# Problem 3.10

Solve the following inequalities:

**(a)** $x^3 - 2x^2 - 3x \geq 0$

**(b)** $x^3 - 2x^2 - 3x > 0$

**(c)** $x^2 - 2x + 1 \leq 0$

**(d)** $x^2 - 2x + 1 \geq 0$

**(e)** $x^2 - 2x + 6 \leq 1$

# Problem 3.11

Solve the following inequalities:

**(a)** $7 \leq |12x + 1|$

**(b)** $\dfrac{x+4}{x+2} < 2$

**(c)** $\dfrac{3(4-x)}{x-5} \leq 2$

**(d)** $25 < (-2x+3)^2 \leq 50$

**(e)** $42 \leq |12x + 6| < 72$

**(f)** $5 \leq \dfrac{(x+4)^2}{|x+4|} \leq 10$

# Summary

- equations and inequalities
- domain and solution set
- transformation into equivalent problem
- possible errors with multiplication and division
- equations with powers and roots
- equations with polynomials and absolute values
- roots of polynomials
- equations with exponents and logarithms
- method for solving inequalities