Chapter 2

Terms

Term

A mathematical expression like

$$B = R \cdot \frac{q^n - 1}{q^n(q-1)} \qquad \text{or} \qquad (x+1)(x-1) = x^2 - 1$$

contains symbols which denote mathematical objects.
These symbols and compositions of symbols are called **terms**.

Terms can be

- **numbers**,
- **constants** (symbols, which represent *fixed* values),
- **variables** (which are placeholders for *arbitrary* values), and
- **compositions** *of terms*.

Domain

We have to take care that a term may not be defined for some values of its variables.

- $\frac{1}{x-1}$ is only defined for $x \in \mathbb{R} \setminus \{1\}$.
- $\sqrt{x+1}$ is only defined for $x \geq -1$.

The set of values for which a term is defined is called the **domain** of the *term*.

Sigma Notation

Sums with many terms that can be generated by some rule can be represented in a compact form called *summation* or **sigma notation**.

$$\sum_{n=1}^{6} a_n = a_1 + a_2 + \cdots + a_6$$

a_n ... formula for the terms

n ... index of summation

1 ... first value of index n

6 ... last value of n

The expression is read as the "sum of a_n as n goes from 1 to 6."

First and last value can (and often are) given by *symbols*.

Sigma Notation

The sum of the first 10 integers greater than 2 can be written in both notations as

$$\sum_{i=1}^{10}(2+i) = (2+1)+(2+2)+(2+3)+\cdots+(2+10)$$

The sigma notation can be seen as a convenient shortcut of the long expression on the r.h.s.

It also avoids ambiguity caused by the *ellipsis* "$\ldots$" that might look like an IQ test rather than an exact mathematical expression.

Sigma Notation – Rules

Keep in mind that all the *usual rules* for multiplication and addition (associativity, commutativity, distributivity) apply:

- $\sum_{i=1}^{n} a_i = \sum_{k=1}^{n} a_k$
- $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$
- $\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$

Sigma Notation – Rules

Simplify

$$\sum_{i=2}^{n}(a_i + b_i)^2 - \sum_{j=2}^{n} a_j^2 - \sum_{k=2}^{n} b_k^2$$

Solution:

$$= \sum_{i=2}^{n}(a_i + b_i)^2 - \sum_{i=2}^{n} a_i^2 - \sum_{i=2}^{n} b_i^2$$

$$= \sum_{i=2}^{n} \left((a_i + b_i)^2 - a_i^2 - b_i^2\right)$$

$$= \sum_{i=2}^{n} 2\, a_i b_i$$

$$= 2 \sum_{i=2}^{n} a_i b_i$$

Problem 2.1

Which of the following expressions is equal to the summation

$$\sum_{i=2}^{10} 5(i+3) \ ?$$

(a) $5(2+3+4+\ldots+9+10+3)$

(b) $5(2+3+3+3+4+3+5+3+6+3+\ldots+10+3)$

(c) $5(2+3+4+\ldots+9+10)+5\cdot 3$

(d) $5(2+3+4+\ldots+9+10)+9\cdot 5\cdot 3$

Problem 2.2

Compute and simplify:

(a) $$\sum_{i=0}^{5} a^i b^{5-i}$$

(b) $$\sum_{i=1}^{5} (a_i - a_{i+1})$$

(c) $$\sum_{i=1}^{n} (a_i - a_{i+1})$$

Remark: The sum in (c) is a so called *telescoping sum*.

Problem 2.3

Simplify the following summations:

(a) $$\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 - \sum_{k=1}^{n} (a_k - b_k)^2$$

(b) $$\sum_{i=1}^{n} (a_i b_{n-i+1} - a_{n-i+1} b_i)$$

(c) $$\sum_{i=1}^{n} (x_i + y_i)^2 + \sum_{j=1}^{n} (x_j - y_j)^2$$

(d) $$\sum_{j=0}^{n-1} x_j - \sum_{i=1}^{n} x_i$$

Problem 2.4

Arithmetic mean (“average”)

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and *variance*

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

are important location and dispersion parameters in statistics.

Variance σ^2 can be computed by means of formula

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2$$

which requires to read data (x_i) only once.

Problem 2.4 / 2

Verify

$$\sigma^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

for

(a) $n = 2$,

(b) $n = 3$,

(c) $n \geq 2$ arbitrary.

Hint: Show that

$$\sum_{i=1}^{n}(x_i - \bar{x})^2 - \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2\right) = 0.$$

Problem 2.5

Let μ be the "true" value of a metric variate and $\{x_i\}$ the results of some measurement with stochastic errors. Then

$$MSE = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

is called the *mean square error* of sample $\{x_i\}$.

Verify:

$$MSE = \sigma^2 + (\bar{x} - \mu)^2$$

i.e., the MSE is the sum of

- the variance of the measurement (*dispersion*), and
- the squared deviation of the sample mean from μ (*bias*).

Absolute value

The **absolute value** (or *modulus*) $|x|$ of a number x is its distance from origin 0 on the number line:

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

$|5| = 5$ and $|-3| = -(-3) = 3$.

We have

$$|x| \cdot |y| = |x \cdot y|$$

Problem 2.6

Find simple equivalent formulas for the following expressions without using absolute values:

(a) $|x^2 + 1|$

(b) $|x| \cdot x^3$

Find simpler expressions by means of absolute values:

(c) $\begin{cases} x^2, & \text{for } x \geq 0, \\ -x^2, & \text{for } x < 0. \end{cases}$

(d) $\begin{cases} x^\alpha, & \text{for } x > 0, \\ -(-x)^\alpha & \text{for } x < 0, \end{cases}$ for some fixed $\alpha \in \mathbb{R}$.

Power

The n-th **power** of x is defined by

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

- x is the **basis**, and
- n is the **exponent** of x^n.

Expression x^n is read as "x raised to the n-th power", "x raised to the power of n", or "the n-th power of x".

Example: $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

For *negative* exponents we define:

$$x^{-n} = \frac{1}{x^n}$$

Root

A number y is called the

- **n-th root** $\sqrt[n]{x}$ of x, if $y^n = x$.

Computing the *n-th root* can be seen as the inverse operation of computing a power.

We just write $\sqrt{x}$ for the *square root* $\sqrt[2]{x}$.

Beware:
Symbol $\sqrt[n]{x}$ is used for the **positive** (real) root of x.

If we need the negative square root of 2 we have to write $\boxed{-\sqrt{2}}$.

Powers with Rational Exponents

Powers with *rational exponents* are defined by

$$x^{\frac{1}{m}} = \sqrt[m]{x} \qquad \text{for } m \in \mathbb{Z} \text{ and } x \geq 0;$$

and

$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad \text{for } m, n \in \mathbb{Z} \text{ and } x \geq 0.$$

Important:
For non-integer exponents the basis must be non-negative!

Powers x^α can also be generalized for $\alpha \in \mathbb{R}$.

Calculation Rule for Powers and Roots

$$x^{-n} = \frac{1}{x^n} \qquad x^0 = 1 \qquad (x \neq 0)$$

$$x^{n+m} = x^n \cdot x^m \qquad x^{\frac{1}{m}} = \sqrt[m]{x} \qquad (x \geq 0)$$

$$x^{n-m} = \frac{x^n}{x^m} \qquad x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad (x \geq 0)$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}} \qquad (x \geq 0)$$

$$(x^n)^m = x^{n \cdot m}$$

Important!

0^0 is *not* defined!

Computations with Powers and Roots

- $\sqrt[3]{5^6} = (5^6)^{\frac{1}{3}} = 5^{(6\cdot\frac{1}{3})} = 5^{\frac{6}{3}} = 5^2 = 25$
- $(\sqrt[3]{5})^6 = (5^{\frac{1}{3}})^6 = 5^{(\frac{1}{3}\cdot 6)} = 5^{\frac{6}{3}} = 5^2 = 25$
- $5^{4-3} = 5^1 = 5$
- $5^{4-3} = \frac{5^4}{5^3} = \frac{625}{125} = 5$
- $5^{2-2} = 5^0 = 1$
- $5^{2-2} = \frac{5^2}{5^2} = \frac{25}{25} = 1$

Computations with Powers and Roots

- $\dfrac{(x \cdot y)^4}{x^{-2} y^3} = x^4\, y^4\, x^{-(-2)}\, y^{-3} = x^6\, y$

- $\dfrac{(2\,x^2)^3\,(3y)^{-2}}{(4\,x^2\,y)^2\,(x^3\,y)} = \dfrac{2^3\,x^{2\cdot 3}\,3^{-2}\,y^{-2}}{4^2\,x^{2\cdot 2}\,y^2\,x^3\,y} = \dfrac{\frac{8}{9}\,x^6\,y^{-2}}{16\,x^7\,y^3}$

  $= \frac{1}{18}\,x^{6-7}\,y^{-2-3} = \frac{1}{18}\,x^{-1} y^{-5} = \dfrac{1}{18\,x\,y^5}$

- $\left(3\,x^{\frac{1}{3}}\,y^{-\frac{4}{3}}\right)^3 = 3^3\,x^{\frac{3}{3}}\,y^{-\frac{12}{3}} = 27\,x\,y^{-4} = \dfrac{27\,x}{y^4}$

Sources of Errors

Important!

- $-x^2$ is **not** equal to $(-x)^2$!
- $(x + y)^n$ is **not** equal to $x^n + y^n$!
- $x^n + y^n$ can**not** be simplified (in general)!

Problem 2.7

Simplify the following expressions:

(a) $$\frac{(xy)^{\frac{1}{3}}}{x^{\frac{1}{6}}y^{\frac{2}{3}}}$$

(b) $$\frac{1}{(\sqrt{x})^{-\frac{3}{2}}}$$

(c) $$\left(\frac{|x|^{\frac{1}{3}}}{|x|^{\frac{1}{6}}}\right)^6$$

Monomial

A **monomial** is a real number, variable, or product of variables raised to positive integer powers.

The **degree** of a monomial is the sum of all exponents of the variables (including a possible implicit exponent 1).

$6\,x^2$ is a monomial of degree 2.

$3\,x^3\,y$ and $x\,y^2\,z$ are monomials of degree 4.

$\sqrt{x}$ and $\frac{2}{3\,x\,y^2}$ are *not* monomials.

Polynomial

A **polynomial** is a sum of (one or more) monomials.

The **degree** of a polynomial is the maximum degree among its monomials.

$4\,x^2\,y^3 - 2\,x^3\,y + 4\,x + 7\,y$ is a polynomial of degree 5.

Polynomials in *one variable* (in sigma notation):

$$P(x) = \sum_{i=0}^{n} a_i\,x^i = a_n\,x^n + a_{n-1}\,x^{n-1} + \cdots + a_1\,x + a_0$$

where x is the variable and $a_i \in \mathbb{R}$ are constants.

Problem 2.8

Which of the following expressions are monomials or polynomials?
What is their degree?

Assume that x, y, and z are variables and all other symbols represent constants.

(a) x^2

(b) $x^{2/3}$

(c) $2x^2 + 3xy + 4y^2$

(d) $(2x^2 + 3xy + 4y^2)\,(x^2 - z^2)$

(e) $(x - a)(y - b)(z + 1)$

(f) $x\sqrt{y} - \sqrt{x}y$

(g) $ab + c$

(h) $x\sqrt{a} - \sqrt{b}y$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

is called the **binomial coefficient** (read: “n choose k”) and

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

denotes the **factorial** of n (read: “n-factorial”).

For convenience we set $\boxed{0! = 1}$.

Binomial Coefficient

Note:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

Computation:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot \quad 1}$$

$\binom{n}{k}$ is the number of ways to choose an (unordered) subset of k elements from a fixed set of n elements.

Binomial Theorem

- $(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = x^2 + 2xy + y^2$

- $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

  $= x^3 + 3x^2y + 3xy^2 + y^3$

Problem 2.9

Compute

(a) $(x+y)^4$

(b) $(x+y)^5$

by means of the binomial theorem.

Problem 2.10

Show by means of the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hint: Use $1 + 1 = 2$.

Multiplication

The product of two polynomials of degree n and m, resp., is a polynomial of degree $n + m$.

$$(2\,x^2 + 3\,x - 5) \cdot (x^3 - 2\,x + 1) =$$

$$\begin{aligned} &= 2\,x^2 \cdot x^3 + 2\,x^2 \cdot (-2\,x) + 2\,x^2 \cdot 1 \\ &\quad + 3\,x \cdot x^3 + 3\,x \cdot (-2\,x) + 3\,x \cdot 1 \\ &\quad + (-5) \cdot x^3 + (-5) \cdot (-2\,x) + (-5) \cdot 1 \\ &= 2\,x^5 + 3\,x^4 - 9\,x^3 - 4\,x^2 + 13\,x - 5 \end{aligned}$$

Problem 2.11

Simplify the following expressions:

(a) $(x+h)^2-(x-h)^2$

(b) $(a+b)c-(a+bc)$

(c) $(A-B)(A^2+AB+B^2)$

(d) $(x+y)^4-(x-y)^4$

Division

Polynomials can be **divided** similarly to the division of integers.

$$
\begin{array}{rl}
 & (x^3 + x^2 \quad + 0\,x - 2) : (x-1) = \mathbf{x^2 + 2\,x + 2} \\
\mathbf{x^2} \cdot (x-1) \longrightarrow & \underline{x^3 - x^2} \\
 & \qquad 2\,x^2 + 0\,x \\
\mathbf{2\,x} \cdot (x-1) \longrightarrow & \qquad \underline{2\,x^2 - 2\,x} \\
 & \qquad\qquad 2\,x - 2 \\
\mathbf{2} \cdot (x-1) \longrightarrow & \qquad\qquad \underline{2\,x - 2} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

We thus yield $x^3 + x^2 - 2 = (x-1) \cdot (x^2 + 2\,x + 2)$.

If the *divisor* is not a factor of the *dividend*, then we obtain a *remainder*.

Factorization

The process of expressing a polynomial as the product of polynomials of smaller degree (**factor**) is called **factorization**.

$$2x^2 + 4xy + 8xy^3 = 2x \cdot (x + 2y + 4y^3)$$

$$x^2 - y^2 = (x + y) \cdot (x - y)$$

$$x^2 - 1 = (x + 1) \cdot (x - 1)$$

$$x^2 + 2xy + y^2 = (x + y) \cdot (x + y) = (x + y)^2$$

$$x^3 + y^3 = (x + y) \cdot (x^2 - xy + y^2)$$

These products can be easily verified by multiplying their factors.

Factorization

The factorization

$$(x^2 - y^2) = (x + y) \cdot (x - y)$$

or equivalently

$$(x - y) = (\sqrt{x} + \sqrt{y}) \cdot (\sqrt{x} - \sqrt{y})$$

can be very useful.

Memorize it!

The “Ausmultiplizierreflex”

Important!

Factorizing a polynomial is often *very hard*
while multiplying its factors is fast and easy.
(The RSA public key encryption is based on this idea.)

Beware!

A factorized expression contains more information than their expanded counterpart.

In my experience many students have an
acquired “Ausmultiplizierreflex”:

Instantaneously (and **without thinking**) they *multiply* all factors
(which often turns a simple problem into a difficult one).

The “Ausmultiplizierreflex”

Suppress your “Ausmultiplizierreflex”!

Think first and multiply factors only when it seems to be useful!

Linear Term

A polynomial of *degree* 1 is called a **linear term**.

- $a + b\,x + y + a\,c$ is a linear term in x and y, if a, b and c are constants.
- $xy + x + y$ is not linear, as xy has degree 2.

Linear Factor

A factor (polynomial) of degree 1 is called a **linear factor**.

A polynomial in one variable x with root x_1 has linear factor $(x - x_1)$.

If a polynomial in x of degree n,

$$P(x) = \sum_{i=0}^{n} a_i \, x^i$$

has n real roots $x_1, x_2, \ldots, x_n$, then it can be written as the product of the n linear factors $(x - x_i)$:

$$P(x) = a_n \prod_{i=1}^{n} (x - x_i)$$

(This is a special case of the Fundamental Theorem of Algebra.)

Problem 2.12

(a) Give a polynomial in x of degree 4 with roots -1, 2, 3, and 4.

(b) What is the set of all such polynomials?

(c) Can such a polynomial have other roots?

Rational Term

A **rational term** is one of the form

$$\boxed{\frac{P(x)}{Q(x)}}$$

where $P(x)$ and $Q(x)$ are polynomials called **numerator** and **denominator**, resp.

Alternatively one can write $P(x)/Q(x)$.

The domain of a rational term is $\mathbb{R}$ without the roots of the denominator.

$\dfrac{x^2+x-4}{x^3+5}$ is a rational term with domain $\mathbb{R} \setminus \{-\sqrt[3]{5}\}$.

Beware! The expression $\dfrac{0}{0}$ is not defined.

Calculation Rule for Fractions and Rational Terms

Let $b, c, e \neq 0$.

$\dfrac{c \cdot a}{c \cdot b} = \dfrac{a}{b}$	*Reduce*
$\dfrac{a}{b} = \dfrac{c \cdot a}{c \cdot b}$	*Expand*
$\dfrac{a}{b} \cdot \dfrac{d}{c} = \dfrac{a \cdot d}{b \cdot c}$	*Multiplying*
$\dfrac{a}{b} : \dfrac{e}{c} = \dfrac{a}{b} \cdot \dfrac{c}{e}$	*Dividing*
$\dfrac{\frac{a}{b}}{\frac{e}{c}} = \dfrac{a \cdot c}{b \cdot e}$	*Compound fraction*

Calculation Rule for Fractions and Rational Terms

Let $b, c \neq 0$.

$$\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$$ *Addition with common denominator*

$$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$$ *Addition*

Very important! *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!

Calculation Rule for Fractions and Rational Terms

- $$\frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x-1$$

- $$\frac{4\,x^3+2\,x^2}{2\,x\,y} = \frac{2\,x^2(2\,x+1)}{2\,x\,y} = \frac{x(2\,x+1)}{y}$$

- $$\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{(x+1)^2+(x-1)^2}{(x-1)(x+1)}$$
  $$= \frac{x^2+2\,x+1+x^2-2\,x+1}{(x-1)(x+1)} = 2\,\frac{x^2+1}{x^2-1}$$

Calculation Rule for Fractions and Rational Terms

I want to stress at this point that there happen *a lot of mistakes* in calculations that involve rational terms.

The following examples of such fallacies are collected from students' exams.

Sources of Errors

Very Important! *Really!*

$\dfrac{a+c}{b+c}$ is **not** equal to $\dfrac{a}{b}$

$\dfrac{x}{a}+\dfrac{y}{b}$ is **not** equal to $\dfrac{x+y}{a+b}$

$\dfrac{a}{b+c}$ is **not** equal to $\dfrac{a}{b}+\dfrac{a}{c}$

$$\frac{x+2}{y+2} \neq \frac{x}{y} \qquad \frac{1}{2}+\frac{1}{3} \neq \frac{1}{5}.$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2}+\frac{1}{y^2}$$

Problem 2.13

Simplify the following expressions:

(a) $$\frac{1}{1+x} + \frac{1}{x-1}$$

(b) $$\frac{s}{st^2 - t^3} - \frac{1}{s^2 - st} - \frac{1}{t^2}$$

(c) $$\frac{\frac{1}{x} + \frac{1}{y}}{xy + xz + y(z-x)}$$

(d) $$\frac{\frac{x+y}{y}}{\frac{x-y}{x}} + \frac{\frac{x+y}{x}}{\frac{x-y}{y}}$$

Problem 2.14

Factorize and reduce the fractions:

(a) $\dfrac{1-x^2}{1-x}$

(b) $\dfrac{1+x^2}{1-x}$

(c) $\dfrac{x^3-x^4}{1-x}$

(d) $\dfrac{x^3-x^5}{1-x}$

(e) $\dfrac{x^2-x^6}{1-x}$

(f) $\dfrac{1-x^3}{1-x}$

Problem 2.15

Simplify the following expressions:

(a) $y(xy + x + 1) - \dfrac{x^2y^2 - 1}{x - \frac{1}{y}}$

(b) $\dfrac{\frac{x^2+y}{2x+1}}{\frac{2xy}{2x+y}}$

(c) $\dfrac{\frac{a}{x} - \frac{b}{x+1}}{\frac{a}{x+1} + \frac{b}{x}}$

(d) $\dfrac{2x^2y - 4xy^2}{x^2 - 4y^2} + \dfrac{x^2}{x + 2y}$

Problem 2.16

Simplify the following expressions:

(a) $$\frac{x^{\frac{1}{4}} - y^{\frac{1}{3}}}{x^{\frac{1}{8}} + y^{\frac{1}{6}}}$$

(b) $$\frac{\sqrt{x} - 4}{x^{\frac{1}{4}} - 2}$$

(c) $$\frac{\frac{2}{x^{-\frac{1}{7}}}}{x^{-\frac{7}{2}}}$$

Problem 2.17

Simplify the following expressions:

(a) $$\frac{(\sqrt{x}+y)^{\frac{1}{3}}}{x^{\frac{1}{6}}}$$

(b) $$\frac{1}{3\sqrt{x}-1}\cdot\frac{1}{1+\frac{1}{3\sqrt{x}}}\cdot\frac{1}{\sqrt{x}}$$

(c) $$\frac{(xy)^{\frac{1}{6}}-3}{(xy)^{\frac{1}{3}}-9}$$

(d) $$\frac{x-y}{\sqrt{x}-\sqrt{y}}$$

Exponential Function

- **Power function**

  $$(0, \infty) \to (0, \infty),\ x \mapsto x^{\alpha} \quad \text{for some } \textit{fixed exponent } \alpha \in \mathbb{R}$$

- **Exponential function**

  $$\mathbb{R} \to (0, \infty),\ x \mapsto a^{x} \quad \text{for some } \textit{fixed basis } a \in (0, \infty)$$

Exponent and Logarithm

A number y is called the **logarithm** to basis a, if $a^y = x$.
The logarithm is the *exponent of a number to basis* a.
We write

$$y = \log_a(x) \qquad \Leftrightarrow \qquad x = a^y$$

Important logarithms:

- **natural logarithm $\ln(x)$** with basis $e = 2.7182818\ldots$ (sometimes called *Euler's number*)
- **common logarithm $\lg(x)$** with basis 10 (sometimes called decadic or decimal logarithm)

Exponent and Logarithm

- $\log_{10}(100) = 2, \quad$ as $10^2 = 100$
- $\log_{10}\left(\frac{1}{1000}\right) = -3, \quad$ as $10^{-3} = \frac{1}{1000}$
- $\log_2(8) = 3, \quad$ as $2^3 = 8$
- $\log_{\sqrt{2}}(16) = 8, \quad$ as $\sqrt{2}^8 = 2^{8/2} = 2^4 = 16$

Calculations with Exponent and Logarithm

Conversion formula:

$$a^x = e^{x \ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$\log_2(123) = \frac{\ln(123)}{\ln(2)} \approx \frac{4.812184}{0.6931472} = 6.942515$$

$$3^7 = e^{7 \ln(3)}$$

Implicit Basis

Important:
Often one can see $\log(x)$ without an explicit basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- In *mathematics*: *natural* logarithm
  financial mathematics, programs like R, *Mathematica*, Maxima, ...

- In *applied sciences*: *common* logarithm
  economics, pocket calculator, Excel, ...

Calculation Rules for Exponent and Logarithm

$$a^{x+y} = a^x \cdot a^y \qquad \log_a(x \cdot y) = \log_a(x) + \log_a(y)$$

$$a^{x-y} = \frac{a^x}{a^y} \qquad \log_a(\tfrac{x}{y}) = \log_a(x) - \log_a(y)$$

$$(a^x)^y = a^{x \cdot y} \qquad \log_a(x^\beta) = \beta \cdot \log_a(x)$$

$$(a \cdot b)^x = a^x \cdot b^x$$

$$a^{\log_a(x)} = x \qquad \log_a(a^x) = x$$

$$a^0 = 1 \qquad \log_a(1) = 0$$

$\log_a(x)$ has (as real-valued function) domain $x > 0$!

Problem 2.18

Compute without a calculator:

(a) $\log_2(2)$

(b) $\log_2(4)$

(c) $\log_2(16)$

(d) $\log_2(0)$

(e) $\log_2(1)$

(f) $\log_2\left(\frac{1}{4}\right)$

(g) $\log_2\left(\sqrt{2}\right)$

(h) $\log_2\left(\frac{1}{\sqrt{2}}\right)$

(i) $\log_2(-4)$

Problem 2.19

Compute without a calculator:

(a) $\log_{10}(300)$

(b) $\log_{10}(3^{10})$

Use $\log_{10}(3) = 0.47712$.

Problem 2.20

Compute (simplify) without a calculator:

(a) $0.01^{-\log_{10}(100)}$

(b) $\log_{\sqrt{5}}\left(\frac{1}{25}\right)$

(c) $10^{3\log_{10}(3)}$

(d) $\frac{\log_{10}(200)}{\log_{\frac{1}{\sqrt{7}}}(49)}$

(e) $\log_{8}\left(\frac{1}{512}\right)$

(f) $\log_{\frac{1}{3}}(81)$

Problem 2.21

Represent the following expression in the form $y = A\,e^{cx}$
(i.e., determine A and c):

(a) $y = 10^{x-1}$

(b) $y = 4^{x+2}$

(c) $y = 3^x 5^{2x}$

(d) $y = 1.08^{x-\frac{x}{2}}$

(e) $y = 0.9 \cdot 1.1^{\frac{x}{10}}$

(f) $y = \sqrt{q}\,2^{x/2}$

Summary

- sigma notation
- absolute value
- powers and roots
- monomials and polynomials
- binomial theorem
- multiplication, factorization, and linear factors
- trap door “Ausmultiplizierreflex”
- fractions, rational terms and many fallacies
- exponent and logarithm