Chapter 2

Terms

Term

A mathematical expression like

$$B = R \cdot \frac{q^n - 1}{q^n(q-1)} \quad \text{or} \quad (x+1)(x-1) = x^2 - 1$$

contains symbols which denote mathematical objects.
These symbols and compositions of symbols are called **terms**.

Terms can be

- **numbers**,
- **constants** (symbols, which represent *fixed* values),
- **variables** (which are placeholders for *arbitrary* values), and
- **compositions** *of terms*.

Domain

We have to take care that a term may not be defined for some values of its variables.

- $\frac{1}{x-1}$ is only defined for $x \in \mathbb{R} \setminus \{1\}$.
- $\sqrt{x+1}$ is only defined for $x \geq -1$.

The set of values for which a term is defined is called the **domain** of the *term*.

Sigma Notation

Sums with many terms that can be generated by some rule can be represented in a compact form called *summation* or **sigma notation**.

$$\sum_{n=1}^{6} a_n = a_1 + a_2 + \cdots + a_6$$

a_n ... formula for the terms
n ... index of summation
1 ... first value of index n
6 ... last value of n

The expression is read as the "sum of a_n as n goes from 1 to 6."

First and last value can (and often are) given by *symbols*.

Sigma Notation

The sum of the first 10 integers greater than 2 can be written in both notations as

$$\sum_{i=1}^{10}(2+i) = (2+1) + (2+2) + (2+3) + \cdots + (2+10)$$

The sigma notation can be seen as a convenient shortcut of the long expression on the r.h.s.

It also avoids ambiguity caused by the *ellipsis* "..." that might look like an IQ test rather than an exact mathematical expression.

Sigma Notation – Rules

Keep in mind that all the *usual rules* for multiplication and addition (associativity, commutativity, distributivity) apply:

- $\sum_{i=1}^{n} a_i = \sum_{k=1}^{n} a_k$
- $\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i = \sum_{i=1}^{n} (a_i + b_i)$
- $\sum_{i=1}^{n} c\, a_i = c \sum_{i=1}^{n} a_i$

Sigma Notation – Rules

Simplify

$$\sum_{i=2}^{n}(a_i+b_i)^2 - \sum_{j=2}^{n} a_j^2 - \sum_{k=2}^{n} b_k^2$$

Solution:

$$= \sum_{i=2}^{n}(a_i+b_i)^2 - \sum_{i=2}^{n} a_i^2 - \sum_{i=2}^{n} b_i^2$$
$$= \sum_{i=2}^{n}\left((a_i+b_i)^2 - a_i^2 - b_i^2\right)$$
$$= \sum_{i=2}^{n} 2\, a_i b_i$$
$$= 2\sum_{i=2}^{n} a_i b_i$$

Problem 2.1

Which of the following expressions is equal to the summation

$$\sum_{i=2}^{10} 5(i+3)\ ?$$

(a) $5(2+3+4+\ldots+9+10+3)$

(b) $5(2+3+3+3+4+3+5+3+6+3+\ldots+10+3)$

(c) $5(2+3+4+\ldots+9+10)+5\cdot 3$

(d) $5(2+3+4+\ldots+9+10)+9\cdot 5\cdot 3$

Problem 2.2

Compute and simplify:

(a) $\sum_{i=0}^{5} a^i b^{5-i}$

(b) $\sum_{i=1}^{5} (a_i - a_{i+1})$

(c) $\sum_{i=1}^{n} (a_i - a_{i+1})$

Remark: The sum in (c) is a so called *telescoping sum*.

Problem 2.3

Simplify the following summations:

(a) $\sum_{i=1}^{n} a_i^2 + \sum_{j=1}^{n} b_j^2 - \sum_{k=1}^{n} (a_k - b_k)^2$

(b) $\sum_{i=1}^{n} (a_i b_{n-i+1} - a_{n-i+1} b_i)$

(c) $\sum_{i=1}^{n} (x_i + y_i)^2 + \sum_{j=1}^{n} (x_j - y_j)^2$

(d) $\sum_{j=0}^{n-1} x_j - \sum_{i=1}^{n} x_i$

Problem 2.4

Arithmetic mean ("average")

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

and *variance*

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2$$

are important location and dispersion parameters in statistics.

Variance σ^2 can be computed by means of formula

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \left(\frac{1}{n} \sum_{i=1}^{n} x_i \right)^2$$

which requires to read data (x_i) only once.

Problem 2.4 / 2

Verify

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2$$

for

(a) $n = 2$,

(b) $n = 3$,

(c) $n \geq 2$ arbitrary.

Hint: Show that

$$\sum_{i=1}^{n} (x_i - \bar{x})^2 - \left(\sum_{i=1}^{n} x_i^2 - n\bar{x}^2 \right) = 0.$$

Problem 2.5

Let μ be the "true" value of a metric variate and $\{x_i\}$ the results of some measurement with stochastic errors. Then

$$MSE = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

is called the *mean square error* of sample $\{x_i\}$.

Verify:

$$MSE = \sigma^2 + (\bar{x} - \mu)^2$$

i.e., the MSE is the sum of

- the variance of the measurement (*dispersion*), and
- the squared deviation of the sample mean from μ (*bias*).

Absolute value

The **absolute value** (or *modulus*) $|x|$ of a number x is its distance from origin 0 on the number line:

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

$|5| = 5$ and $|-3| = -(-3) = 3$.

We have

$$|x| \cdot |y| = |x \cdot y|$$

Problem 2.6

Find simple equivalent formulas for the following expressions without using absolute values:

(a) $|x^2 + 1|$

(b) $|x| \cdot x^3$

Find simpler expressions by means of absolute values:

(c) $\begin{cases} x^2, & \text{for } x \geq 0, \\ -x^2, & \text{for } x < 0. \end{cases}$

(d) $\begin{cases} x^\alpha, & \text{for } x > 0, \\ -(-x)^\alpha & \text{for } x < 0, \end{cases}$ for some fixed $\alpha \in \mathbb{R}$.

Power

The n-th **power** of x is defined by

$$x^n = \underbrace{x \cdot x \cdot \ldots \cdot x}_{n \text{ factors}}$$

- x is the **basis**, and
- n is the **exponent** of x^n.

Expression x^n is read as "x raised to the n-th power", "x raised to the power of n", or "the n-th power of x".

Example: $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

For *negative* exponents we define:

$$x^{-n} = \frac{1}{x^n}$$

Root

A number y is called the

- **n-th root** $\sqrt[n]{x}$ of x, if $y^n = x$.

Computing the *n-th root* can be seen as the inverse operation of computing a power.

We just write $\sqrt{x}$ for the *square root* $\sqrt[2]{x}$.

Beware:
Symbol $\sqrt[n]{x}$ is used for the **positive** (real) root of x.

If we need the negative square root of 2 we have to write $\boxed{-\sqrt{2}}$.

Powers with Rational Exponents

Powers with *rational exponents* are defined by

$$\boxed{x^{\frac{1}{m}} = \sqrt[m]{x} \qquad \text{for } m \in \mathbb{Z} \text{ and } x \geq 0;}$$

and

$$\boxed{x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad \text{for } m, n \in \mathbb{Z} \text{ and } x \geq 0.}$$

Important:
For non-integer exponents the basis must be non-negative!

Powers x^α can also be generalized for $\alpha \in \mathbb{R}$.

Calculation Rule for Powers and Roots

$x^{-n} = \frac{1}{x^n}$	$x^0 = 1$	$(x \neq 0)$
$x^{n+m} = x^n \cdot x^m$	$x^{\frac{1}{m}} = \sqrt[m]{x}$	$(x \geq 0)$
$x^{n-m} = \frac{x^n}{x^m}$	$x^{\frac{n}{m}} = \sqrt[m]{x^n}$	$(x \geq 0)$
$(x \cdot y)^n = x^n \cdot y^n$	$x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}}$	$(x \geq 0)$
$(x^n)^m = x^{n \cdot m}$		

Important!

0^0 is *not* defined!

Computations with Powers and Roots

- $\sqrt[3]{5^6} = (5^6)^{\frac{1}{3}} = 5^{(6 \cdot \frac{1}{3})} = 5^{\frac{6}{3}} = 5^2 = 25$
- $(\sqrt[3]{5})^6 = (5^{\frac{1}{3}})^6 = 5^{(\frac{1}{3} \cdot 6)} = 5^{\frac{6}{3}} = 5^2 = 25$
- $5^{4-3} = 5^1 = 5$
- $5^{4-3} = \frac{5^4}{5^3} = \frac{625}{125} = 5$
- $5^{2-2} = 5^0 = 1$
- $5^{2-2} = \frac{5^2}{5^2} = \frac{25}{25} = 1$

Computations with Powers and Roots

- $\frac{(x \cdot y)^4}{x^{-2} y^3} = x^4 y^4 x^{-(-2)} y^{-3} = x^6 y$
- $\frac{(2x^2)^3 (3y)^{-2}}{(4x^2 y)^2 (x^3 y)} = \frac{2^3 x^{2 \cdot 3} 3^{-2} y^{-2}}{4^2 x^{2 \cdot 2} y^2 x^3 y} = \frac{\frac{8}{9} x^6 y^{-2}}{16 x^7 y^3}$

  $= \frac{1}{18} x^{6-7} y^{-2-3} = \frac{1}{18} x^{-1} y^{-5} = \frac{1}{18 x y^5}$
- $\left(3 x^{\frac{1}{3}} y^{-\frac{4}{3}}\right)^3 = 3^3 x^{\frac{3}{3}} y^{-\frac{12}{3}} = 27 x y^{-4} = \frac{27x}{y^4}$

Sources of Errors

Important!

- $-x^2$ is **not** equal to $(-x)^2$!
- $(x+y)^n$ is **not** equal to $x^n + y^n$!
- $x^n + y^n$ can**not** be simplified (in general)!

Problem 2.7

Simplify the following expressions:

(a) $\frac{(xy)^{\frac{1}{3}}}{x^{\frac{1}{6}} y^{\frac{2}{3}}}$

(b) $\frac{1}{(\sqrt{x})^{-\frac{3}{2}}}$

(c) $\left(\frac{|x|^{\frac{1}{3}}}{|x|^{\frac{1}{6}}}\right)^6$

Monomial

A **monomial** is a real number, variable, or product of variables raised to positive integer powers.

The **degree** of a monomial is the sum of all exponents of the variables (including a possible implicit exponent 1).

$6x^2$ is a monomial of degree 2.

$3x^3 y$ and $x y^2 z$ are monomials of degree 4.

$\sqrt{x}$ and $\frac{2}{3 x y^2}$ are *not* monomials.

Polynomial

A **polynomial** is a sum of (one or more) monomials.

The **degree** of a polynomial is the maximum degree among its monomials.

$4\,x^2\,y^3 - 2\,x^3\,y + 4\,x + 7\,y$ is a polynomial of degree 5.

Polynomials in *one variable* (in sigma notation):

$$P(x) = \sum_{i=0}^{n} a_i\, x^i = a_n\, x^n + a_{n-1}\, x^{n-1} + \cdots + a_1\, x + a_0$$

where x is the variable and $a_i \in \mathbb{R}$ are constants.

Problem 2.8

Which of the following expressions are monomials or polynomials? What is their degree?

Assume that x, y, and z are variables and all other symbols represent constants.

(a) x^2

(b) $x^{2/3}$

(c) $2x^2 + 3xy + 4y^2$

(d) $(2x^2 + 3xy + 4y^2)\,(x^2 - z^2)$

(e) $(x-a)(y-b)(z+1)$

(f) $x\sqrt{y} - \sqrt{x}y$

(g) $ab + c$

(h) $x\sqrt{a} - \sqrt{b}y$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k}\, y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

is called the **binomial coefficient** (read: "n choose k") and

$$n! = 1 \cdot 2 \cdot \ldots \cdot n$$

denotes the **factorial** of n (read: "n-factorial").

For convenience we set $0! = 1$.

Binomial Coefficient

Note:

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!} = \binom{n}{n-k}$$

Computation:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \ldots \cdot (n-k+1)}{k \cdot (k-1) \cdot \ldots \cdot \quad 1}$$

$\binom{n}{k}$ is the number of ways to choose an (unordered) subset of k elements from a fixed set of n elements.

Binomial Theorem

- $(x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = x^2 + 2xy + y^2$
- $(x+y)^3 = \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3$

  $= x^3 + 3x^2y + 3xy^2 + y^3$

Problem 2.9

Compute

(a) $(x+y)^4$

(b) $(x+y)^5$

by means of the binomial theorem.

Problem 2.10

Show by means of the binomial theorem that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Hint: Use $1 + 1 = 2$.

Multiplication

The product of two polynomials of degree n and m, resp., is a polynomial of degree $n + m$.

$$\begin{aligned}(2\,x^2 + 3\,x - 5) \cdot (x^3 - 2\,x + 1) = \\ &= 2\,x^2 \cdot x^3 + 2\,x^2 \cdot (-2\,x) + 2\,x^2 \cdot 1 \\ &\quad + 3\,x \cdot x^3 + 3\,x \cdot (-2\,x) + 3\,x \cdot 1 \\ &\quad + (-5) \cdot x^3 + (-5) \cdot (-2\,x) + (-5) \cdot 1 \\ &= 2\,x^5 + 3\,x^4 - 9\,x^3 - 4\,x^2 + 13\,x - 5\end{aligned}$$

Problem 2.11

Simplify the following expressions:

(a) $(x+h)^2 - (x-h)^2$

(b) $(a+b)c - (a+bc)$

(c) $(A-B)(A^2+AB+B^2)$

(d) $(x+y)^4 - (x-y)^4$

Division

Polynomials can be **divided** similarly to the division of integers.

$$
\begin{array}{rl}
 & (x^3 + x^2 \quad + 0\,x - 2) : (x-1) = \mathbf{x^2 + 2\,x + 2} \\
\mathbf{x^2} \cdot (x-1) \longrightarrow & \underline{x^3 - x^2} \\
 & \quad 2\,x^2 + 0\,x \\
\mathbf{2\,x} \cdot (x-1) \longrightarrow & \quad \underline{2\,x^2 - 2\,x} \\
 & \qquad 2\,x - 2 \\
\mathbf{2} \cdot (x-1) \longrightarrow & \qquad \underline{2\,x - 2} \\
 & \qquad\qquad 0
\end{array}
$$

We thus yield $x^3 + x^2 - 2 = (x-1)\cdot(x^2 + 2\,x + 2)$.

If the *divisor* is not a factor of the *dividend*, then we obtain a *remainder*.

Factorization

The process of expressing a polynomial as the product of polynomials of smaller degree (**factor**) is called **factorization**.

$$2\,x^2 + 4\,x\,y + 8\,x\,y^3 = 2\,x \cdot (x + 2\,y + 4\,y^3)$$

$$x^2 - y^2 = (x+y)\cdot(x-y)$$

$$x^2 - 1 = (x+1)\cdot(x-1)$$

$$x^2 + 2\,x\,y + y^2 = (x+y)\cdot(x+y) = (x+y)^2$$

$$x^3 + y^3 = (x+y)\cdot(x^2 - x\,y + y^2)$$

These products can be easily verified by multiplying their factors.

Factorization

The factorization

$$\boxed{(x^2 - y^2) = (x+y)\cdot(x-y)}$$

or equivalently

$$\boxed{(x-y) = (\sqrt{x} + \sqrt{y})\cdot(\sqrt{x} - \sqrt{y})}$$

can be very useful.

Memorize it!

The “Ausmultiplizierreflex”

Important!

Factorizing a polynomial is often *very hard*
while multiplying its factors is fast and easy.
(The RSA public key encryption is based on this idea.)

Beware!

A factorized expression contains more information than their expanded counterpart.

In my experience many students have an
acquired “Ausmultiplizierreflex”:
Instantaneously (and **without thinking**) they *multiply* all factors
(which often turns a simple problem into a difficult one).

The “Ausmultiplizierreflex”

Suppress your “Ausmultiplizierreflex”!

Think first and multiply factors only when it seems to be useful!

Linear Term

A polynomial of *degree* 1 is called a **linear term**.

- $a + b\,x + y + a\,c$ is a linear term in x and y, if a, b and c are constants.
- $xy + x + y$ is not linear, as xy has degree 2.

Linear Factor

A factor (polynomial) of degree 1 is called a **linear factor**.

A polynomial in one variable x with root x_1 has linear factor $(x - x_1)$.

If a polynomial in x of degree n,

$$P(x) = \sum_{i=0}^{n} a_i\,x^i$$

has n real roots $x_1, x_2, \ldots, x_n$, then it can be written as the product of the n linear factors $(x - x_i)$:

$$\boxed{P(x) = a_n \prod_{i=1}^{n} (x - x_i)}$$

(This is a special case of the Fundamental Theorem of Algebra.)

Problem 2.12

(a) Give a polynomial in x of degree 4 with roots $-1, 2, 3$, and 4.

(b) What is the set of all such polynomials?

(c) Can such a polynomial have other roots?

Rational Term

A **rational term** is one of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials called **numerator** and **denominator**, resp.

Alternatively one can write $P(x)/Q(x)$.

The domain of a rational term is $\mathbb{R}$ without the roots of the denominator.

$\frac{x^2+x-4}{x^3+5}$ is a rational term with domain $\mathbb{R} \setminus \{-\sqrt[3]{5}\}$.

Beware! The expression $\frac{0}{0}$ is not defined.

Calculation Rule for Fractions and Rational Terms

Let $b, c, e \neq 0$.

$\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$	*Reduce*
$\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$	*Expand*
$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$	*Multiplying*
$\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e}$	*Dividing*
$\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e}$	*Compound fraction*

Calculation Rule for Fractions and Rational Terms

Let $b, c \neq 0$.

$\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$	*Addition with common denominator*
$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$	*Addition*

Very important! *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!

Calculation Rule for Fractions and Rational Terms

- $\frac{x^2-1}{x+1} = \frac{(x+1)(x-1)}{x+1} = x-1$
- $\frac{4x^3+2x^2}{2xy} = \frac{2x^2(2x+1)}{2xy} = \frac{x(2x+1)}{y}$
- $\frac{x+1}{x-1} + \frac{x-1}{x+1} = \frac{(x+1)^2+(x-1)^2}{(x-1)(x+1)}$
  $= \frac{x^2+2x+1+x^2-2x+1}{(x-1)(x+1)} = 2\frac{x^2+1}{x^2-1}$

Calculation Rule for Fractions and Rational Terms

I want to stress at this point that there happen *a lot of mistakes* in calculations that involve rational terms.

The following examples of such fallacies are collected from students' exams.

Sources of Errors

Very Important! *Really!*

$\frac{a+c}{b+c}$ is **not** equal to $\frac{a}{b}$

$\frac{x}{a} + \frac{y}{b}$ is **not** equal to $\frac{x+y}{a+b}$

$\frac{a}{b+c}$ is **not** equal to $\frac{a}{b} + \frac{a}{c}$

$\frac{x+2}{y+2} \neq \frac{x}{y}$ $\qquad$ $\frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}.$

$\frac{1}{x^2+y^2} \neq \frac{1}{x^2} + \frac{1}{y^2}$

Problem 2.13

Simplify the following expressions:

(a) $\frac{1}{1+x} + \frac{1}{x-1}$

(b) $\frac{s}{st^2-t^3} - \frac{1}{s^2-st} - \frac{1}{t^2}$

(c) $\frac{\frac{1}{x}+\frac{1}{y}}{xy+xz+y(z-x)}$

(d) $\frac{\frac{x+y}{y}}{\frac{x-y}{x}} + \frac{\frac{x+y}{x}}{\frac{x-y}{y}}$

Problem 2.14

Factorize and reduce the fractions:

(a) $\dfrac{1-x^2}{1-x}$

(b) $\dfrac{1+x^2}{1-x}$

(c) $\dfrac{x^3-x^4}{1-x}$

(d) $\dfrac{x^3-x^5}{1-x}$

(e) $\dfrac{x^2-x^6}{1-x}$

(f) $\dfrac{1-x^3}{1-x}$

Problem 2.15

Simplify the following expressions:

(a) $y(xy+x+1) - \dfrac{x^2y^2-1}{x-\frac{1}{y}}$

(b) $\dfrac{\frac{x^2+y}{2x+1}}{\frac{2xy}{2x+y}}$

(c) $\dfrac{\frac{a}{x}-\frac{b}{x+1}}{\frac{a}{x+1}+\frac{b}{x}}$

(d) $\dfrac{2x^2y-4xy^2}{x^2-4y^2} + \dfrac{x^2}{x+2y}$

Problem 2.16

Simplify the following expressions:

(a) $\dfrac{x^{\frac{1}{4}}-y^{\frac{1}{3}}}{x^{\frac{1}{8}}+y^{\frac{1}{6}}}$

(b) $\dfrac{\sqrt{x}-4}{x^{\frac{1}{4}}-2}$

(c) $\dfrac{\frac{2}{x^{-\frac{1}{7}}}}{x^{-\frac{7}{2}}}$

Problem 2.17

Simplify the following expressions:

(a) $\dfrac{(\sqrt{x}+y)^{\frac{1}{3}}}{x^{\frac{1}{6}}}$

(b) $\dfrac{1}{3\sqrt{x}-1} \cdot \dfrac{1}{1+\frac{1}{3\sqrt{x}}} \cdot \dfrac{1}{\sqrt{x}}$

(c) $\dfrac{(xy)^{\frac{1}{6}}-3}{(xy)^{\frac{1}{3}}-9}$

(d) $\dfrac{x-y}{\sqrt{x}-\sqrt{y}}$

Exponential Function

- **Power function**

  $(0,\infty) \to (0,\infty),\ x \mapsto x^\alpha$ for some *fixed exponent* $\alpha \in \mathbb{R}$

- **Exponential function**

  $\mathbb{R} \to (0,\infty),\ x \mapsto a^x$ for some *fixed basis* $a \in (0,\infty)$

Exponent and Logarithm

A number y is called the **logarithm** to basis a, if $a^y = x$.
The logarithm is the *exponent of a number to basis a*.
We write

$$y = \log_a(x) \quad \Leftrightarrow \quad x = a^y$$

Important logarithms:

- **natural logarithm $\ln(x)$** with basis $e = 2.7182818\ldots$ (sometimes called *Euler's number*)
- **common logarithm $\lg(x)$** with basis 10 (sometimes called decadic or decimal logarithm)

Exponent and Logarithm

- $\log_{10}(100) = 2,$ as $10^2 = 100$
- $\log_{10}\left(\frac{1}{1000}\right) = -3,$ as $10^{-3} = \frac{1}{1000}$
- $\log_2(8) = 3,$ as $2^3 = 8$
- $\log_{\sqrt{2}}(16) = 8,$ as $\sqrt{2}^8 = 2^{8/2} = 2^4 = 16$

Calculations with Exponent and Logarithm

Conversion formula:

$$a^x = e^{x\ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$\log_2(123) = \frac{\ln(123)}{\ln(2)} \approx \frac{4.812184}{0.6931472} = 6.942515$$

$$3^7 = e^{7\ln(3)}$$

Implicit Basis

Important:
Often one can see $\log(x)$ without an explicit basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- In *mathematics*: *natural* logarithm
  financial mathematics, programs like R, *Mathematica*, Maxima, ...
- In *applied sciences*: *common* logarithm
  economics, pocket calculator, Excel, ...

Calculation Rules for Exponent and Logarithm

$a^{x+y} = a^x \cdot a^y$	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
$a^{x-y} = \frac{a^x}{a^y}$	$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
$(a^x)^y = a^{x \cdot y}$	$\log_a(x^\beta) = \beta \cdot \log_a(x)$
$(a \cdot b)^x = a^x \cdot b^x$	
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$
$a^0 = 1$	$\log_a(1) = 0$

$\log_a(x)$ has (as real-valued function) domain $x > 0$!

Problem 2.18

Compute without a calculator:

(a) $\log_2(2)$

(b) $\log_2(4)$

(c) $\log_2(16)$

(d) $\log_2(0)$

(e) $\log_2(1)$

(f) $\log_2\left(\frac{1}{4}\right)$

(g) $\log_2\left(\sqrt{2}\right)$

(h) $\log_2\left(\frac{1}{\sqrt{2}}\right)$

(i) $\log_2(-4)$

Problem 2.19

Compute without a calculator:

(a) $\log_{10}(300)$

(b) $\log_{10}(3^{10})$

Use $\log_{10}(3) = 0.47712$.

Problem 2.20

Compute (simplify) without a calculator:

(a) $0.01^{-\log_{10}(100)}$

(b) $\log_{\sqrt{5}}\left(\frac{1}{25}\right)$

(c) $10^{3\log_{10}(3)}$

(d) $\frac{\log_{10}(200)}{\log_{\frac{1}{\sqrt{7}}}(49)}$

(e) $\log_8\left(\frac{1}{512}\right)$

(f) $\log_{\frac{1}{3}}(81)$

Problem 2.21

Represent the following expression in the form $y = A\,e^{cx}$
(i.e., determine A and c):

(a) $y = 10^{x-1}$

(b) $y = 4^{x+2}$

(c) $y = 3^x 5^{2x}$

(d) $y = 1.08^{x-\frac{x}{2}}$

(e) $y = 0.9 \cdot 1.1^{\frac{x}{10}}$

(f) $y = \sqrt{q}\,2^{x/2}$

Summary

- sigma notation
- absolute value
- powers and roots
- monomials and polynomials
- binomial theorem
- multiplication, factorization, and linear factors
- trap door “Ausmultiplizierreflex”
- fractions, rational terms and many fallacies
- exponent and logarithm