

Chapter 2

Terms

Term

A mathematical expression like

$$B = R \cdot \frac{q^n - 1}{q^n(q - 1)} \quad \text{or} \quad (x + 1)(x - 1) = x^2 - 1$$

contains symbols which denote mathematical objects.

These symbols and compositions of symbols are called **terms**.

Terms can be

- ▶ **numbers**,
- ▶ **constants** (symbols, which represent *fixed* values),
- ▶ **variables** (which are placeholders for *arbitrary* values), and
- ▶ **compositions of terms**.

Domain

We have to take care that a term may not be defined for some values of its variables.

- ▶ $\frac{1}{x-1}$ is only defined for $x \in \mathbb{R} \setminus \{1\}$.
- ▶ $\sqrt{x+1}$ is only defined for $x \geq -1$.

The set of values for which a term is defined is called the **domain** of the *term*.

Sigma Notation

Sums with many terms that can be generated by some rule can be represented in a compact form called *summation* or **sigma notation**.

$$\sum_{n=1}^6 a_n = a_1 + a_2 + \cdots + a_6$$

a_n ... formula for the terms

n ... index of summation

1 ... first value of index n

6 ... last value of n

The expression is read as the “sum of a_n as n goes from 1 to 6.”

First and last value can (and often are) given by *symbols*.

Sigma Notation

The sum of the first 10 integers greater than 2 can be written in both notations as

$$\sum_{i=1}^{10} (2 + i) = (2 + 1) + (2 + 2) + (2 + 3) + \cdots + (2 + 10)$$

The sigma notation can be seen as a convenient shortcut of the long expression on the r.h.s.

It also avoids ambiguity caused by the *ellipsis* “...” that might look like an IQ test rather than an exact mathematical expression.

Sigma Notation – Rules

Keep in mind that all the *usual rules* for multiplication and addition (associativity, commutativity, distributivity) apply:

$$\blacktriangleright \sum_{i=1}^n a_i = \sum_{k=1}^n a_k$$

$$\blacktriangleright \sum_{i=1}^n a_i + \sum_{i=1}^n b_i = \sum_{i=1}^n (a_i + b_i)$$

$$\blacktriangleright \sum_{i=1}^n c a_i = c \sum_{i=1}^n a_i$$

Sigma Notation – Rules

Simplify

$$\sum_{i=2}^n (a_i + b_i)^2 - \sum_{j=2}^n a_j^2 - \sum_{k=2}^n b_k^2$$

Solution:

$$\begin{aligned} &= \sum_{i=2}^n (a_i + b_i)^2 - \sum_{i=2}^n a_i^2 - \sum_{i=2}^n b_i^2 \\ &= \sum_{i=2}^n ((a_i + b_i)^2 - a_i^2 - b_i^2) \\ &= \sum_{i=2}^n 2a_i b_i \\ &= 2 \sum_{i=2}^n a_i b_i \end{aligned}$$

Problem 2.1

Which of the following expressions is equal to the summation

$$\sum_{i=2}^{10} 5(i+3) ?$$

- (a) $5(2 + 3 + 4 + \dots + 9 + 10 + 3)$
- (b) $5(2 + 3 + 3 + 3 + 4 + 3 + 5 + 3 + 6 + 3 + \dots + 10 + 3)$
- (c) $5(2 + 3 + 4 + \dots + 9 + 10) + 5 \cdot 3$
- (d) $5(2 + 3 + 4 + \dots + 9 + 10) + 9 \cdot 5 \cdot 3$

Problem 2.2

Compute and simplify:

- (a) $\sum_{i=0}^5 a^i b^{5-i}$
- (b) $\sum_{i=1}^5 (a_i - a_{i+1})$
- (c) $\sum_{i=1}^n (a_i - a_{i+1})$

Remark: The sum in (c) is a so called *telescoping sum*.

Problem 2.3

Simplify the following summations:

$$(a) \sum_{i=1}^n a_i^2 + \sum_{j=1}^n b_j^2 - \sum_{k=1}^n (a_k - b_k)^2$$

$$(b) \sum_{i=1}^n (a_i b_{n-i+1} - a_{n-i+1} b_i)$$

$$(c) \sum_{i=1}^n (x_i + y_i)^2 + \sum_{j=1}^n (x_j - y_j)^2$$

$$(d) \sum_{j=0}^{n-1} x_j - \sum_{i=1}^n x_i$$

Problem 2.4

Arithmetic mean (“average”)

$$\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$$

and *variance*

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2$$

are important location and dispersion parameters in statistics.

Variance σ^2 can be computed by means of formula

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \left(\frac{1}{n} \sum_{i=1}^n x_i \right)^2$$

which requires to read data (x_i) only once.

Problem 2.4 / 2

Verify

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i - \bar{x})^2 = \frac{1}{n} \sum_{i=1}^n x_i^2 - \bar{x}^2$$

for

- (a) $n = 2$,
- (b) $n = 3$,
- (c) $n \geq 2$ arbitrary.

Hint: Show that

$$\sum_{i=1}^n (x_i - \bar{x})^2 - \left(\sum_{i=1}^n x_i^2 - n\bar{x}^2 \right) = 0.$$

Problem 2.5

Let μ be the “true” value of a metric variate and $\{x_i\}$ the results of some measurement with stochastic errors. Then

$$MSE = \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2$$

is called the *mean square error* of sample $\{x_i\}$.

Verify:

$$MSE = \sigma^2 + (\bar{x} - \mu)^2$$

i.e., the MSE is the sum of

- ▶ the variance of the measurement (*dispersion*), and
- ▶ the squared deviation of the sample mean from μ (*bias*).

Absolute value

The **absolute value** (or *modulus*) $|x|$ of a number x is its distance from origin 0 on the number line:

$$|x| = \begin{cases} x, & \text{for } x \geq 0, \\ -x, & \text{for } x < 0. \end{cases}$$

$$|5| = 5 \text{ and } |-3| = -(-3) = 3.$$

We have

$$|x| \cdot |y| = |x \cdot y|$$

Problem 2.6

Find simple equivalent formulas for the following expressions without using absolute values:

(a) $|x^2 + 1|$

(b) $|x| \cdot x^3$

Find simpler expressions by means of absolute values:

(c) $\begin{cases} x^2, & \text{for } x \geq 0, \\ -x^2, & \text{for } x < 0. \end{cases}$

(d) $\begin{cases} x^\alpha, & \text{for } x > 0, \\ -(-x)^\alpha & \text{for } x < 0, \end{cases}$ for some fixed $\alpha \in \mathbb{R}$.

Power

The n -th **power** of x is defined by

$$x^n = \underbrace{x \cdot x \cdot \dots \cdot x}_{n \text{ factors}}$$

- ▶ x is the **basis**, and
- ▶ n is the **exponent** of x^n .

Expression x^n is read as “ x raised to the n -th power”, “ x raised to the power of n ”, or “the n -th power of x ”.

Example: $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$

For *negative* exponents we define:

$$x^{-n} = \frac{1}{x^n}$$

Root

A number y is called the

- ▶ **n -th root** $\sqrt[n]{x}$ of x , if $y^n = x$.

Computing the n -th root can be seen as the inverse operation of computing a power.

We just write \sqrt{x} for the *square root* $\sqrt[2]{x}$.

Beware:

Symbol $\sqrt[n]{x}$ is used for the **positive** (real) root of x .

If we need the negative square root of 2 we have to write $-\sqrt{2}$.

Powers with Rational Exponents

Powers with *rational exponents* are defined by

$$x^{\frac{1}{m}} = \sqrt[m]{x} \quad \text{for } m \in \mathbb{Z} \text{ and } x \geq 0;$$

and

$$x^{\frac{n}{m}} = \sqrt[m]{x^n} \quad \text{for } m, n \in \mathbb{Z} \text{ and } x \geq 0.$$

Important:

For non-integer exponents the basis must be non-negative!

Powers x^α can also be generalized for $\alpha \in \mathbb{R}$.

Calculation Rule for Powers and Roots

$$x^{-n} = \frac{1}{x^n} \qquad x^0 = 1 \qquad (x \neq 0)$$

$$x^{n+m} = x^n \cdot x^m \qquad x^{\frac{1}{m}} = \sqrt[m]{x} \qquad (x \geq 0)$$

$$x^{n-m} = \frac{x^n}{x^m} \qquad x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad (x \geq 0)$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}} \qquad (x \geq 0)$$

$$(x^n)^m = x^{n \cdot m}$$

Important!

0^0 is *not* defined!

Computations with Powers and Roots

$$\blacktriangleright \sqrt[3]{5^6} = (5^6)^{\frac{1}{3}} = 5^{(6 \cdot \frac{1}{3})} = 5^{\frac{6}{3}} = 5^2 = 25$$

$$\blacktriangleright (\sqrt[3]{5})^6 = (5^{\frac{1}{3}})^6 = 5^{(\frac{1}{3} \cdot 6)} = 5^{\frac{6}{3}} = 5^2 = 25$$

$$\blacktriangleright 5^{4-3} = 5^1 = 5$$

$$\blacktriangleright 5^{4-3} = \frac{5^4}{5^3} = \frac{625}{125} = 5$$

$$\blacktriangleright 5^{2-2} = 5^0 = 1$$

$$\blacktriangleright 5^{2-2} = \frac{5^2}{5^2} = \frac{25}{25} = 1$$

Computations with Powers and Roots

$$\blacktriangleright \frac{(x \cdot y)^4}{x^{-2}y^3} = x^4 y^4 x^{-(-2)} y^{-3} = x^6 y$$

$$\begin{aligned} \blacktriangleright \frac{(2x^2)^3 (3y)^{-2}}{(4x^2y)^2 (x^3y)} &= \frac{2^3 x^{2 \cdot 3} 3^{-2} y^{-2}}{4^2 x^{2 \cdot 2} y^2 x^3 y} = \frac{8 x^6 y^{-2}}{16 x^7 y^3} \\ &= \frac{1}{18} x^{6-7} y^{-2-3} = \frac{1}{18} x^{-1} y^{-5} = \frac{1}{18 x y^5} \end{aligned}$$

$$\blacktriangleright \left(3x^{\frac{1}{3}}y^{-\frac{4}{3}}\right)^3 = 3^3 x^{\frac{3}{3}} y^{-\frac{12}{3}} = 27 x y^{-4} = \frac{27 x}{y^4}$$

Sources of Errors

Important!

- ▶ $-x^2$ is **not** equal to $(-x)^2$!
- ▶ $(x + y)^n$ is **not** equal to $x^n + y^n$!
- ▶ $x^n + y^n$ **cannot** be simplified (in general)!

Problem 2.7

Simplify the following expressions:

(a) $\frac{(xy)^{\frac{1}{3}}}{x^{\frac{1}{6}}y^{\frac{2}{3}}}$

(b) $\frac{1}{(\sqrt{x})^{-\frac{3}{2}}}$

(c) $\left(\frac{|x|^{\frac{1}{3}}}{|x|^{\frac{1}{6}}}\right)^6$

Monomial

A **monomial** is a real number, variable, or product of variables raised to positive integer powers.

The **degree** of a monomial is the sum of all exponents of the variables (including a possible implicit exponent 1).

$6x^2$ is a monomial of degree 2.

$3x^3y$ and xy^2z are monomials of degree 4.

\sqrt{x} and $\frac{2}{3xy^2}$ are *not* monomials.

Polynomial

A **polynomial** is a sum of (one or more) monomials.

The **degree** of a polynomial is the maximum degree among its monomials.

$4x^2y^3 - 2x^3y + 4x + 7y$ is a polynomial of degree 5.

Polynomials in *one variable* (in sigma notation):

$$P(x) = \sum_{i=0}^n a_i x^i = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

where x is the variable and $a_i \in \mathbb{R}$ are constants.

Problem 2.8

Which of the following expressions are monomials or polynomials?
What is their degree?

Assume that x , y , and z are variables and all other symbols represent constants.

- (a) x^2
- (b) $x^{2/3}$
- (c) $2x^2 + 3xy + 4y^2$
- (d) $(2x^2 + 3xy + 4y^2)(x^2 - z^2)$
- (e) $(x - a)(y - b)(z + 1)$
- (f) $x\sqrt{y} - \sqrt{xy}$
- (g) $ab + c$
- (h) $x\sqrt{a} - \sqrt{by}$

Binomial Theorem

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is called the **binomial coefficient** (read: “ n choose k ”) and

$$n! = 1 \cdot 2 \cdot \dots \cdot n$$

denotes the **factorial** of n (read: “ n -factorial”).

For convenience we set $0! = 1$.

Binomial Coefficient

Note:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \binom{n}{n-k}$$

Computation:

$$\binom{n}{k} = \frac{n \cdot (n-1) \cdot \dots \cdot (n-k+1)}{k \cdot (k-1) \cdot \dots \cdot 1}$$

$\binom{n}{k}$ is the number of ways to choose an (unordered) subset of k elements from a fixed set of n elements.

Binomial Theorem

$$\blacktriangleright (x+y)^2 = \binom{2}{0}x^2 + \binom{2}{1}xy + \binom{2}{2}y^2 = x^2 + 2xy + y^2$$

$$\begin{aligned}\blacktriangleright (x+y)^3 &= \binom{3}{0}x^3 + \binom{3}{1}x^2y + \binom{3}{2}xy^2 + \binom{3}{3}y^3 \\ &= x^3 + 3x^2y + 3xy^2 + y^3\end{aligned}$$

Problem 2.9

Compute

(a) $(x+y)^4$

(b) $(x+y)^5$

by means of the binomial theorem.

Problem 2.10

Show by means of the binomial theorem that

$$\sum_{k=0}^n \binom{n}{k} = 2^n$$

Hint: Use $1 + 1 = 2$.

Multiplication

The product of two polynomials of degree n and m , resp., is a polynomial of degree $n + m$.

$$\begin{aligned}(2x^2 + 3x - 5) \cdot (x^3 - 2x + 1) &= \\ &= 2x^2 \cdot x^3 + 2x^2 \cdot (-2x) + 2x^2 \cdot 1 \\ &\quad + 3x \cdot x^3 + 3x \cdot (-2x) + 3x \cdot 1 \\ &\quad + (-5) \cdot x^3 + (-5) \cdot (-2x) + (-5) \cdot 1 \\ &= 2x^5 + 3x^4 - 9x^3 - 4x^2 + 13x - 5\end{aligned}$$

Problem 2.11

Simplify the following expressions:

(a) $(x + h)^2 - (x - h)^2$

(b) $(a + b)c - (a + bc)$

(c) $(A - B)(A^2 + AB + B^2)$

(d) $(x + y)^4 - (x - y)^4$

Division

Polynomials can be **divided** similarly to the division of integers.

$$\begin{array}{r} (x^3 + x^2 + 0x - 2) : (x - 1) = x^2 + 2x + 2 \\ x^2 \cdot (x - 1) \longrightarrow \begin{array}{r} x^3 - x^2 \\ \hline 2x^2 + 0x \end{array} \\ 2x \cdot (x - 1) \longrightarrow \begin{array}{r} 2x^2 - 2x \\ \hline 2x - 2 \\ \hline 2x - 2 \\ \hline 0 \end{array} \end{array}$$

We thus yield $x^3 + x^2 - 2 = (x - 1) \cdot (x^2 + 2x + 2)$.

If the *divisor* is not a factor of the *dividend*, then we obtain a *remainder*.

Factorization

The process of expressing a polynomial as the product of polynomials of smaller degree (**factor**) is called **factorization**.

$$2x^2 + 4xy + 8xy^3 = 2x \cdot (x + 2y + 4y^3)$$

$$x^2 - y^2 = (x + y) \cdot (x - y)$$

$$x^2 - 1 = (x + 1) \cdot (x - 1)$$

$$x^2 + 2xy + y^2 = (x + y) \cdot (x + y) = (x + y)^2$$

$$x^3 + y^3 = (x + y) \cdot (x^2 - xy + y^2)$$

These products can be easily verified by multiplying their factors.

Factorization

The factorization

$$(x^2 - y^2) = (x + y) \cdot (x - y)$$

or equivalently

$$(x - y) = (\sqrt{x} + \sqrt{y}) \cdot (\sqrt{x} - \sqrt{y})$$

can be very useful.

Memorize it!

The “Ausmultiplizierreflex”

Important!

Factorizing a polynomial is often *very hard* while multiplying its factors is fast and easy. (The RSA public key encryption is based on this idea.)

Beware!

A factorized expression contains more information than their expanded counterpart.

In my experience many students have an

acquired “Ausmultiplizierreflex”:

Instantaneously (and **without thinking**) they *multiply* all factors (which often turns a simple problem into a difficult one).

The “Ausmultiplizierreflex”

Suppress your “Ausmultiplizierreflex”!

Think first and multiply factors only when it seems to be useful!

Linear Term

A polynomial of *degree 1* is called a **linear term**.

- ▶ $a + bx + y + ac$ is a linear term in x and y , if a , b and c are constants.
- ▶ $xy + x + y$ is not linear, as xy has degree 2.

Linear Factor

A factor (polynomial) of degree 1 is called a **linear factor**.

A polynomial in one variable x with root x_1 has linear factor $(x - x_1)$.

If a polynomial in x of degree n ,

$$P(x) = \sum_{i=0}^n a_i x^i$$

has n real roots x_1, x_2, \dots, x_n , then it can be written as the product of the n linear factors $(x - x_i)$:

$$P(x) = a_n \prod_{i=1}^n (x - x_i)$$

(This is a special case of the Fundamental Theorem of Algebra.)

Problem 2.12

- (a) Give a polynomial in x of degree 4 with roots $-1, 2, 3$, and 4 .
- (b) What is the set of all such polynomials?
- (c) Can such a polynomial have other roots?

Rational Term

A **rational term** is one of the form

$$\frac{P(x)}{Q(x)}$$

where $P(x)$ and $Q(x)$ are polynomials called **numerator** and **denominator**, resp.

Alternatively one can write $P(x)/Q(x)$.

The domain of a rational term is \mathbb{R} without the roots of the denominator.

$\frac{x^2 + x - 4}{x^3 + 5}$ is a rational term with domain $\mathbb{R} \setminus \{-\sqrt[3]{5}\}$.

Beware! The expression $\frac{0}{0}$ is not defined.

Calculation Rule for Fractions and Rational Terms

Let $b, c, e \neq 0$.

$$\frac{c \cdot a}{c \cdot b} = \frac{a}{b} \quad \text{Reduce}$$

$$\frac{a}{b} = \frac{c \cdot a}{c \cdot b} \quad \text{Expand}$$

$$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c} \quad \text{Multiplying}$$

$$\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e} \quad \text{Dividing}$$

$$\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e} \quad \text{Compound fraction}$$

Calculation Rule for Fractions and Rational Terms

Let $b, c \neq 0$.

$$\frac{a}{b} + \frac{d}{b} = \frac{a + d}{b} \quad \text{Addition with common denominator}$$

$$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c} \quad \text{Addition}$$

Very important! Really!

You have to expand fractions such that they have a **common denominator** *before* you add them!

Calculation Rule for Fractions and Rational Terms

$$\blacktriangleright \frac{x^2 - 1}{x + 1} = \frac{(x + 1)(x - 1)}{x + 1} = x - 1$$

$$\blacktriangleright \frac{4x^3 + 2x^2}{2xy} = \frac{2x^2(2x + 1)}{2xy} = \frac{x(2x + 1)}{y}$$

$$\begin{aligned} \blacktriangleright \frac{x + 1}{x - 1} + \frac{x - 1}{x + 1} &= \frac{(x + 1)^2 + (x - 1)^2}{(x - 1)(x + 1)} \\ &= \frac{x^2 + 2x + 1 + x^2 - 2x + 1}{(x - 1)(x + 1)} = 2 \frac{x^2 + 1}{x^2 - 1} \end{aligned}$$

Calculation Rule for Fractions and Rational Terms

I want to stress at this point that there happen *a lot of mistakes* in calculations that involve rational terms.

The following examples of such fallacies are collected from students' exams.

Sources of Errors

Very Important! Really!

$$\frac{a+c}{b+c} \text{ is not equal to } \frac{a}{b}$$

$$\frac{x}{a} + \frac{y}{b} \text{ is not equal to } \frac{x+y}{a+b}$$

$$\frac{a}{b+c} \text{ is not equal to } \frac{a}{b} + \frac{a}{c}$$

$$\frac{x+2}{y+2} \neq \frac{x}{y} \qquad \frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2} + \frac{1}{y^2}$$

Problem 2.13

Simplify the following expressions:

(a) $\frac{1}{1+x} + \frac{1}{x-1}$

(b) $\frac{s}{st^2 - t^3} - \frac{1}{s^2 - st} - \frac{1}{t^2}$

(c) $\frac{\frac{1}{x} + \frac{1}{y}}{xy + xz + y(z-x)}$

(d) $\frac{\frac{x+y}{y}}{\frac{x-y}{x}} + \frac{\frac{x+y}{x}}{\frac{x-y}{y}}$

Problem 2.14

Factorize and reduce the fractions:

(a) $\frac{1 - x^2}{1 - x}$

(b) $\frac{1 + x^2}{1 - x}$

(c) $\frac{x^3 - x^4}{1 - x}$

(d) $\frac{x^3 - x^5}{1 - x}$

(e) $\frac{x^2 - x^6}{1 - x}$

(f) $\frac{1 - x^3}{1 - x}$

Problem 2.15

Simplify the following expressions:

(a) $y(xy + x + 1) - \frac{x^2y^2 - 1}{x - \frac{1}{y}}$

(b) $\frac{\frac{x^2+y}{2x+1}}{\frac{2xy}{2x+y}}$

(c) $\frac{\frac{a}{x} - \frac{b}{x+1}}{\frac{a}{x+1} + \frac{b}{x}}$

(d) $\frac{2x^2y - 4xy^2}{x^2 - 4y^2} + \frac{x^2}{x + 2y}$

Problem 2.16

Simplify the following expressions:

(a) $\frac{x^{\frac{1}{4}} - y^{\frac{1}{3}}}{x^{\frac{1}{8}} + y^{\frac{1}{6}}}$

(b) $\frac{\sqrt{x} - 4}{x^{\frac{1}{4}} - 2}$

(c) $\frac{\frac{2}{x^{-\frac{1}{7}}}}{x^{-\frac{7}{2}}}$

Problem 2.17

Simplify the following expressions:

$$(a) \frac{(\sqrt{x} + y)^{\frac{1}{3}}}{x^{\frac{1}{6}}}$$

$$(b) \frac{1}{3\sqrt{x} - 1} \cdot \frac{1}{1 + \frac{1}{3\sqrt{x}}} \cdot \frac{1}{\sqrt{x}}$$

$$(c) \frac{(xy)^{\frac{1}{6}} - 3}{(xy)^{\frac{1}{3}} - 9}$$

$$(d) \frac{x - y}{\sqrt{x} - \sqrt{y}}$$

Exponential Function

► **Power function**

$(0, \infty) \rightarrow (0, \infty)$, $x \mapsto x^\alpha$ for some *fixed exponent* $\alpha \in \mathbb{R}$

► **Exponential function**

$\mathbb{R} \rightarrow (0, \infty)$, $x \mapsto a^x$ for some *fixed basis* $a \in (0, \infty)$

Exponent and Logarithm

A number y is called the **logarithm** to basis a , if $a^y = x$.

The logarithm is the *exponent of a number to basis a* .

We write

$$y = \log_a(x) \quad \Leftrightarrow \quad x = a^y$$

Important logarithms:

► **natural logarithm** $\ln(x)$ with basis $e = 2.7182818\dots$
(sometimes called *Euler's number*)

► **common logarithm** $\lg(x)$ with basis 10
(sometimes called *decadic* or *decimal logarithm*)

Exponent and Logarithm

- ▶ $\log_{10}(100) = 2$, as $10^2 = 100$
- ▶ $\log_{10}\left(\frac{1}{1000}\right) = -3$, as $10^{-3} = \frac{1}{1000}$
- ▶ $\log_2(8) = 3$, as $2^3 = 8$
- ▶ $\log_{\sqrt{2}}(16) = 8$, as $\sqrt{2}^8 = 2^{8/2} = 2^4 = 16$

Calculations with Exponent and Logarithm

Conversion formula:

$$a^x = e^{x \ln(a)} \quad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

$$\log_2(123) = \frac{\ln(123)}{\ln(2)} \approx \frac{4.812184}{0.6931472} = 6.942515$$

$$3^7 = e^{7 \ln(3)}$$

Implicit Basis

Important:

Often one can see $\log(x)$ without an explicit basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- ▶ In *mathematics*: *natural* logarithm
financial mathematics, programs like R, *Mathematica*, Maxima, ...
- ▶ In *applied sciences*: *common* logarithm
economics, pocket calculator, Excel, ...

Calculation Rules for Exponent and Logarithm

$$a^{x+y} = a^x \cdot a^y$$

$$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$$

$$a^{x-y} = \frac{a^x}{a^y}$$

$$\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$$

$$(a^x)^y = a^{x \cdot y}$$

$$\log_a(x^\beta) = \beta \cdot \log_a(x)$$

$$(a \cdot b)^x = a^x \cdot b^x$$

$$a^{\log_a(x)} = x$$

$$\log_a(a^x) = x$$

$$a^0 = 1$$

$$\log_a(1) = 0$$

$\log_a(x)$ has (as real-valued function) domain $x > 0$!

Problem 2.18

Compute without a calculator:

(a) $\log_2(2)$

(f) $\log_2\left(\frac{1}{4}\right)$

(b) $\log_2(4)$

(g) $\log_2(\sqrt{2})$

(c) $\log_2(16)$

(h) $\log_2\left(\frac{1}{\sqrt{2}}\right)$

(d) $\log_2(0)$

(i) $\log_2(-4)$

(e) $\log_2(1)$

Problem 2.19

Compute without a calculator:

(a) $\log_{10}(300)$

(b) $\log_{10}(3^{10})$

Use $\log_{10}(3) = 0.47712$.

Problem 2.20

Compute (simplify) without a calculator:

(a) $0.01^{-\log_{10}(100)}$

(b) $\log_{\sqrt{5}}\left(\frac{1}{25}\right)$

(c) $10^{3\log_{10}(3)}$

(d) $\frac{\log_{10}(200)}{\log_{\frac{1}{\sqrt{7}}}(49)}$

(e) $\log_8\left(\frac{1}{512}\right)$

(f) $\log_{\frac{1}{3}}(81)$

Problem 2.21

Represent the following expression in the form $y = A e^{cx}$
(i.e., determine A and c):

(a) $y = 10^{x-1}$

(b) $y = 4^{x+2}$

(c) $y = 3^x 5^{2x}$

(d) $y = 1.08^{x-\frac{x}{2}}$

(e) $y = 0.9 \cdot 1.1^{\frac{x}{10}}$

(f) $y = \sqrt{q} 2^{x/2}$

Summary

- ▶ sigma notation
- ▶ absolute value
- ▶ powers and roots
- ▶ monomials and polynomials
- ▶ binomial theorem
- ▶ multiplication, factorization, and linear factors
- ▶ trap door “Ausmultiplizierreflex”
- ▶ fractions, rational terms and many fallacies
- ▶ exponent and logarithm