

Chapter 1

Sets and Maps

Set

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

A **set** is a collection of *distinct* objects.

An object a of a set A is called an **element** of the set. We write:

$a \in A$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\dots\}$.

$$A = \{1, 2, 3, 4, 5, 6\} \quad B = \{x \mid x \text{ is an integer divisible by } 2\}$$

Important Sets

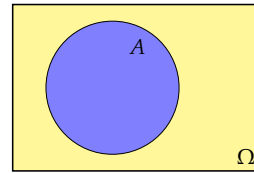
| Symbol | Description |
|--------------|---|
| \emptyset | empty set sometimes: $\{\}$ |
| \mathbb{N} | natural numbers $\{1, 2, 3, \dots\}$ |
| \mathbb{Z} | integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ |
| \mathbb{Q} | rational numbers $\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$ |
| \mathbb{R} | real numbers |
| $[a, b]$ | closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ |
| (a, b) | open interval ^a $\{x \in \mathbb{R} \mid a < x < b\}$ |
| $[a, b)$ | half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$ |
| \mathbb{C} | complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$ |

^aalso: $]a, b[$

Venn Diagram

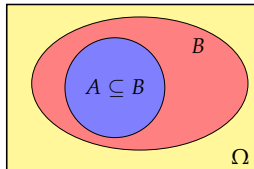
We assume that all *sets* are subsets of some universal **superset** Ω .

Sets can be represented by **Venn diagrams** where Ω is a rectangle and sets are depicted as circles or ovals.



Subset and Superset

Set A is a **subset** of B , $A \subseteq B$, if each element of A is also an element of B , i.e., $x \in A \Rightarrow x \in B$.



Vice versa, B is then called a **superset** of A , $B \supseteq A$.

Set A is a **proper subset** of B , $A \subset B$ (or: $A \subsetneq B$), if $A \subseteq B$ and $A \neq B$.

Problem 1.1

Which of the the following sets is a subset of

$$A = \{x \mid x \in \mathbb{R} \text{ and } 10 < x < 200\}$$

- (a) $\{x \mid x \in \mathbb{R} \text{ and } 10 < x \leq 200\}$
- (b) $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 121\}$
- (c) $\{x \mid x \in \mathbb{R} \text{ and } 4\pi < x < \sqrt{181}\}$
- (d) $\{x \mid x \in \mathbb{R} \text{ and } 20 < |x| < 100\}$

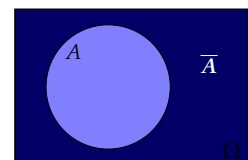
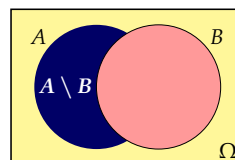
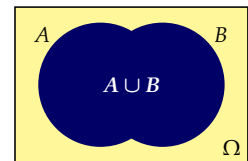
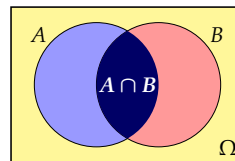
Basic Set Operations

| Symbol | Definition | Name |
|-----------------|--|---|
| $A \cap B$ | $\{x \mid x \in A \text{ and } x \in B\}$ | intersection |
| $A \cup B$ | $\{x \mid x \in A \text{ or } x \in B\}$ | union |
| $A \setminus B$ | $\{x \mid x \in A \text{ and } x \notin B\}$ | set-theoretic difference^a |
| \bar{A} | $\Omega \setminus A$ | complement |

^aalso: $A - B$

Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Basic Set Operations



Problem 1.2

The set $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ has subsets $A = \{1, 3, 6, 9\}$, $B = \{2, 4, 6, 10\}$ and $C = \{3, 6, 7, 9, 10\}$.

Draw the Venn diagram and give the following sets:

- (a) $A \cup C$
- (b) $A \cap B$
- (c) $A \setminus C$
- (d) \bar{A}
- (e) $(A \cup C) \cap B$
- (f) $(\bar{A} \cup B) \setminus C$
- (g) $\overline{(A \cup C)} \cap B$
- (h) $(\bar{A} \setminus B) \cap (\bar{A} \setminus C)$
- (i) $(A \cap B) \cup (A \cap C)$

Problem 1.3

Mark the following set in the corresponding Venn diagram:

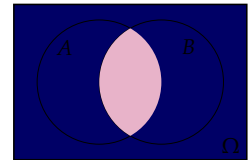
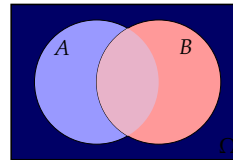
$$(A \cap \bar{B}) \cup (A \cap B)$$

Rules for Basic Operations

| Rule | Name |
|---|----------------|
| $A \cup A = A \cap A = A$ | Idempotence |
| $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ | Identity |
| $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ | Associativity |
| $A \cup B = B \cup A$ and $A \cap B = B \cap A$ | Commutativity |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributivity |
| $\bar{\bar{A}} = A$ and $\bar{A} \cap A = \emptyset$ | |

De Morgan's Law

$$\overline{(A \cup B)} = \bar{A} \cap \bar{B} \quad \text{and} \quad \overline{(A \cap B)} = \bar{A} \cup \bar{B}$$



Problem 1.4

Simplify the following set-theoretic expression:

$$(A \cap \bar{B}) \cup (A \cap B)$$

Problem 1.5

Simplify the following set-theoretic expressions:

- (a) $\overline{(A \cup B)} \cap \bar{B}$
- (b) $(A \cup \bar{B}) \cap (A \cup B)$
- (c) $((\bar{A} \cup \bar{B}) \cap \overline{(A \cap B)}) \cap A$
- (d) $(C \cup B) \cap (\overline{C \cap B}) \cap (C \cup \bar{B})$

Cartesian Product

The set

$$A \times B = \{(x, y) | x \in A, y \in B\}$$

is called the **Cartesian product** of sets A and B .

Given two sets A and B the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set A and the second element is from set B .

In general we have $A \times B \neq B \times A$.

Cartesian Product

The Cartesian product of $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ is

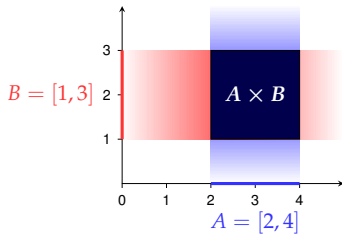
$$A \times B = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}.$$

| | | | |
|--------------|--------|--------|--------|
| $A \times B$ | 2 | 3 | 4 |
| 0 | (0, 2) | (0, 3) | (0, 4) |
| 1 | (1, 2) | (1, 3) | (1, 4) |

Cartesian Product

The Cartesian product of $A = [2, 4]$ and $B = [1, 3]$ is

$$A \times B = \{(x, y) \mid x \in [2, 4] \text{ and } y \in [1, 3]\}.$$



Problem 1.6

Describe the Cartesian products of

- $A = [0, 1]$ and $P = \{2\}$.
- $A = [0, 1]$ and $Q = \{(x, y) : 0 \leq x, y \leq 1\}$.
- $A = [0, 1]$ and $O = \{(x, y) : 0 < x, y < 1\}$.
- $A = [0, 1]$ and $C = \{(x, y) : x^2 + y^2 \leq 1\}$.
- $A = [0, 1]$ and \mathbb{R} .
- $Q_1 = \{(x, y) : 0 \leq x, y \leq 1\}$ and $Q_2 = \{(x, y) : 0 \leq x, y \leq 1\}$.

Map

A **map** (or **mapping**) f is defined by

- a **domain** D_f ,
- a **codomain (target set)** W_f and
- a **rule**, that maps each element of D to *exactly one* element of W .

$$f: D \rightarrow W, \quad x \mapsto y = f(x)$$

- x is called the **independent** variable, y the **dependent** variable.
- y is the **image** of x , x is the **preimage** of y .
- $f(x)$ is the **function term**, x is called the **argument** of f .
- $f(D) = \{y \in W : y = f(x) \text{ for some } x \in D\}$ is the **image (or range)** of f .

Other names: *function, transformation*

Problem 1.7

We are given map

$$\varphi: [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto y = x^\alpha \quad \text{for some } \alpha > 0$$

What are

- function name,
- domain,
- codomain,
- image (range),
- function term,
- argument,
- independent and dependent variable?

Injective · Surjective · Bijective

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

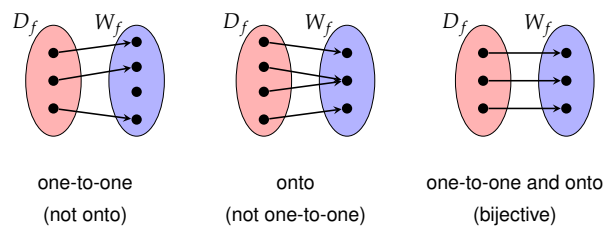
- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Injections have the important property

$$f(x) \neq f(y) \Leftrightarrow x \neq y$$

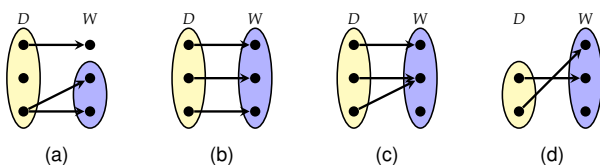
Injective · Surjective · Bijective

Maps can be visualized by means of arrows.



Problem 1.8

Which of these diagrams represent maps?
Which of these maps are one-to-one, onto, both or neither?



Problem 1.9

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

- $f: [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto x^2$
- $f: [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto x^{-2}$
- $f: [0, \infty) \rightarrow [0, \infty), \quad x \mapsto x^2$
- $f: [0, \infty) \rightarrow \mathbb{R}, \quad x \mapsto \sqrt{x}$
- $f: [0, \infty) \rightarrow [0, \infty), \quad x \mapsto \sqrt{x}$
- $f: [0, \infty) \rightarrow [0, \infty), \quad x \mapsto \{y \in [0, \infty) : x = y^2\}$

Problem 1.10

Let $\mathcal{P}_n = \{\sum_{i=0}^n a_i x^i : a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n .

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

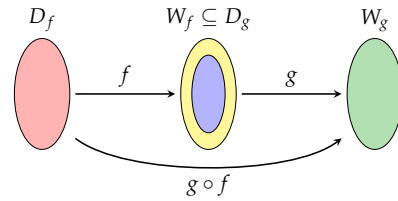
- (a) $D: \mathcal{P}_n \rightarrow \mathcal{P}_n, p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of p)
- (b) $D: \mathcal{P}_n \rightarrow \mathcal{P}_{n-1}, p(x) \mapsto \frac{dp(x)}{dx}$
- (c) $D: \mathcal{P}_n \rightarrow \mathcal{P}_{n-2}, p(x) \mapsto \frac{dp(x)}{dx}$

Function Composition

Let $f: D_f \rightarrow W_f$ and $g: D_g \rightarrow W_g$ be functions with $W_f \subseteq D_g$.
Function

$$g \circ f: D_f \rightarrow W_g, x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function**.
(read: "g composed with f", "g circle f", or "g after f")



Inverse Map

If $f: D_f \rightarrow W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.

Thus we get a map

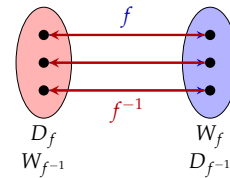
$$f^{-1}: W_f \rightarrow D_f, y \mapsto x = f^{-1}(y)$$

which is called the **inverse map** of f .

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y.$$

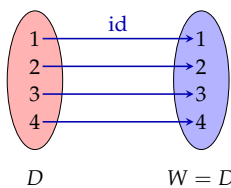
Inverse Map



Identity

The most elementary function is the **identity map** id , which maps its argument to itself, i.e.,

$$\text{id}: D \rightarrow W = D, x \mapsto x$$



Identity

The identity map has a similar role for compositions of functions as 1 has for multiplications of numbers:

$$f \circ \text{id} = f \quad \text{and} \quad \text{id} \circ f = f$$

Moreover,

$$f^{-1} \circ f = \text{id}: D_f \rightarrow D_f \quad \text{and} \quad f \circ f^{-1} = \text{id}: W_f \rightarrow W_f$$

Real-valued Functions

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$f: \mathbb{R} \rightarrow \mathbb{R}, x \mapsto f(x)$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.

Summary

- ▶ sets, subsets and supersets
- ▶ Venn diagram
- ▶ basic set operations
- ▶ de Morgan's law
- ▶ Cartesian product
- ▶ maps
- ▶ one-to-one and onto
- ▶ inverse map and identity