Chapter 1

# Sets and Maps



## Set

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

A **set** is a collection of *distinct* objects.

An object $a$ of a set $A$ is called an **element** of the set. We write:

$$\boxed{a \in A}$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\dots\}$.

$A = \{1, 2, 3, 4, 5, 6\}$ $\qquad B = \{x \mid x \text{ is an integer divisible by } 2\}$



## Important Sets

| Symbol | Description |
|---|---|
| $\emptyset$ | empty set sometimes: $\{\}$ |
| $\mathbb{N}$ | natural numbers $\{1, 2, 3, \dots\}$ |
| $\mathbb{Z}$ | integers $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ |
| $\mathbb{Q}$ | rational numbers $\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$ |
| $\mathbb{R}$ | real numbers |
| $[a, b]$ | closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ |
| $(a, b)$ | open interval[a] $\{x \in \mathbb{R} \mid a < x < b\}$ |
| $[a, b)$ | half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$ |
| $\mathbb{C}$ | complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$ |

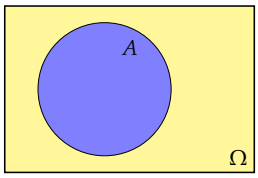
[a] also: $]a, b[$



## Venn Diagram

We assume that all *sets* are subsets of some universal **superset** $\Omega$.

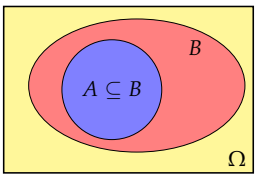
Sets can be represented by **Venn diagrams** where $\Omega$ is a rectangle and sets are depicted as circles or ovals.






## Subset and Superset

Set $A$ is a **subset** of $B$, $\boxed{A \subseteq B}$, if each element of $A$ is also an element of $B$, i.e., $x \in A \Rightarrow x \in B$.




Vice versa, $B$ is then called a **superset** of $A$, $\boxed{B \supseteq A}$.

Set $A$ is a **proper subset** of $B$, $\boxed{A \subset B}$ (or: $A \subsetneq B$), if $A \subseteq B$ and $A \neq B$.



## Problem 1.1

Which of the the following sets is a subset of

$$A = \{x \mid x \in \mathbb{R} \text{ and } 10 < x < 200\}$$

**(a)** $\{x \mid x \in \mathbb{R} \text{ and } 10 < x \leq 200\}$

**(b)** $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 121\}$

**(c)** $\{x \mid x \in \mathbb{R} \text{ and } 4\pi < x < \sqrt{181}\}$

**(d)** $\{x \mid x \in \mathbb{R} \text{ and } 20 < |x| < 100\}$



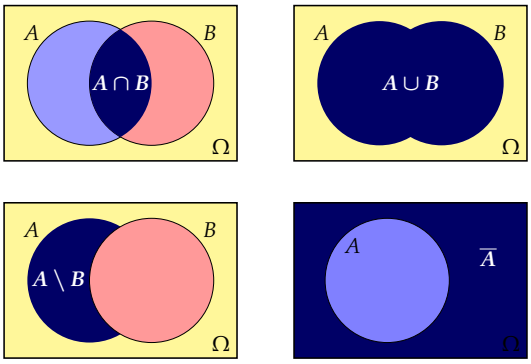
## Basic Set Operations

| Symbol | Definition | Name |
|---|---|---|
| $A \cap B$ | $\{x \mid x \in A \text{ and } x \in B\}$ | **intersection** |
| $A \cup B$ | $\{x \mid x \in A \text{ or } x \in B\}$ | **union** |
| $A \setminus B$ | $\{x \mid x \in A \text{ and } x \notin B\}$ | **set-theoretic difference**[a] |
| $\overline{A}$ | $\Omega \setminus A$ | **complement** |

[a] also: $A - B$

Two sets $A$ and $B$ are **disjoint** if $A \cap B = \emptyset$.

## Basic Set Operations






## Problem 1.2

The set $\Omega = \{1,2,3,4,5,6,7,8,9,10\}$ has subsets $A = \{1,3,6,9\}$, $B = \{2,4,6,10\}$ and $C = \{3,6,7,9,10\}$.

Draw the Venn diagram and give the following sets:

**(a)** $A \cup C$
**(b)** $A \cap B$
**(c)** $A \setminus C$
**(d)** $\overline{A}$
**(e)** $(A \cup C) \cap B$
**(f)** $(\overline{A} \cup B) \setminus C$
**(g)** $\overline{(A \cup C)} \cap B$
**(h)** $(\overline{A} \setminus B) \cap (\overline{A} \setminus C)$
**(i)** $(A \cap B) \cup (A \cap C)$



## Problem 1.3

Mark the following set in the corresponding Venn diagram:

$$(A \cap \overline{B}) \cup (A \cap B)$$



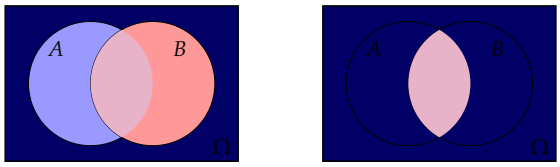
## Rules for Basic Operations

| Rule | Name |
|---|---|
| $A \cup A = A \cap A = A$ | Idempotence |
| $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ | Identity |
| $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ | Associativity |
| $A \cup B = B \cup A$ and $A \cap B = B \cap A$ | Commutativity |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | Distributivity |
| $\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$ | |



## De Morgan's Law

$$\boxed{\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}}$$






## Problem 1.4

Simplify the following set-theoretic expression:

$$(A \cap \overline{B}) \cup (A \cap B)$$



## Problem 1.5

Simplify the following set-theoretic expressions:

**(a)** $\overline{(A \cup B)} \cap \overline{B}$

**(b)** $(A \cup \overline{B}) \cap (A \cup B)$

**(c)** $((\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{B})}) \cap A$

**(d)** $(C \cup B) \cap \overline{(\overline{C} \cap \overline{B})} \cap (C \cup \overline{B})$



## Cartesian Product

The set

$$\boxed{A \times B = \{(x,y) | x \in A,\, y \in B\}}$$

is called the **Cartesian product** of *sets* $A$ and $B$.

Given two sets $A$ and $B$ the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set $A$ and the second element is from set $B$.

In general we have $A \times B \neq B \times A$.

## Cartesian Product

The Cartesian product of $A = \{0,1\}$ and $B = \{2,3,4\}$ is

$$A \times B = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}.$$

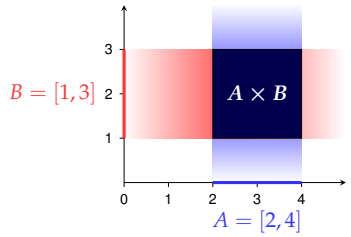
| $A \times B$ | 2 | 3 | 4 |
|---|---|---|---|
| 0 | $(0,2)$ | $(0,3)$ | $(0,4)$ |
| 1 | $(1,2)$ | $(1,3)$ | $(1,4)$ |



## Cartesian Product

The Cartesian product of $A = [2,4]$ and $B = [1,3]$ is

$$A \times B = \{(x,y) \mid x \in [2,4] \text{ and } y \in [1,3]\}.$$






## Problem 1.6

Describe the Cartesian products of

**(a)** $A = [0,1]$ and $P = \{2\}$.

**(b)** $A = [0,1]$ and $Q = \{(x,y) \colon 0 \le x,y \le 1\}$.

**(c)** $A = [0,1]$ and $O = \{(x,y) \colon 0 < x,y < 1\}$.

**(d)** $A = [0,1]$ and $C = \{(x,y) \colon x^2 + y^2 \le 1\}$.

**(e)** $A = [0,1]$ and $\mathbb{R}$.

**(f)** $Q_1 = \{(x,y) \colon 0 \le x,y \le 1\}$ and $Q_2 = \{(x,y) \colon 0 \le x,y \le 1\}$.



## Map

A **map** (or **mapping**) $f$ is defined by

**(i)** a **domain** $D_f$,

**(ii)** a **codomain** (**target set**) $W_f$ and

**(iii)** a **rule**, that maps each element of $D$ to *exactly one* element of $W$.

$$\boxed{f \colon D \to W, \quad x \mapsto y = f(x)}$$

- $x$ is called the **independent** variable, $y$ the **dependent** variable.
- $y$ is the **image** of $x$, $x$ is the **preimage** of $y$.
- $f(x)$ is the **function term**, $x$ is called the **argument** of $f$.
- $f(D) = \{y \in W \colon y = f(x) \text{ for some } x \in D\}$ is the **image** (or **range**) of $f$.

Other names: *function*, *transformation*



## Problem 1.7

We are given map

$$\varphi \colon [0,\infty) \to \mathbb{R},\ x \mapsto y = x^\alpha \qquad \text{for some } \alpha > 0$$

What are

- function name,
- domain,
- codomain,
- image (range),
- function term,
- argument,
- independent and dependent variable?



## Injective · Surjective · Bijective

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

- A map $f$ is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

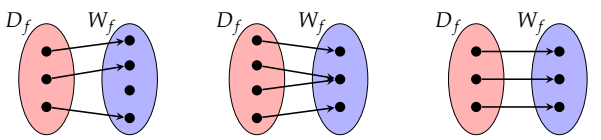
*Injections* have the important property

$$f(x) \ne f(y) \quad \Leftrightarrow \quad x \ne y$$



## Injective · Surjective · Bijective

Maps can be visualized by means of arrows.

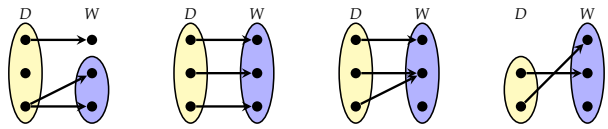



one-to-one (not onto) | onto (not one-to-one) | one-to-one and onto (bijective)



## Problem 1.8

Which of these diagrams represent maps?
Which of these maps are one-to-one, onto, both or neither?




(a) (b) (c) (d)

## Problem 1.9

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

**(a)** $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^2$

**(b)** $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^{-2}$

**(c)** $f\colon [0,\infty) \to [0,\infty),\ x \mapsto x^2$

**(d)** $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$

**(e)** $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$

**(f)** $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \{y \in [0,\infty)\colon x = y^2\}$



## Problem 1.10

Let $\mathcal{P}_n = \{\sum_{i=0}^{n} a_i x^i \colon a_i \in \mathbb{R}\}$ be the set of all polynomials in $x$ of degree less than or equal to $n$.

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

**(a)** $D\colon \mathcal{P}_n \to \mathcal{P}_n,\ p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of $p$)

**(b)** $D\colon \mathcal{P}_n \to \mathcal{P}_{n-1},\ p(x) \mapsto \frac{dp(x)}{dx}$

**(c)** $D\colon \mathcal{P}_n \to \mathcal{P}_{n-2},\ p(x) \mapsto \frac{dp(x)}{dx}$

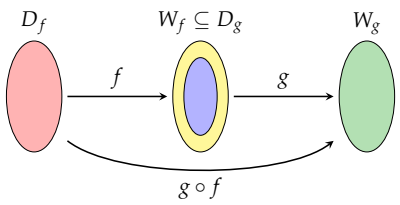


## Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.
Function

$$\boxed{g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))}$$

is called **composite function**.
(read: "$g$ composed with $f$", "$g$ circle $f$", or "$g$ after $f$")






## Inverse Map

If $f\colon D_f \to W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.
Thus we get a map

$$\boxed{f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)}$$

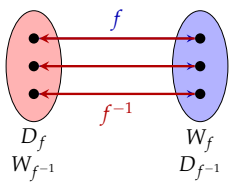
which is called the **inverse map** of $f$.

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\,.$$



## Inverse Map

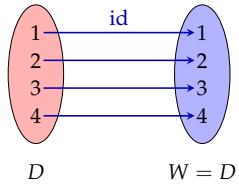





## Identity

The most elementary function is the **identity map** $\mathrm{id}$,
which maps its argument to itself, i.e.,

$$\boxed{\mathrm{id}\colon D \to W = D,\ x \mapsto x}$$






## Identity

The identity map has a similar role for compositions of functions as $1$ has for multiplications of numbers:

$$f \circ \mathrm{id} = f \qquad \text{and} \qquad \mathrm{id} \circ f = f$$

Moreover,

$$\boxed{f^{-1} \circ f = \mathrm{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}\colon W_f \to W_f}$$



## Real-valued Functions

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$\boxed{f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(x)}$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.

## Summary

- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan's law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity