Chapter 1

Sets and Maps

Set

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

A **set** is a collection of *distinct* objects.

An object a of a set A is called an **element** of the set. We write:

$$\boxed{a \in A}$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\dots\}$.

$A = \{1, 2, 3, 4, 5, 6\}$ $\qquad$ $B = \{x \mid x \text{ is an integer divisible by } 2\}$

Important Sets

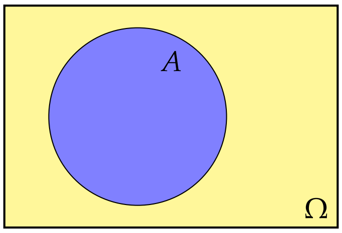
Symbol	Description	
$\emptyset$	empty set	sometimes: $\{\}$
$\mathbb{N}$	natural numbers	$\{1, 2, 3, \dots\}$
$\mathbb{Z}$	integers	$\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$
$\mathbb{Q}$	rational numbers	$\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$
$\mathbb{R}$	real numbers	
$[a, b]$	closed interval	$\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a, b)	open interval[a]	$\{x \in \mathbb{R} \mid a < x < b\}$
$[a, b)$	half-open interval	$\{x \in \mathbb{R} \mid a \leq x < b\}$
$\mathbb{C}$	complex numbers	$\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$

[a] also: $]a, b[$

Venn Diagram

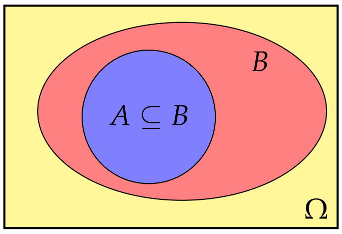
We assume that all *sets* are subsets of some universal **superset** Ω.

Sets can be represented by **Venn diagrams** where Ω is a rectangle and sets are depicted as circles or ovals.



Subset and Superset

Set A is a **subset** of B, $A \subseteq B$, if each element of A is also an element of B, i.e., $x \in A \Rightarrow x \in B$.



Vice versa, B is then called a **superset** of A, $B \supseteq A$.

Set A is a **proper subset** of B, $A \subset B$ (or: $A \subsetneq B$), if $A \subseteq B$ and $A \neq B$.

Problem 1.1

Which of the the following sets is a subset of

$$A = \{x \mid x \in \mathbb{R} \text{ and } 10 < x < 200\}$$

(a) $\{x \mid x \in \mathbb{R} \text{ and } 10 < x \leq 200\}$

(b) $\{x \mid x \in \mathbb{R} \text{ and } x^2 = 121\}$

(c) $\{x \mid x \in \mathbb{R} \text{ and } 4\pi < x < \sqrt{181}\}$

(d) $\{x \mid x \in \mathbb{R} \text{ and } 20 < |x| < 100\}$

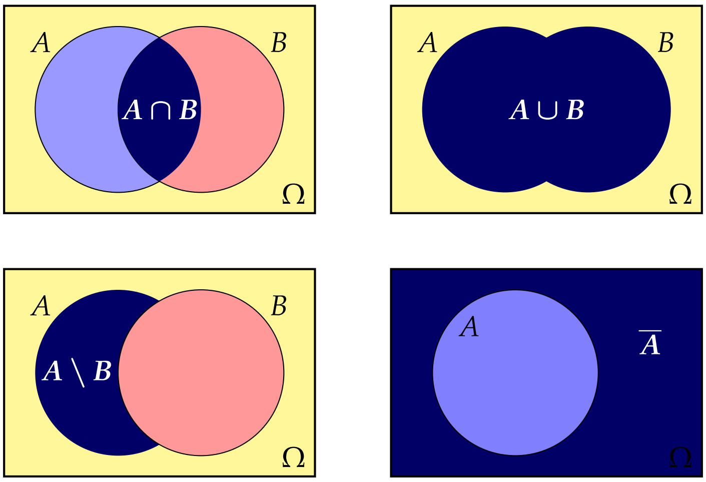
Basic Set Operations

Symbol	Definition	Name
$A \cap B$	$\{x \mid x \in A$ and $x \in B\}$	**intersection**
$A \cup B$	$\{x \mid x \in A$ or $x \in B\}$	**union**
$A \setminus B$	$\{x \mid x \in A$ and $x \notin B\}$	**set-theoretic difference**[a]
$\bar{A}$	$\Omega \setminus A$	**complement**

[a]also: $A - B$

Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Basic Set Operations



Problem 1.2

The set $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$ has subsets $A = \{1, 3, 6, 9\}$, $B = \{2, 4, 6, 10\}$ and $C = \{3, 6, 7, 9, 10\}$.

Draw the Venn diagram and give the following sets:

(a) $A \cup C$
(b) $A \cap B$
(c) $A \setminus C$
(d) $\bar{A}$
(e) $(A \cup C) \cap B$
(f) $(\bar{A} \cup B) \setminus C$
(g) $\overline{(A \cup C)} \cap B$
(h) $(\bar{A} \setminus B) \cap (\bar{A} \setminus C)$
(i) $(A \cap B) \cup (A \cap C)$

Problem 1.3

Mark the following set in the corresponding Venn diagram:

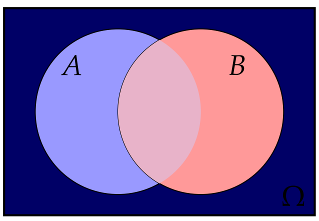
$$(A \cap \overline{B}) \cup (A \cap B)$$

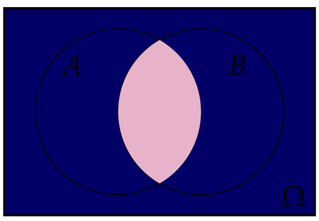
Rules for Basic Operations

Rule	Name
$A \cup A = A \cap A = A$	Idempotence
$A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$	Identity
$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$	Associativity
$A \cup B = B \cup A$ and $A \cap B = B \cap A$	Commutativity
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributivity
$\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$	

De Morgan's Law

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$





Problem 1.4

Simplify the following set-theoretic expression:

$$(A \cap \overline{B}) \cup (A \cap B)$$

Problem 1.5

Simplify the following set-theoretic expressions:

(a) $\overline{(A \cup B)} \cap \overline{B}$

(b) $(A \cup \overline{B}) \cap (A \cup B)$

(c) $((\overline{A} \cup \overline{B}) \cap \overline{(A \cap \overline{B})}) \cap A$

(d) $(C \cup B) \cap \overline{(\overline{C} \cap \overline{B})} \cap (C \cup \overline{B})$

Cartesian Product

The set

$$A \times B = \{(x,y) | x \in A, y \in B\}$$

is called the **Cartesian product** of *sets* A and B.

Given two sets A and B the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set A and the second element is from set B.

In general we have $A \times B \neq B \times A$.

Cartesian Product

The Cartesian product of $A = \{0, 1\}$ and $B = \{2, 3, 4\}$ is

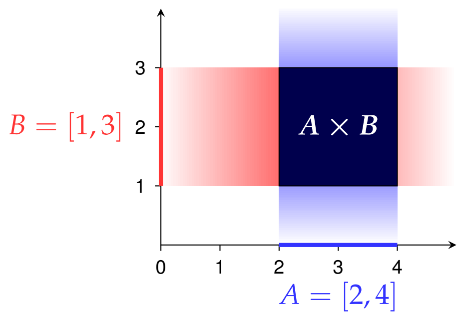
$$A \times B = \{(0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 4)\}.$$

$A \times B$	2	3	4
0	$(0, 2)$	$(0, 3)$	$(0, 4)$
1	$(1, 2)$	$(1, 3)$	$(1, 4)$

Cartesian Product

The Cartesian product of $A = [2, 4]$ and $B = [1, 3]$ is

$$A \times B = \{(x, y) \mid x \in [2, 4] \text{ and } y \in [1, 3]\}.$$



Problem 1.6

Describe the Cartesian products of

(a) $A = [0, 1]$ and $P = \{2\}$.

(b) $A = [0, 1]$ and $Q = \{(x, y) : 0 \le x, y \le 1\}$.

(c) $A = [0, 1]$ and $O = \{(x, y) : 0 < x, y < 1\}$.

(d) $A = [0, 1]$ and $C = \{(x, y) : x^2 + y^2 \le 1\}$.

(e) $A = [0, 1]$ and $\mathbb{R}$.

(f) $Q_1 = \{(x, y) : 0 \le x, y \le 1\}$ and $Q_2 = \{(x, y) : 0 \le x, y \le 1\}$.

Map

A **map** (or **mapping**) f is defined by

(i) a **domain** D_f,

(ii) a **codomain** (**target set**) W_f and

(iii) a **rule**, that maps each element of D to *exactly one* element of W.

$$f: D \to W, \quad x \mapsto y = f(x)$$

- x is called the **independent** variable, y the **dependent** variable.
- y is the **image** of x, x is the **preimage** of y.
- $f(x)$ is the **function term**, x is called the **argument** of f.
- $f(D) = \{y \in W : y = f(x) \text{ for some } x \in D\}$ is the **image** (or **range**) of f.

Other names: *function*, *transformation*

Problem 1.7

We are given map

$$\varphi: [0, \infty) \to \mathbb{R},\ x \mapsto y = x^\alpha \qquad \text{for some } \alpha > 0$$

What are

- function name,
- domain,
- codomain,
- image (range),
- function term,
- argument,
- independent and dependent variable?

Injective · Surjective · Bijective

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

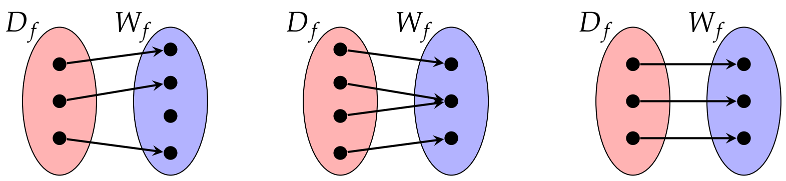
- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Injections have the important property

$$f(x) \neq f(y) \quad \Leftrightarrow \quad x \neq y$$

Injective · Surjective · Bijective

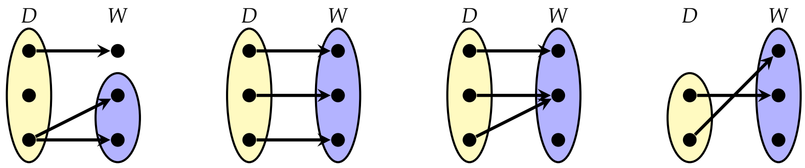
Maps can be visualized by means of arrows.



one-to-one	onto	one-to-one and onto
(not onto)	(not one-to-one)	(bijective)

Problem 1.8

Which of these diagrams represent maps?
Which of these maps are one-to-one, onto, both or neither?



(a) (b) (c) (d)

Problem 1.9

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

(a) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^2$

(b) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto x^{-2}$

(c) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto x^2$

(d) $f\colon [0,\infty) \to \mathbb{R},\ x \mapsto \sqrt{x}$

(e) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$

(f) $f\colon [0,\infty) \to [0,\infty),\ x \mapsto \{y \in [0,\infty)\colon x = y^2\}$

Problem 1.10

Let $\mathcal{P}_n = \{\sum_{i=0}^{n} a_i x^i \colon a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n.

Which of the following are proper definitions of mappings?
Which of the maps are one-to-one, onto, both or neither?

(a) $D\colon \mathcal{P}_n \to \mathcal{P}_n,\ p(x) \mapsto \frac{dp(x)}{dx}$ (derivative of p)

(b) $D\colon \mathcal{P}_n \to \mathcal{P}_{n-1},\ p(x) \mapsto \frac{dp(x)}{dx}$

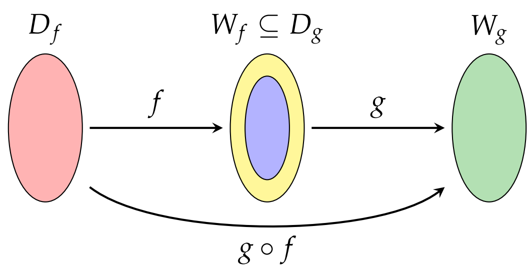
(c) $D\colon \mathcal{P}_n \to \mathcal{P}_{n-2},\ p(x) \mapsto \frac{dp(x)}{dx}$

Function Composition

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.
Function

$$g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function**.
(read: "g composed with f", "g circle f", or "g after f")



Inverse Map

If $f\colon D_f \to W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.
Thus we get a map

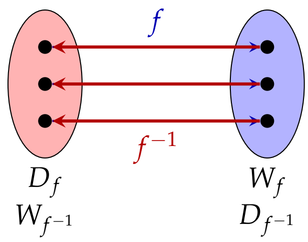
$$f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

which is called the **inverse map** of f.

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\,.$$

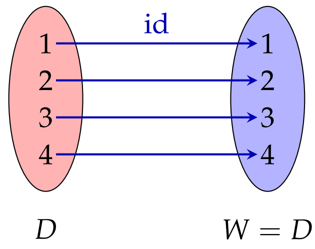
Inverse Map



Identity

The most elementary function is the **identity map** id,
which maps its argument to itself, i.e.,

$$\text{id}\colon D \to W = D,\ x \mapsto x$$



Identity

The identity map has a similar role for compositions of functions as 1 has for multiplications of numbers:

$$f \circ \text{id} = f \qquad \text{and} \qquad \text{id} \circ f = f$$

Moreover,

$$f^{-1} \circ f = \text{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \text{id}\colon W_f \to W_f$$

Real-valued Functions

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$f\colon \mathbb{R} \to \mathbb{R},\; x \mapsto f(x)$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.

Summary

- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan's law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity