# Foundations of Mathematics

**(MSc Economics)**

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# Introduction



## Learning Outcomes

- Fundamental mathematical methods

Depending on your prior knowledge:

- *Repetition* of mathematical notions and methods.
- Learning of new methods.



## Static Analysis of Equilibria

- At which price do we have market equilibrium?
  Find a price where demand and supply function coincide.
- Which amounts of goods have to be produced in a national economy such that consumers' needs are satisfied?
  Find the inverse of the matrix in a Leontief input-output model.
- How can a consumer optimize his or her utility?
  Find the absolute maximum of a utility function.
- What is the optimal production program for a company?
  Find the absolute maximum of a revenue function.



## Comparative-Statistic Analysis

- When market equilibrium is distorted, what happens to the price?
  Determine the derivative of the price as a function of time.
- What is the marginal production vector when demand changes in a Leontief model?
  Compute the derivative of a vector-valued function.
- How does the optimal utility of a consumer change, if income or prices change?
  Compute the derivative of the maximal utility w.r.t. exogenous parameters.



## Dynamic Analysis

- Assume we know the rate of change of a price w.r.t. time.
  How does the price evolve?
  Solve a difference equation or differential equation, resp.
- Which political program optimizes economic growth of a state?
  Determine the parameters of a differential equation, such that the terminal point of a solution curve is maximal.
- What is the optimal investment and consumption strategy of a consumer who wants to maximize her intertemporal utility?
  Determine the rate of savings (as a function of time) which maximizes the sum of discounted consumption.



## Learning Outcomes – Basic Concepts

- Linear Algebra:
  matrix and vector · matrix algebra · vector space · rank and linear dependency · inverse matrix · determinant · eigenvalues · quadratic form · definiteness and principle minors
- Univariate Analysis:
  function · graph · one-to-one and onto · limit · continuity · differential quotient and derivative · monotonicity · convex and concave
- Multivariate Analysis:
  partial derivative · gradient and Jacobian matrix · total differential · implicit and inverse function · Hessian matrix · Taylor series

## Learning Outcomes – Optimization

- Static Optimization:
  local and global extremum · saddle point · convex and concave · Lagrange function · Kuhn-Tucker conditions · envelope theorem
- Dynamic Analysis:
  integration · differential equation · difference equation · stable and unstable equilibrium point · difference equations · cobweb diagram · control theory · Hamiltonian and transversality condition



## Course Organization

- Course based on *slides*.
  Download for handouts available.
- *Reading* and *preparation* of new chapters in self-study (handouts).
- *Presentation* of new concepts and examples.
- *Homework problems*.
- *Discussion* of students' solutions of homework problems.
- *Short online quizzes* in each course unit.
- *Question time* for final test.
- *Final test*.



## Course Material

All information and course materials can be found and downloaded via the the CANVAS (see *Downloads*).



## Literature

- ALPHA C. CHIANG, KEVIN WAINWRIGHT
  *Fundamental Methods of Mathematical Economics*
  *McGraw-Hill, 2005*.
- KNUT SYDSÆTER, PETER HAMMOND
  *Essential Mathematics for Economics Analysis*
  Prentice Hall, 3rd ed., 2008.
- KNUT SYDSÆTER, PETER HAMMOND, ATLE SEIERSTAD, ARNE STRØM
  *Further Mathematics for Economics Analysis*
  Prentice Hall, 2005.
- JOSEF LEYDOLD
  *Mathematik* für Ökonomen
  3. Auflage, Oldenbourg Verlag, München, 2003 (in German).



## Further Exercises

Books from *Schaum's Outline Series* (McGraw Hill) offer many example problems with detailed explanations. In particular:

- SEYMOUR LIPSCHUTZ, MARC LIPSON
  *Linear Algebra*, 4th ed., McGraw Hill, 2009.
- RICHARD BRONSON
  *Matrix Operations*, 2nd ed., McGraw Hill, 2011.
- ELLIOT MENDELSON
  *Beginning Calculus*, 3rd ed., McGraw Hill, 2003.
- ROBERT WREDE, MURRAY R. SPIEGEL
  *Advanced Calculus*, 3rd ed., McGraw Hill, 2010.
- ELLIOTT MENDELSON
  *3,000 Solved Problems in Calculus*, McGraw Hill, 1988.



## Prerequisites*

Knowledge about fundamental concepts and tools (like terms, sets, equations, sequences, limits, univariate functions, derivatives, integration) is obligatory for this course. These are (should have been) already known from high school and mathematical courses in your Bachelor program.

For the case of knowledge gaps we refer to the *Bridging Course Mathematics*. A link to learning materials for that course can be found on the web page.

Some slides still cover these topics and are marked by symbol * in the title of the slide.
However, we will discuss these slide only on request.



## Prerequisites – Issues*

The following problems may cause issues:

- Drawing (or sketching) of graphs of functions.
- Transform equations into equivalent ones.
- Handling inequalities.
- Correct handling of fractions.
- Calculations with exponents and logarithms.
- Obstructive multiplying of factors.
- Usage of mathematical notation.

Presented *"solutions"* of such calculation subtasks are surprisingly often *wrong*.



## Über die mathematische Methode

*Man **kann** also gar nicht prinzipieller Gegner der mathematischen Denkformen sein, sonst müßte man das Denken auf diesem Gebiete überhaupt aufgeben. Was man meint, wenn man die mathematische Methode ablehnt, ist vielmehr die höhere Mathematik. Man hilft sich, wo es absolut nötig ist, lieber mit schematischen Darstellungen und ähnlichen primitiven Behelfen, als mit der angemessenen Methode.*
*Das ist nun aber natürlich unzulässig.*

*Joseph Schumpeter (1906)*

Über die mathematische Methode der theoretischen Ökonomie, Zeitschrift für Volkswirtschaft, Sozialpolitik und Verwaltung Bd. 15, S. 30–49 (1906).

## About the Mathematical Method

*One **cannot** be an opponent of mathematical forms of thought as a matter of principle, since otherwise one has to stop thinking in this field at all. What one means, if someone refuses the mathematical method, is in fact higher mathematics. One uses a schematic representation or other primitive makeshift methods where absolutely required rather than the appropriate method.*
*However, this is of course not allowed.*

*Joseph Schumpeter (1906)*

Über die mathematische Methode der theoretischen Ökonomie, Zeitschrift für Volkswirtschaft, Sozialpolitik und Verwaltung Bd. 15, S. 30–49 (1906).
Translation by JL.



## Science Track

- Discuss basics of mathematical reasoning.
- Extend our tool box of mathematical methods for static optimization and dynamic optimization.
- For more information see the corresponding web pages for the courses *Mathematics I* and *Mathematics II*.



## Computer Algebra System (CAS)

*Maxima* is a so called **Computer Algebra System** (CAS), i.e., one can

- manipulate algebraic expressions,
- solve equations,
- differentiate and integrate functions symbolically,
- perform abstract matrix algebra,
- draw graphs of functions in one or two variables,
- ...

*wxMaxima* is an IDE for this system:

`http://wxmaxima.sourceforge.net/`

You find an *Introduction to Maxima for Economics* on the web page of this course.



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## Table of Contents – II – Linear Algebra





## Table of Contents – II – Linear Algebra / 2





## Table of Contents – III – Analysis





## Table of Contents – III – Analysis / 2

## Table of Contents – III – Analysis / 3





## Table of Contents – III – Analysis / 4





## Table of Contents – IV – Static Optimization





## Table of Contents – IV – Static Optimization / 2





## Table of Contents – V – Dynamic Analysis





## Table of Contents – V – Dynamic Analysis / 2





***May you do well!***

**Viel Erfolg!**



Chapter 1

# Logic, Sets and Maps

## Proposition

We need some elementary knowledge about **logic** for doing mathematics. The central notion is "proposition".

> A **proposition** is a sentence with is either **true** (T) or **false** (F).

- *"Vienna is located at river Danube."* is a true proposition.
- *"Bill Clinton was president of Austria."* is a false proposition.
- *"19 is a prime number."* is a true proposition.
- *"This statement is false."* is not a proposition.



## Logical Connectives

We get compound propositions by connecting (simpler) propositions by using **logical connectives**.

This is done by means of words *"and"*, *"or"*, *"not"*, or *"if . . . then"*, known from everyday language.

| Connective | Symbol | Name |
|---|---|---|
| not $P$ | $\neg P$ | negation |
| $P$ and $Q$ | $P \wedge Q$ | conjunction |
| $P$ or $Q$ | $P \vee Q$ | disjunction |
| if $P$ then $Q$ | $P \Rightarrow Q$ | implication |
| $P$ if and only if $Q$ | $P \Leftrightarrow Q$ | equivalence |



## Truth Table

Truth values of logical connectives.

| $P$ | $Q$ | $\neg P$ | $P \wedge Q$ | $P \vee Q$ | $P \Rightarrow Q$ | $P \Leftrightarrow Q$ |
|---|---|---|---|---|---|---|
| T | T | F | T | T | T | T |
| T | F | F | F | T | F | F |
| F | T | T | F | T | T | F |
| F | F | T | F | F | T | T |

Let $P =$ "$x$ is divisible by 2" and $Q =$ "$x$ is divisible by 3".
Proposition $P \wedge Q$ is true if and only if $x$ is divisible by 2 and 3 (i.e., by 6).



## Negation and Disjunction

- *Negation* $\neg P$ is not the "opposite" of proposition $P$.

  Negation of $P =$ "all cats are black"
  is $\neg P =$ "Not all cats are black"

  (And not "all cats are not black" or even "all cats are white"!)

- *Disjunction* $P \vee Q$ is in a non-exclusive sense:

  $P \vee Q$ is true if and only if
  - $P$ is true, or
  - $Q$ is true, or
  - both $P$ and $Q$ are true.



## Implication

The truth value of *implication* $P \Rightarrow Q$ seems a bit mysterious.

Note that $P \Rightarrow Q$ does not make any proposition about the truth value of $P$ or $Q$!

Which of the following propositions is true?

- "*If* Bill Clinton is Austrian citizen, *then* he can be elected for Austrian president."
- "*If* Karl (born 1970) is Austrian citizen, *then* he can be elected for Austrian president."
- "*If* $x$ is a prime number larger than 2, *then* $x$ is odd."

Implication $P \Rightarrow Q$ is *equivalent* to $\neg P \vee Q$:

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$



## A Simple Logical Proof

We can derive the truth value of proposition $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ by means of a truth table:

| $P$ | $Q$ | $\neg P$ | $(\neg P \vee Q)$ | $(P \Rightarrow Q)$ | $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

That is, proposition $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ is always true independently from the truth values for $P$ and $Q$.

It is a so called *tautology*.



## Theorems

> *Mathematics consists of propositions of the form: P implies Q, but you never ask whether P is true.* (Bertrand Russell)

A **mathematical statement** (*theorem*, *proposition*, *lemma*, *corollary*) is a proposition of the form $P \Rightarrow Q$.

$P$ is called a **sufficient** condition for $Q$.

A *sufficient* condition $P$ guarantees that proposition $Q$ is true. However, $Q$ can be true even if $P$ is false.

$Q$ is called a **necessary** condition for $P$, $\quad Q \Leftarrow P$.

A *necessary* condition $Q$ must be true to allow $P$ to be true. It does not guarantee that $P$ is true.

Necessary conditions often are used to find *candidates* for valid answers to our problems.



## Quantors

Mathematical texts often use the expressions *"for all"* and *"there exists"*, resp.

In formal notation the following symbols are used:

| Quantor | Symbol |
|---|---|
| for all | $\forall$ |
| there exists a | $\exists$ |
| there exists exactly one | $\exists!$ |
| there does not exists | $\nexists$ |

## Set*

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

A **set** is a collection of *distinct* objects.

An object $a$ of a set $A$ is called an **element** of the set. We write:

$$\boxed{a \in A}$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\ldots\}$.

$A = \{1, 2, 3, 4, 5, 6\}$ $\qquad$ $B = \{x \mid x$ is an integer divisible by $2\}$



## Important Sets*

| Symbol | Description |
|---|---|
| $\emptyset$ | empty set sometimes: $\{\}$ |
| $\mathbb{N}$ | natural numbers $\{1, 2, 3, \ldots\}$ |
| $\mathbb{Z}$ | integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ |
| $\mathbb{Q}$ | rational numbers $\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$ |
| $\mathbb{R}$ | real numbers |
| $[a, b]$ | closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$ |
| $(a, b)$ | open interval[a] $\{x \in \mathbb{R} \mid a < x < b\}$ |
| $[a, b)$ | half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$ |
| $\mathbb{C}$ | complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$ |

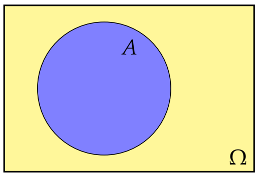
[a]also: $]a, b[$



## Venn Diagram*

We assume that all *sets* are subsets of some universal **superset** $\Omega$.

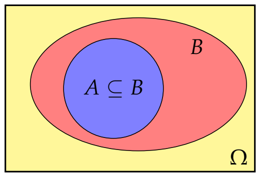
Sets can be represented by **Venn diagrams** where $\Omega$ is a rectangle and sets are depicted as circles or ovals.






## Subset and Superset*

Set $A$ is a **subset** of $B$, $\boxed{A \subseteq B}$, if all elements of $A$ also belong to $B$, $x \in A \Rightarrow x \in B$.




Vice versa, $B$ is then called a **superset** of $A$, $\boxed{B \supseteq A}$.

Set $A$ is a **proper subset** of $B$, $\boxed{A \subset B}$ (or: $A \subsetneq B$), if $A \subseteq B$ and $A \neq B$.



## Basic Set Operations*

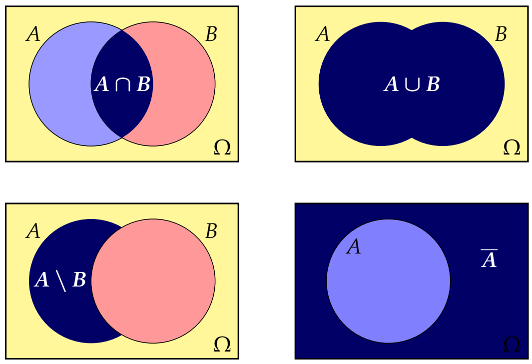
| Symbol | Definition | Name |
|---|---|---|
| $A \cap B$ | $\{x \mid x \in A$ and $x \in B\}$ | **intersection** |
| $A \cup B$ | $\{x \mid x \in A$ or $x \in B\}$ | **union** |
| $A \setminus B$ | $\{x \mid x \in A$ and $x \notin B\}$ | **set-theoretic difference**[a] |
| $\overline{A}$ | $\Omega \setminus A$ | **complement** |

[a]also: $A - B$

Two sets $A$ and $B$ are **disjoint** if $A \cap B = \emptyset$.



## Basic Set Operations*





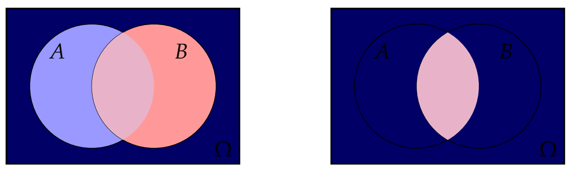

## Rules for Basic Operations*

| Rule | Name |
|---|---|
| $A \cup A = A \cap A = A$ | *Idempotence* |
| $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$ | *Identity* |
| $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ | *Associativity* |
| $A \cup B = B \cup A$ and $A \cap B = B \cap A$ | *Commutativity* |
| $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ | *Distributivity* |
| $\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$ | |



## De Morgan's Law*

$$\boxed{\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}}$$




$A$ union $B$ complemented is the equivalent of $A$ complemented intersected with $B$ complemented.

## Cartesian Product*

The set

$$\boxed{A \times B = \{(x,y) | x \in A,\ y \in B\}}$$

is called the **Cartesian product** of *sets* $A$ and $B$.

Given two sets $A$ and $B$ the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set $A$ and the second element is from set $B$.

In general we have $\quad A \times B \neq B \times A.$



## Cartesian Product*

The Cartesian product of $A = \{0,1\}$ and $B = \{2,3,4\}$ is

$$A \times B = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}.$$

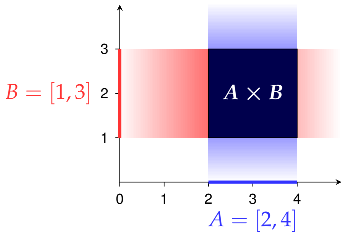
| $A \times B$ | 2 | 3 | 4 |
|---|---|---|---|
| 0 | $(0,2)$ | $(0,3)$ | $(0,4)$ |
| 1 | $(1,2)$ | $(1,3)$ | $(1,4)$ |



## Cartesian Product*

The Cartesian product of $A = [2,4]$ and $B = [1,3]$ is

$$A \times B = \{(x,y) \mid x \in [2,4] \text{ and } y \in [1,3]\}.$$






## Map*

A **map** (or **mapping**) $f$ is defined by

**(i)** a **domain** $D_f$,

**(ii)** a **codomain** (**target set**) $W_f$ and

**(iii)** a **rule**, that maps each element of $D$ to *exactly one* element of $W$.

$$\boxed{f\colon D \to W, \quad x \mapsto y = f(x)}$$

- $x$ is called the **independent** variable, $\quad y$ the **dependent** variable.
- $y$ is the **image** of $x$, $\quad x$ is the **preimage** of $y$.
- $f(x)$ is the **function term**, $\quad x$ is called the **argument** of $f$.
- $f(D) = \{y \in W\colon y = f(x) \text{ for some } x \in D\}$ is the **image** (or **range**) of $f$.

Other names: *function*, *transformation*



## Injective · Surjective · Bijective*

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

- A map $f$ is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

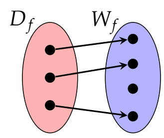
*Injections* have the important property

$$f(x) \neq f(y) \quad \Leftrightarrow \quad x \neq y$$

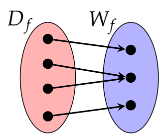


## Injective · Surjective · Bijective*

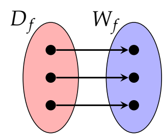
Maps can be visualized by means of arrows.




one-to-one
(not onto)




onto
(not one-to-one)




one-to-one and onto
(bijective)

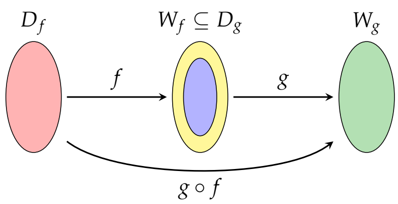


## Function Composition*

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.
Function

$$\boxed{g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))}$$

is called **composite function**.
(read: "$g$ composed with $f$", "$g$ circle $f$", or "$g$ after $f$")






## Inverse Map*

If $f\colon D_f \to W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.

Thus we get a map

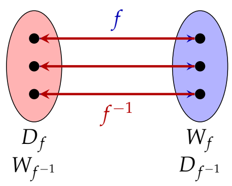
$$\boxed{f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)}$$

which is called the **inverse map** of $f$.

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\,.$$

## Inverse Map*

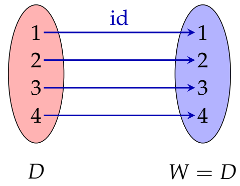





## Identity*

The most elementary function is the **identity map** $\mathrm{id}$, which maps its argument to itself, i.e.,

$$\boxed{\mathrm{id}\colon D \to W = D,\; x \mapsto x}$$






## Identity*

The identity map has a similar role for compositions of functions as $1$ has for multiplications of numbers:

$$f \circ \mathrm{id} = f \qquad \text{and} \qquad \mathrm{id} \circ f = f$$

Moreover,

$$\boxed{f^{-1} \circ f = \mathrm{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}\colon W_f \to W_f}$$



## Real-valued Functions*

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$\boxed{f\colon \mathbb{R} \to \mathbb{R},\; x \mapsto f(x)}$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.



## Summary

- mathematical logic
- theorem
- necessary and sufficient condition
- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan's law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity



Chapter 2

# Matrix Algebra



## A Very Simplistic Leontief Model

A community operates the services PUBLIC TRANSPORT, ELECTRICITY and GAS.

Technology matrix and weekly demand (in unit values):

| expenditure of \ for | transport | electricity | gas | demand |
|---|---|---|---|---|
| transport | 0.0 | 0.2 | 0.2 | 7.0 |
| electricity | 0.4 | 0.2 | 0.1 | 12.5 |
| gas | 0.0 | 0.5 | 0.1 | 16.5 |

What is the weekly production that satisfies the demand (but does not create excess)?



## A Very Simplistic Leontief Model

We denote the unknown units of production of TRANSPORT, ELECTRICITY and GAS by $x_1$, $x_2$,and $x_3$, resp.
For our production we must have:

$$\text{demand} = \text{production} - \text{internal expenditur}$$

$$\begin{aligned}
7.0 &= x_1 - (0.0\,x_1 + 0.2\,x_2 + 0.2\,x_3) \\
12.5 &= x_2 - (0.4\,x_1 + 0.2\,x_2 + 0.1\,x_3) \\
16.5 &= x_3 - (0.0\,x_1 + 0.5\,x_2 + 0.1\,x_3)
\end{aligned}$$

Transformation into an equivalent system of equations yields:

$$\begin{aligned}
1.0\,x_1 - 0.2\,x_2 - 0.2\,x_3 &= 7.0 \\
-0.4\,x_1 + 0.8\,x_2 - 0.1\,x_3 &= 12.5 \\
0.0\,x_1 - 0.5\,x_2 + 0.9\,x_3 &= 16.5
\end{aligned}$$

Which values for $x_1$, $x_2$, and $x_3$ solves these equations simultaneously?

## Matrix

An $m \times n$ **matrix** is a rectangular array of mathematical expressions (e.g., numbers) that consists of $m$ *rows* and $n$ *columns*.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})$$

Alternative notation: square brackets $[a_{ij}]$.

The terms $a_{ij}$ are called **elements** or **coefficients** of matrix $\mathbf{A}$, the integers $i$ and $j$ are called **row index** and **column index**, resp.

Matrices are denoted by bold upper case Latin letters, its coefficients by the corresponding lower case Latin letters.



## Vector

- A (column) **vector** is an $n \times 1$ matrix: $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$
- A **row vector** is a $1 \times n$-Matrix: $\mathbf{x}^\mathsf{T} = (x_1, \dots, x_n)$
- The $i$-th **unit vector** $\mathbf{e}_i$ is a vector where the $i$-th component is equal to $1$ and all other components are $0$.

Vectors are denoted by bold *lower case* Latin letters.

We write $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ for a matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$.



## Elements of a Matrix

We use the symbol

$$[\mathbf{A}]_{ij} = a_{ij}$$

to denote the coefficient with respective row and column index $i$ and $j$.

The convenient symbol

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

is called the **Kronecker symbol**.

Example of its usage: $[\mathbf{I}]_{ij} = \delta_{ij}$.



## Special Matrices

- An $n \times n$ matrix is called **square matrix**.
- An **upper triangular matrix** is a square matrix where all elements *below* the main diagonal are zero.

$$\mathbf{U} = \begin{pmatrix} -1 & -3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

Formally:
Matrix $\mathbf{U}$ is an upper triangular matrix if

$$[\mathbf{U}]_{ij} = 0 \text{ whenever } i > j.$$



## Special Matrices

- A **lower triangular matrix** is a square matrix where all elements *above* the main diagonal are zero.

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 0 \end{pmatrix}$$

Formally:
Matrix $\mathbf{L}$ is a lower triangular matrix if

$$[\mathbf{L}]_{ij} = 0 \text{ whenever } i < j.$$



## Special Matrices

- A **diagonal matrix** is a square matrix where all elements outside the main diagonal are zero.

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Formally:
Matrix $\mathbf{D}$ is a diagonal matrix if

$$[\mathbf{D}]_{ij} = 0 \text{ whenever } i \neq j.$$



## Special Matrices

- A matrix where all its coefficients are zero is called a **zero matrix** and is denoted by $\mathbf{O}_{n,m}$ or $\mathbf{0}$.
- An **identity matrix** is a diagonal matrix where all its diagonal entries are equal to $1$. It is denoted by $\mathbf{I}_n$ or $\mathbf{I}$.
(In German literature also symbol $\mathbf{E}$ is used.)

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark: Both identity matrix $\mathbf{I}_n$ and zero matrix $\mathbf{O}_{n,n}$ are examples of upper and lower triangular matrices and of a diagonal matrix.



## Transposed Matrix

We get the **transposed** $\mathbf{A}^\mathsf{T}$ of matrix $\mathbf{A}$ by exchanging rows and columns:

$$\left[\mathbf{A}^\mathsf{T}\right]_{ij} = [\mathbf{A}]_{ji}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Alternative notation: $\mathbf{A}'$

## Symmetric Matrix

A matrix $\mathbf{A}$ is called symmetric if

$$\boxed{\mathbf{A}^\mathsf{T} = \mathbf{A}}$$

i.e., if

$$\boxed{[\mathbf{A}]_{ij} = [\mathbf{A}]_{ji} \quad \text{for all } i, j.}$$

Obviously every symmetric matrix is a square matrix.

Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric.



## Scalar Multiplication

A matrix $\mathbf{A}$ can be multiplied by a constant (scalar) $\alpha \in \mathbb{R}$ *component-wise*:

$$\boxed{[\alpha \cdot \mathbf{A}]_{ij} = \alpha\, [\mathbf{A}]_{ij}}$$

$$3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$



## Addition of Matrices

Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are added *component-wise*:

$$\boxed{[\mathbf{A} + \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij}}$$

Addition of two matrices is only possible if their numbers of rows and columns coincide!

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$



## Multiplication of Matrices

The *product* $\mathbf{A} \cdot \mathbf{B}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined only if the number of columns of the first factor $\mathbf{A}$ coincides with the number of rows of the second factor $\mathbf{B}$.

That is, if $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{B}$ must be an $n \times k$ matrix.
The product $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ then is an $m \times k$ matrix.

Element $[\mathbf{A} \cdot \mathbf{B}]_{ij}$ is then the product of the $i$th row of $\mathbf{A}$ and the $j$th column of $\mathbf{B}$ (in the sense of a scalar product):

$$\boxed{[\mathbf{A} \cdot \mathbf{B}]_{ij} = \sum_{s=1}^{n} a_{is} \cdot b_{sj}}$$

Matrix multiplication is **not commutative!**



## Falk's Scheme

$$\begin{array}{ccc|cc}
\multicolumn{3}{c|}{\mathbf{A} \cdot \mathbf{B} \rightarrow} & 1 & 2 \\
\multicolumn{3}{c|}{\downarrow} & 3 & 4 \\
 & & & 5 & 6 \\
\hline
1 & 2 & 3 & c_{11} & c_{12} \\
4 & 5 & 6 & c_{21} & c_{22} \\
7 & 8 & 9 & c_{31} & c_{32}
\end{array}$$

$$c_{21} = 1 \cdot 4 + 5 \cdot 3 + 6 \cdot 5 = 49$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \end{pmatrix}$$



## Non-Commutativity

**Beware!**

Matrix multiplication is **not commutative!**

In general we have

$$\boxed{\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}}$$



## Non-Commutativity

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{is not defined}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}$$



## Non-Commutativity

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 22 & 29 \end{pmatrix}$$

while

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}$$

## Powers of a Matrix

$$\boxed{\begin{aligned} \mathbf{A}^2 &= \mathbf{A}\cdot\mathbf{A} \\ \mathbf{A}^3 &= \mathbf{A}\cdot\mathbf{A}\cdot\mathbf{A} \\ &\vdots \\ \mathbf{A}^n &= \underbrace{\mathbf{A}\cdot\ldots\cdot\mathbf{A}}_{n \text{ times}} \end{aligned}}$$



## Inverse Matrix

Let $\mathbf{A}$ be some square matrix.
If there exists a matrix $\mathbf{A}^{-1}$ with property

$$\boxed{\mathbf{A}\cdot\mathbf{A}^{-1} = \mathbf{A}^{-1}\cdot\mathbf{A} = \mathbf{I}}$$

then $\mathbf{A}^{-1}$ is called the **inverse matrix** of $\mathbf{A}$.

Matrix $\mathbf{A}$ is called **invertible** if it has an *inverse* matrix.
Otherwise it is called **singular**.

**Beware!**
Our definition implies that every invertible matrix must be a *square* matrix.

**Remark:** For any two square matrices $\mathbf{A}$ and $\mathbf{B}$,

$\mathbf{A}\cdot\mathbf{B} = \mathbf{I}$ implies $\mathbf{B}\cdot\mathbf{A} = \mathbf{I}$.



## Calculation Rules for Matrices

$$\mathbf{A}+\mathbf{B} = \mathbf{B}+\mathbf{A}$$
$$(\mathbf{A}+\mathbf{B})+\mathbf{C} = \mathbf{A}+(\mathbf{B}+\mathbf{C})$$
$$\mathbf{A}+\mathbf{0} = \mathbf{A}$$

$$(\mathbf{A}\cdot\mathbf{B})\cdot\mathbf{C} = \mathbf{A}\cdot(\mathbf{B}\cdot\mathbf{C})$$
$$\mathbf{I}\cdot\mathbf{A} = \mathbf{A}\cdot\mathbf{I} = \mathbf{A}$$

$$(\alpha\,\mathbf{A})\cdot\mathbf{B} = \alpha(\mathbf{A}\cdot\mathbf{B})$$
$$\mathbf{A}\cdot(\alpha\,\mathbf{B}) = \alpha(\mathbf{A}\cdot\mathbf{B})$$

$$\mathbf{C}\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{C}\cdot\mathbf{A}+\mathbf{C}\cdot\mathbf{B}$$
$$(\mathbf{A}+\mathbf{B})\cdot\mathbf{D} = \mathbf{A}\cdot\mathbf{D}+\mathbf{B}\cdot\mathbf{D}$$

$\mathbf{A}$ and $\mathbf{B}$ invertible
$\Rightarrow$ $\mathbf{A}\cdot\mathbf{B}$ invertible

$$(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$$
$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{A}\cdot\mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}}\cdot\mathbf{A}^{\mathsf{T}}$$
$$(\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A}$$
$$(\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}}$$

**Beware!**
In general we have
$$\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$$



## Computations with Matrices

For *appropriate* matrices we have similar calculation rules as for real numbers.
However, we have to keep in mind:

- A *zero matrix* $\mathbf{0}$ is the analog to number $0$.
- An *identity matrix* $\mathbf{I}$ corresponds to number $1$.
- Matrix multiplication is **not commutative!**
  In general we have $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$.
- There is **no** such thing like **division** by matrices!
  Use multiplication by the *inverse matrix* instead.



## Example – Computations with Matrices

$$(\mathbf{A}+\mathbf{B})^2 = (\mathbf{A}+\mathbf{B})\cdot(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2+\mathbf{A}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{A}+\mathbf{B}^2$$

$$\begin{aligned} &\mathbf{A}^{-1}\cdot(\mathbf{A}+\mathbf{B})\cdot\mathbf{B}^{-1}\mathbf{x} = \\ &\quad = (\mathbf{A}^{-1}\cdot\mathbf{A}+\mathbf{A}^{-1}\mathbf{B})\cdot\mathbf{B}^{-1}\mathbf{x} \\ &\quad = (\mathbf{I}+\mathbf{A}^{-1}\mathbf{B})\cdot\mathbf{B}^{-1}\mathbf{x} = \\ &\quad = (\mathbf{B}^{-1}+\mathbf{A}^{-1}\cdot\mathbf{B}\,\mathbf{B}^{-1})\mathbf{x} \\ &\quad = (\mathbf{B}^{-1}+\mathbf{A}^{-1})\mathbf{x} \\ &\quad = \mathbf{B}^{-1}\mathbf{x}+\mathbf{A}^{-1}\mathbf{x} \end{aligned}$$



## Equations with Matrices

If we multiply an equation with matrices by some matrix $\mathbf{A}$ we have to take care that multiplication is *not commutative*.
That is, $\mathbf{A}$ must be either the first or the second factor of the multiplication on either side of the equality sign!

**Beware!**
There is **no** such thing like **division** by matrices!

We have to multiply by the inverse matrix instead.



## Example – Equations with Matrices

Let $\mathbf{B}+\mathbf{A}\,\mathbf{X} = 2\mathbf{A}$ where $\mathbf{A}$ and $\mathbf{B}$ are known matrices.
Find matrix $\mathbf{X}$?

$$\begin{aligned} \mathbf{B}+\mathbf{A}\,\mathbf{X} &= 2\,\mathbf{A} && \mid -\mathbf{B} \\ \mathbf{A}\,\mathbf{X} &= 2\,\mathbf{A}-\mathbf{B} && \mid \mathbf{A}^{-1}\cdot \\ \mathbf{A}^{-1}\cdot\mathbf{A}\,\mathbf{X} &= \mathbf{A}^{-1}\cdot(2\,\mathbf{A}-\mathbf{B}) \\ \mathbf{I}\cdot\mathbf{X} &= 2\,\mathbf{A}^{-1}\mathbf{A}-\mathbf{A}^{-1}\cdot\mathbf{B} \\ \mathbf{X} &= 2\,\mathbf{I}-\mathbf{A}^{-1}\cdot\mathbf{B} \end{aligned}$$

We have to take care that all matrix operations are defined.

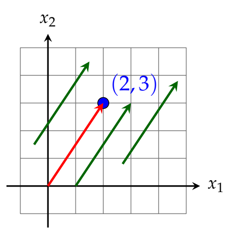


## Geometric Interpretation I

We have introduced vectors as special cases of matrices.

However, vector $\binom{x_1}{x_2}$ can also be seen as a geometrical object.
It can be interpreted as

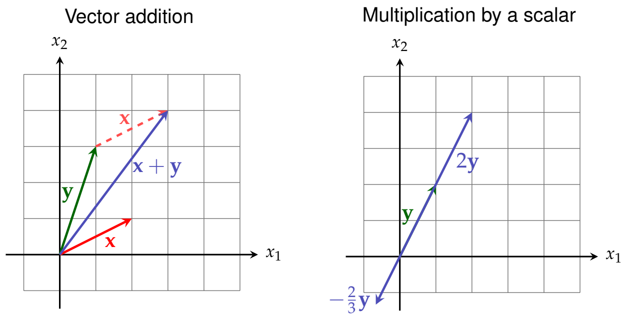
- a **point** $(x_1, x_2)$ in the $xy$-plain.
- an arrow from the origin $(0,0)$ to point $(x_1, x_2)$ (**position vector**).
- any arrow of the same length, direction and orientation as the position vector. (equivalence class of arrows)




We always choose the representation that fits our needs.

These pictures help us to think about these objects ("thinking crutch").
However, we need formulas to verify our conjectures!

## Geometric Interpretation II






## Scalar Product

The **inner product** (or **scalar product**) of two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\boxed{\mathbf{x}^\mathsf{T}\,\mathbf{y} = \sum_{i=1}^{n} x_i\, y_i}$$

Two vectors are called **orthogonal** to each other, if $\boxed{\mathbf{x}^\mathsf{T}\,\mathbf{y} = 0}$.

We also say that these vectors are *normal* or *perpendicular* or *in a right angle* to each other.

The inner product of $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix}4\\5\\6\end{pmatrix}$ is given by

$$\mathbf{x}^\mathsf{T}\,\mathbf{y} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32$$



## Norm

The (Euclidean) **norm** $\|\mathbf{x}\|$ of vector $\mathbf{x}$:

$$\boxed{\|\mathbf{x}\| = \sqrt{\mathbf{x}^\mathsf{T}\,\mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}}$$

A vector $\mathbf{x}$ is called **normalized**, if $\|\mathbf{x}\| = 1$.

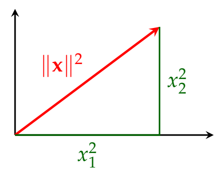
The norm of $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ is given by

$$\|\mathbf{x}\| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$$



## Geometric Interpretation

The *norm* of a vector can be interpreted as its *length*:




Pythagorean theorem:

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2$$

The *inner product* measures *angles* between two vectors:

$$\boxed{\cos\sphericalangle(\mathbf{x},\mathbf{y}) = \frac{\mathbf{x}^\mathsf{T}\,\mathbf{y}}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}}$$



## Properties of the Norm

**(i)** $\|\mathbf{x}\| \geq 0$.

**(ii)** $\|\mathbf{x}\| = 0 \quad\Leftrightarrow\quad \mathbf{x} = 0$.

**(iii)** $\|\alpha\mathbf{x}\| = |\alpha|\cdot\|\mathbf{x}\|$ for all $\alpha \in \mathbb{R}$.

**(iv)** $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (Triangle inequality)



## Inequalities

- **Cauchy-Schwarz inequality**

  $$\boxed{|\mathbf{x}^\mathsf{T}\mathbf{y}| \leq \|\mathbf{x}\|\cdot\|\mathbf{y}\|}$$

- **Minkowski inequality** (triangle inequality)

  $$\boxed{\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|}$$

- **Pythagorean theorem**

  For orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ we have

  $$\boxed{\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$$



## Leontief Model

$\mathbf{A}$ ... technology matrix    $\mathbf{p}$ ... prices for goods
$\mathbf{x}$ ... production vector    $\mathbf{w}$ ... wages
$\mathbf{b}$ ... demand vector

Prices must cover production costs:

$$p_j = \textstyle\sum_{i=1}^{n} a_{ij}p_i + w_j = a_{1j}p_1 + a_{2j}p_2 + \cdots + a_{nj}p_n + w_j$$
$$\mathbf{p} = \mathbf{A}^\mathsf{T}\mathbf{p} + \mathbf{w}$$

So for fixed wages we find:

$$\mathbf{p} = (\mathbf{I} - \mathbf{A}^\mathsf{T})^{-1}\mathbf{w}$$

Moreover, for the input-output model we have:

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$$



## Leontief Model

Demand is given by the wages for produced goods:

$$\text{demand} = w_1x_1 + w_2x_2 + \cdots + w_nx_n = \mathbf{w}^\mathsf{T}\mathbf{x}$$

Supply is given by prices for demanded goods:

$$\text{supply} = p_1b_1 + p_2b_2 + \cdots + p_nb_n = \mathbf{p}^\mathsf{T}\mathbf{b}$$

If the following equations hold in a input-output model

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b} \quad\text{and}\quad \mathbf{p} = \mathbf{A}^\mathsf{T}\mathbf{p} + \mathbf{w}$$

then we have market equilibrium, i.e., $\mathbf{w}^\mathsf{T}\mathbf{x} = \mathbf{p}^\mathsf{T}\mathbf{b}$.

**Proof:**

$$\mathbf{w}^\mathsf{T}\mathbf{x} = (\mathbf{p}^\mathsf{T} - \mathbf{p}^\mathsf{T}\mathbf{A})\mathbf{x} = \mathbf{p}^\mathsf{T}(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{p}^\mathsf{T}(\mathbf{x} - \mathbf{A}\mathbf{x}) = \mathbf{p}^\mathsf{T}\mathbf{b}$$

## Summary

- matrix and vector
- triangular and diagonal matrix
- zero matrix and identity matrix
- transposed and symmetric matrix
- inverse matrix
- computations with matrices (matrix algebra)
- equations with matrices
- norm and inner product of vectors



Chapter 3

# Linear Equations



## System of Linear Equations

System of $m$ linear equations in $n$ unknowns:

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + & \cdots & + & a_{mn}\,x_n & = & b_m \end{array}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{variables}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\text{vector of constants}}$$

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \mathbf{b}}$$



## Matrix Representation

Advantages of matrix representation:

- Short and compact notation.

  Compare
  $$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$
  to
  $$\sum_{j=1}^{n} a_{ij}x_j = b_i\,, \quad \text{for } i = 1, \dots, m$$

- We can transform equations by means of matrix algebra.
- We can use names for parts of the equation, like PRODUCTION VECTOR, DEMAND VECTOR, TECHNOLOGY MATRIX, etc. in the case of a Leontief model.



## Leontief Model

Input-output model with

$\mathbf{A}$ … technology matrix

$\mathbf{x}$ … production vector

$\mathbf{b}$ … demand vector

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

For a given output $\mathbf{b}$ we get the corresponding input $\mathbf{x}$ by

$$\begin{aligned} \mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{b} && \mid\ -\mathbf{A}\mathbf{x} \\ \mathbf{x} - \mathbf{A}\mathbf{x} &= \mathbf{b} \\ (\mathbf{I} - \mathbf{A})\mathbf{x} &= \mathbf{b} && \mid\ (\mathbf{I} - \mathbf{A})^{-1} \cdot \\ \mathbf{x} &= (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} \end{aligned}$$



## Solutions of a System of Linear Equations

Three possibilities:

- The system of equations has *exactly one* solution.
- The system of equations is *inconsistent*(not solvable).
- The system of equations has *infinitely many* solutions.

In **Gaussian elimination** the augmented coefficient matrix $(\mathbf{A}, \mathbf{b})$ is transformed into **row echelon form**.

Then the solution set is obtained by **back substitution**.

It is not possible to determine the number of solutions from the numbers of equations and unknowns. We have to transform the system first.



## Row Echelon Form

In **row echelon form** the number of *leading zeros* strictly increases from one row to the row below.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 1 & a_{35} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For our purposes it is not required that the first nonzero entries are equal to $1$.



## Steps in Gaussian Elimination

We (have to) obtain the row echelon form (only) by means of following transformations *which do not change the set of solutions*:

- Multiplication of a row by some *nonzero* constant.
- Addition of the multiple of some row to another row.
- Exchange of two rows.

## Example – Gaussian Elimination

$$\begin{array}{rrr|r} 1.0 & -0.2 & -0.2 & 7.0 \\ -0.4 & 0.8 & -0.1 & 12.5 \\ 0.0 & -0.5 & 0.9 & 16.5 \end{array}$$

We first add $0.4$ times the first row to the second row.
We denote this operation by

$$R_2 \leftarrow R_2 + 0.4 \times R_1$$

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & -0.50 & 0.90 & 16.5 \end{array}$$



## Example – Gaussian Elimination

$$R_3 \leftarrow R_3 + \frac{0.5}{0.72} \times R_2$$

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & 0 & 0.775 & 27.125 \end{array}$$



## Example – Back Substitution

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & 0 & 0.775 & 27.125 \end{array}$$

From the third row we immediately get:

$$0.775 \cdot x_3 = 27.125 \quad \Rightarrow \quad x_3 = 35$$

We obtain the remaining variables $x_2$ and $x_1$ by **back substitution**:

$$0.72 \cdot x_2 - 0.18 \cdot 35 = 15.3 \quad \Rightarrow \quad x_2 = 30$$

$$x_1 - 0.2 \cdot 30 - 0.2 \cdot 35 = 7 \quad \Rightarrow \quad x_1 = 20$$

The solution is unique: $\mathbf{x} = (20, 30, 35)^{\mathsf{T}}$



## Example 2

Find the solution of equation

$$\begin{aligned} 3\,x_1 + 4\,x_2 + 5\,x_3 &= 1 \\ x_1 + \phantom{4\,}x_2 - \phantom{5\,}x_3 &= 2 \\ 5\,x_1 + 6\,x_2 + 3\,x_3 &= 4 \end{aligned}$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 1 & 1 & -1 & 2 \\ 5 & 6 & 3 & 4 \end{array}$$

$$R_2 \leftarrow 3 \times R_2 - R_1, \quad R_3 \leftarrow 3 \times R_3 - 5 \times R_1$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 0 & -1 & -8 & 5 \\ 0 & -2 & -16 & 7 \end{array}$$



## Example 2

$$R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 0 & -1 & -8 & 5 \\ 0 & 0 & 0 & -3 \end{array}$$

The third row implies $0 = -3$, a **contradiction**.

This system of equations is **inconsistent**; solution set $L = \emptyset$.



## Example 3

Find the solution of equation

$$\begin{aligned} 2\,x_1 + \phantom{1}8\,x_2 + 10\,x_3 + 10\,x_4 &= \phantom{-}0 \\ x_1 + \phantom{1}5\,x_2 + \phantom{1}2\,x_3 + \phantom{1}9\,x_4 &= \phantom{-}1 \\ -3\,x_1 - 10\,x_2 - 21\,x_3 - \phantom{1}6\,x_4 &= -4 \end{aligned}$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 1 & 5 & 2 & 9 & 1 \\ -3 & -10 & -21 & -6 & -4 \end{array}$$

$$R_2 \leftarrow 2 \times R_2 - R_1, \quad R_3 \leftarrow 2 \times R_3 + 3 \times R_1$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 4 & -12 & 18 & -8 \end{array}$$



## Example 3

$$R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 0 & 0 & 2 & -12 \end{array}$$

This equation has **infinitely many** solutions.
This can be seen from the *row echelon form* as there are *more* variables than nonzero rows.



## Example 3

The third row immediately implies

$$2 \cdot x_4 = -12 \quad \Rightarrow \quad x_4 = -6$$

Back substitution yields

$$2 \cdot x_2 - 6 \cdot x_3 + 8 \cdot (-6) = 2$$

In this case we use *pseudo solution* $x_3 = \alpha$, $\alpha \in \mathbb{R}$, and get

$$x_2 - 3 \cdot \alpha + 4 \cdot (-6) = 1 \quad \Rightarrow \quad x_2 = 25 + 3\,\alpha$$

$$2 \cdot x_1 + 8 \cdot (25 + 3 \cdot \alpha) + 10 \cdot \alpha + 10 \cdot (-6) = 0$$

$$\Rightarrow \quad x_1 = -70 - 17 \cdot \alpha$$

## Example 3

We obtain a solution for each value of $\alpha$. Using vector notation we obtain

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -70 - 17 \cdot \alpha \\ 25 + 3\,\alpha \\ \alpha \\ -6 \end{pmatrix} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

Thus the solution set of this equation is

$$L = \left\{ \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$



## Equivalent Representation of Solutions

In Example 3 we also could use $x_2 = \alpha'$ (instead of $x_3 = \alpha$). Then back substitution yields

$$L' = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{215}{3} \\ 0 \\ -\frac{25}{3} \\ -6 \end{pmatrix} + \alpha' \begin{pmatrix} -\frac{17}{3} \\ 1 \\ \frac{1}{3} \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$

However, these two solution sets are equal, $L' = L$!

We thus have two different – *but equivalent* – representations of the same set.

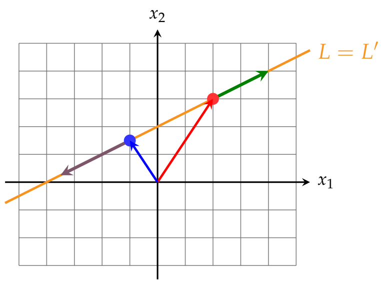
*The solution set is unique, its representation is not!*



## Equivalent Representation of Solutions

The set of solution points in Example 3 can be interpreted as a *line* in a (4-dimensional) space.

The representations in $L$ and $L'$ are thus parametric curves in $\mathbb{R}^4$ with the same image.






## A Non-Example

Find the solution of equation

$$\begin{aligned} 2\,x_1 + \phantom{2\,}x_2 \phantom{{}-2\,x_3} &= 1 \\ -2\,x_1 + 2\,x_2 - 2\,x_3 &= 4 \\ 4\,x_1 + 9\,x_2 - 3\,x_3 &= 9 \end{aligned}$$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ -2 & 2 & -2 & 4 \\ 4 & 9 & -3 & 9 \end{array}$$

$$R_2 \leftarrow R_2 + R_1, \quad R_3 \leftarrow R_3 - 2 \times R_1$$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ 0 & 7 & -3 & 7 \end{array}$$



## A Non-Example

Now one could find $R_3 \leftarrow R_3 - 7 \times R_1$ convenient. However,

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ -14 & 0 & -3 & 0 \end{array}$$

destroys the already created row echelon form in the first column!

Much better: $R_3 \leftarrow 3 \times R_3 - 7 \times R_2$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ 0 & 0 & 5 & -14 \end{array}$$



## Reduced Row Echelon Form

In **Gauss-Jordan elimination** the augmented matrix is transformed into **reduce row echelon form**, i.e.,

- It is in row echelon form.
- The leading entry in each nonzero row is a 1.
- Each column containing a leading 1 has 0s everywhere else.

$$\begin{pmatrix} 1 & 0 & a_{13} & 0 & 0 \\ 0 & 1 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Back substitution is then simpler.



## Gauss-Jordan Elimination / Example 3

Find the solution of equation

$$\begin{aligned} 2\,x_1 + \phantom{1}8\,x_2 + 10\,x_3 + 10\,x_4 &= \phantom{-}0 \\ x_1 + \phantom{1}5\,x_2 + \phantom{1}2\,x_3 + \phantom{1}9\,x_4 &= \phantom{-}1 \\ -3\,x_1 - 10\,x_2 - 21\,x_3 - \phantom{1}6\,x_4 &= -4 \end{aligned}$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 1 & 5 & 2 & 9 & 1 \\ -3 & -10 & -21 & -6 & -4 \end{array}$$

$$R_1 \leftarrow \frac{1}{2} \times R_1, \quad R_2 \leftarrow 2 \times R_2 - R_1, \quad R_3 \leftarrow 2 \times R_3 + 3 \times R_1$$

$$\begin{array}{rrrr|r} 1 & 4 & 5 & 5 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 4 & -12 & 18 & -8 \end{array}$$



## Gauss-Jordan Elimination / Example 3

$$R_1 \leftarrow R_1 - 2 \times R_2, \quad R_2 \leftarrow \frac{1}{2} \times R_2, \quad R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrrr|r} 1 & 0 & 17 & -11 & -4 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 0 & 0 & 2 & -12 \end{array}$$

$$R_1 \leftarrow R_1 + \frac{11}{2} \times R_3, \quad R_2 \leftarrow R_2 - 2 \times R_3, \quad R_3 \leftarrow \frac{1}{2} \times R_3,$$

$$\begin{array}{rrrr|r} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}$$

## Gauss-Jordan Elimination / Example 3

$$\left(\begin{array}{cccc|c} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right)$$

The third row immediately implies $x_4 = -6$

Set *pseudo solution* $x_3 = \alpha$, $\alpha \in \mathbb{R}$.

Back substitution yields $x_2 = 25 + 3\,\alpha$

and $x_1 = -70 - 17 \cdot \alpha$



## Gauss-Jordan Elimination / Example 3

Thus the solution set of this equation is

$$L = \left\{ \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$



## Gauss-Jordan Elimination / Example 3

Compare

$$\left(\begin{array}{cccc|c} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right) \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

The positional vector $(-70, 25, 0, -6)^{\mathsf{T}}$ follows from the r.h.s. of the reduced row echelon while the direction vector $(-17, 3, 1, 0)^{\mathsf{T}}$ is given by the column without leading $1$.



## Inverse of a Matrix

Computation of the inverse $\mathbf{A}^{-1}$ of matrix $\mathbf{A}$ by *Gauss-Jordan elimination*:

**(1)** Augment matrix $\mathbf{A}$ by the corresponding *identity matrix* to the right.

**(2)** Transform the augmented matrix such that the identity matrix appears on the left hand side by means of the transformation steps of Gaussian elimination.

**(3)** Either the procedure is successful. Then we obtain the *inverse matrix* $\mathbf{A}^{-1}$ on the right hand side.

**(4)** Or the procedure aborts (because we obtain a row of zeros on the l.h.s.). Then the matrix is *singular*.



## Example 1

Compute the inverse of

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 3 \\ -3 & -2 & -5 \end{pmatrix}$$

(1) Augment matrix $\mathbf{A}$:

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 6 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ -3 & -2 & -5 & 0 & 0 & 1 \end{array}\right)$$



## Example 1

(2) Transform:

$$R_1 \leftarrow \tfrac{1}{3} \times R_1, \quad R_2 \leftarrow 3 \times R_2 - R_1, \quad R_3 \leftarrow R_3 + R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \tfrac{2}{3} \times R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$



## Example 1

$$R_2 \leftarrow R_2 - 3 \times R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -4 & 3 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

(3) Matrix $\mathbf{A}$ is invertible with inverse

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ -4 & 3 & -3 \\ 1 & 0 & 1 \end{pmatrix}$$



## Example 2

Compute the inverse of

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix}$$

(1) Augment matrix $\mathbf{A}$:

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 5 & 5 & 4 & 0 & 0 & 1 \end{array}\right)$$

## Example 2

(2) Transform:

$$R_1 \leftarrow \tfrac{1}{3} \times R_1, \quad R_2 \leftarrow 3 \times R_2 - 2 \times R_1, \quad R_3 \leftarrow 3 \times R_3 - 5 \times R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 10 & -3 & -2 & 3 & 0 \\ 0 & 10 & -3 & -5 & 0 & 5 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \tfrac{1}{30} \times R_2, \quad R_2 \leftarrow \tfrac{1}{10} \times R_2, \quad R_3 \leftarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{11}{10} & \frac{4}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{3}{10} & -\frac{2}{10} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & -3 & -3 & 5 \end{array}\right)$$

(4) Matrix $\mathbf{A}$ is *not* invertible.



## Summary

- system of linear equations
- Gaussian elimination
- Gauss-Jordan elimination
- computation of inverse matrix



Chapter 4

# Vector Space



## Real Vector Space

The set of all vectors $\mathbf{x}$ with $n$ components is denoted by

$$\boxed{\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \middle| \, x_i \in \mathbb{R}, 1 \leq i \leq n \right\}}$$

It is *the* prototype example of an *$n$-dimensional (real) vector space*.

**Definition:**
A **vector space** $\mathcal{V}$ is a *set* of objects which may be *added* together and *multiplied* by numbers, called *scalars*.
Elements of a vector space are called **vectors**.

For details see course “Mathematics 1”.



## Example – Vector Space

The set of all $2 \times 2$ matrices

$$\mathbb{R}^{2\times 2} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R}, \; i,j \in \{1,2\} \right\}$$

together with matrix addition and scalar multiplication forms a vector space.

Similarly the set of all $m \times n$ matrices

$$\mathbb{R}^{m\times n} = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R}, \; i = 1,\dots,m, \; j = 1,\dots n \right\}$$

forms a vector space.



## A More Abstract Example

Let $\mathcal{P}_n = \{\sum_{i=0}^n a_i x^i | a_i \in \mathbb{R}\}$ be the set of all polynomials in $x$ of degree less than or equal to $n$.

Obviously we can multiply a polynomial by a scalar:

$$3 \cdot (4x^2 - 2x + 5) = 12x^2 - 6x + 15 \in \mathcal{P}_2$$

and add them point-wise:

$$(4x^2 - 2x + 5) + (-4x^2 + 5x - 2) = 3x + 3 \in \mathcal{P}_2$$

So for every $p(x), q(x) \in \mathcal{P}_n$ and $\alpha \in \mathbb{R}$ we find

$$\alpha p(x) \in \mathcal{P}_n \quad \text{and} \quad p(x) + q(x) \in \mathcal{P}_n \,.$$

Thus $\mathcal{P}_n$ together with point-wise addition and scalar multiplication forms a vector space.



## Linear Combination

Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ be vectors and $c_1, \dots, c_k \in \mathbb{R}$ arbitrary numbers.
Then we get a new vector by a **linear combination** of these vectors:

$$\boxed{\mathbf{x} = c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \sum_{i=1}^k c_i \mathbf{v}_i}$$

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$.

Then the following are linear combinations of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:

$\mathbf{x} = 1\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + 3\,\mathbf{v}_3 - 2\,\mathbf{v}_4 = (-3, -4, 3)^\mathsf{T}$,
$\mathbf{y} = -\mathbf{v}_1 + \mathbf{v}_2 - 2\,\mathbf{v}_3 + 3\,\mathbf{v}_4 = (4, 7, -2)^\mathsf{T}$, and
$\mathbf{z} = 2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = (0, 0, 0)^\mathsf{T} = 0$



## Subspace

A **Subspace** $\mathcal{S}$ of a vector space $\mathcal{V}$ is a *subset* of $\mathcal{V}$ which itself forms a *vector space* (with the same rules for addition and scalar multiplication).

In order to verify that a *subset* $\mathcal{S} \subseteq \mathcal{V}$ is a *subspace* of $\mathcal{V}$ we have to verify that for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and all $\alpha, \beta \in \mathbb{R}$

$$\boxed{\alpha \mathbf{x} + \beta \mathbf{y} \in \mathcal{S}}$$

*We say that $\mathcal{S}$ is closed under linear combinations.*

Equivalently: We have to verify that

**(i)** if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, then $\mathbf{x} + \mathbf{y} \in \mathcal{S}$; and

**(ii)** if $\mathbf{x} \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha \mathbf{x} \in \mathcal{S}$.

## Example – Subspace

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_i \in \mathbb{R},\, 1 \le i \le 2 \right\} \subset \mathbb{R}^3 \quad \text{is a subspace of } \mathbb{R}^3.$$

$$\left\{ \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \subset \mathbb{R}^3 \quad \text{is a subspace of } \mathbb{R}^3.$$

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_i \ge 0,\, 1 \le i \le 3 \right\} \subset \mathbb{R}^3 \quad \text{is **not** a subspace of } \mathbb{R}^3.$$



## Example – Homogeneous Linear Equation

Let $\mathbf{A}$ be an $m \times n$ matrix.
The solution set $\mathcal{L}$ of the *homogeneous* linear equation

$$\mathbf{A}\mathbf{x} = 0$$

forms a subspace of $\mathbb{R}^n$:

Let $\mathbf{x}, \mathbf{y} \in \mathcal{L} \subseteq \mathbb{R}^n$, i.e., $\mathbf{A}\mathbf{x} = 0$ and $\mathbf{A}\mathbf{y} = 0$, and $\alpha, \beta \in \mathbb{R}$.

Then a straightforward computation yields

$$\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} = \alpha 0 + \beta 0 = 0$$

i.e., $\alpha\mathbf{x} + \beta\mathbf{y}$ solves the linear equation and hence $\alpha\mathbf{x} + \beta\mathbf{y} \in \mathcal{L}$.

Therefore $\mathcal{L}$ is a subspace of $\mathbb{R}^n$.



## Example – Subspace

$$\left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \middle|\, a_{ii} \in \mathbb{R},\ i \in \{1,2\} \right\} \quad \text{is a subspace of } \mathbb{R}^{2\times 2}.$$

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle|\, a, b \in \mathbb{R} \right\} \quad \text{is a subspace of } \mathbb{R}^{2\times 2}.$$

$$\left\{ \mathbf{A} \in \mathbb{R}^{2\times 2} \,\middle|\, \mathbf{A} \text{ is invertible} \right\} \quad \text{is **not** a subspace of } \mathbb{R}^{2\times 2}.$$



## Linear Span

The set of all *linear combinations* of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathcal{V}$

$$\boxed{\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \left\{ \sum_{i=1}^{k} c_i \mathbf{v}_i \,\middle|\, c_i \in \mathbb{R},\ i = 1, \dots, k \right\}}$$

forms a subspace of $\mathcal{V}$ and is called the **linear span** of $\mathbf{v}_1, \dots, \mathbf{v}_k$.



## Linear Span

Let $\mathbf{x}, \mathbf{y} \in \mathcal{S} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ and $\alpha, \beta \in \mathbb{R}$.
Then there exist $a_i, b_i \in \mathbb{R}$, $i = 1, \dots, k$, such that

$$\mathbf{x} = \sum_{i=1}^{k} a_i \mathbf{v}_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{k} b_i \mathbf{v}_i\,.$$

But then

$$\alpha\mathbf{x} + \beta\mathbf{y} = \alpha \sum_{i=1}^{k} a_i \mathbf{v}_i + \beta \sum_{i=1}^{k} b_i \mathbf{v}_i = \sum_{i=1}^{k} \underbrace{(\alpha a_i + \beta b_i)}_{\in \mathbb{R}} \mathbf{v}_i \in \mathcal{S}$$

as the last summation is a linear combination of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Hence $\mathcal{S} = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k)$ is a subspace of $\mathcal{V}$.



## Example – Linear Span

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$.

$\operatorname{span}(\mathbf{v}_1) = \{c\,\mathbf{v}_1 : c \in \mathbb{R}\}$ is a straight line in $\mathbb{R}^3$ through the origin.

$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ is a plane in $\mathbb{R}^3$ through the origin.

$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$

$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^3$.



## Linear Independency

Every vector $\mathbf{x} \in \operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$.

$$\mathbf{x} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix} = 1\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + 3\,\mathbf{v}_3 - 2\,\mathbf{v}_4 = -1\,\mathbf{v}_1 + 2\,\mathbf{v}_2 + 6\,\mathbf{v}_3 - 2\,\mathbf{v}_4$$

The representation in this example is not unique!

**Reason:** $2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = 0$

One of the vectors seems to be needless:
$\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$



## Linear Independency

Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are called **linearly independent** if the homogeneous system of equations

$$c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \cdots + c_k\,\mathbf{v}_k = 0$$

has the unique solution $c_1 = c_2 = \cdots = c_k = 0$. They are called **linearly dependent** if these equations have other (non-zero) solutions.

If vectors are linearly dependent then *some* vector (but *not* necessarily *each* of these!) can be written as a linear combination of the other vectors.

$$2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = 0 \quad \Leftrightarrow \quad \mathbf{v}_3 = \tfrac{2}{3}\,\mathbf{v}_1 - \tfrac{2}{3}\,\mathbf{v}_2$$

Hence $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$.

## Linear Independency

***Determine linear (in)dependency***

**(1)** Create matrix $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$.

**(2)** Transform $\mathbf{V}$ into row echelon form by means of Gaussian elimination.

**(3)** Count the number of non-zero rows.

**(4)** If this is equal to $k$ (the number of vectors), then these vectors are linearly Independence.

If it is smaller, then the vectors or linearly dependent.

This procedure checks whether the linear equation $\mathbf{V} \cdot \mathbf{c} = 0$ has a unique solution.



## Example – Linearly Independent

Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

linearly independent?

(1) Create a matrix:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$



## Example – Linearly Independent

(2) Transform:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & -27 \end{pmatrix}$$

(3) We count 3 non-zero rows.

(4) The number of non-zero rows coincides with the number of vectors ($= 3$).

Thus the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly independent*.



## Example – Linearly Dependent

Are vectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

linearly independent?

(1) Create a matrix ... (2) and transform:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 10 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

(3) We count 2 non-zero rows.

(4) The number of non-zero rows ($= 2$) is less than the number of vectors ($= 3$).

Thus the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly dependent*.



## Rank of a Matrix

The **rank** of matrix $\mathbf{A}$ is the maximal number of linearly independent columns.

We have: $\boxed{\text{rank}(\mathbf{A}^\mathsf{T}) = \text{rank}(\mathbf{A})}$

The rank of an $n \times k$ matrix is at most $\min(n, k)$.

An $n \times n$ matrix is called **regular**, if it has **full rank**, i.e. if $\text{rank}(\mathbf{A}) = n$.



## Rank of a Matrix

***Computation of the rank:***

**(1)** Transform matrix $\mathbf{A}$ into row echelon form by means of Gaussian elimination.

**(2)** Then $\text{rank}(\mathbf{A})$ is given by the number of non-zero rows.

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & -27 \end{pmatrix} \Rightarrow \text{rank}(\mathbf{A}) = 3$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \text{rank}(\mathbf{A}) = 2$$



## Invertible and Regular

An $n \times n$ matrix $\mathbf{A}$ is *invertible*, if and only if it is *regular*.

The following $3 \times 3$ matrix $\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ has full rank (3).

Thus it is regular and hence invertible.

The following $3 \times 3$ matrix $\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix}$ has only rank 2.

Thus it is *not* regular and hence singular (i.e., not invertible).



## Basis

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ **spans** (or *generates*) a vector space $\mathcal{V}$, if

$$\text{span}(\mathbf{v}_1, \dots, \mathbf{v}_d) = \mathcal{V}$$

This set is thus called a **generating set** for the vector space.

If these vectors are *linearly independent*, then this set is called a **basis** of the vector space.

The basis of a vector space is not uniquely determined!

However, the number of vectors in a basis is uniquely determined.
It is called the **dimension** of the vector space.

$$\dim(\mathcal{V}) = d$$

## Characterizations of a Basis

There are several *equivalent* characterizations of a basis.

A basis $B$ of vector space $\mathcal{V}$ is a

- linearly independent generating set of $\mathcal{V}$
- minimal generating set of $\mathcal{V}$
  (i.e., every proper subset of $B$ does not span $\mathcal{V}$)
- maximal linearly independent set
  (i.e., every proper superset of $B$ is linearly dependent)



## Example – Basis

The so called **canonical basis** of the $\mathbb{R}^n$ consists of the $n$ unit vectors:

$$\boxed{B_0 = \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \subset \mathbb{R}^n}$$

Thus we can conclude that

$$\dim(\mathbb{R}^n) = n$$

an that every basis of $\mathbb{R}^n$ consists of $n$ (linearly independent) vectors.

Another basis of the $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$



## Non-Example – Basis

The following are *not* bases of the $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \right\}$$

is not linearly independent (because it has too many vectors).

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

does not span $\mathbb{R}^3$ (because it has too few vectors).

**Beware:** Three vectors need not necessarily form a basis of $\mathbb{R}^3$. They might be linearly dependent.



## Example – Basis

The *canonical basis* of $\mathbb{R}^{2\times 2}$ consists of the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence

$$\dim(\mathbb{R}^{2\times 2}) = 4 \,.$$



## Example – Basis

The simplest basis of vector space $\mathcal{P}_2 = \{\sum_{i=0}^{2} a_i x^i | a_i \in \mathbb{R}\}$ is given by

$$\{1, x, x^2\}$$

and hence

$$\dim(\mathcal{P}_2) = 3 \,.$$



## Coordinates of a Vector

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of vector space $\mathcal{V}$.
Then for every $c_i \in \mathbb{R}$ we get a vector

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$$

On the other hand for a given vector $\mathbf{x}$ we can find (unique) numbers $c_i(\mathbf{x}) \in \mathbb{R}$ such that

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x}) \mathbf{v}_i$$

The numbers $c_i(\mathbf{x})$ are called the **coefficients** of $\mathbf{x}$ w.r.t. basis $B$.
The vector

$$\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}, \dots, c_n(\mathbf{x}))$$

is called the **coefficient vector** of $\mathbf{x}$ w.r.t. basis $B$.



## Space of Coordinate Vectors

For a fixed basis $B$ the coefficient vector $\mathbf{c}(\mathbf{x})$ of $\mathbf{x}$ is unique and

$$\mathbf{c}(\mathbf{x}) \in \mathbb{R}^n = \mathbb{R}^{\dim(\mathcal{V})} \,.$$

So we have a bijection

$$\mathcal{V} \to \mathbb{R}^n, \ \mathbf{x} \mapsto \mathbf{c}(\mathbf{x})$$

with the nice (structure preserving) property

- $\mathbf{c}(\alpha \mathbf{x}) = \alpha \mathbf{c}(\mathbf{x})$
- $\mathbf{c}(\mathbf{x} + \mathbf{y}) = \mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{y})$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$.

That is, instead of dealing with vectors in $\mathcal{V}$ we can fix a basis $B$ and do all computations with coefficient vectors in $\mathbb{R}^n$.

Thus every $n$-dimensional vector space $\mathcal{V}$ is *isomorphic* to (i.e., looks like) an $\mathbb{R}^n$.



## Coordinates of Vectors in $\mathbb{R}^n$

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$.
We obtain the coordinate vector $\mathbf{c}(\mathbf{x})$ of $\mathbf{x} \in \mathbb{R}^n$ w.r.t. $B$ by solving the linear equation

$$c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \dots + c_n \mathbf{v}_n = \mathbf{x} \,.$$

In matrix notation with $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$:

$$\mathbf{V} \cdot \mathbf{c} = \mathbf{x} \qquad \Rightarrow \qquad \boxed{\mathbf{c} = \mathbf{V}^{-1} \mathbf{x}}$$

By construction $\mathbf{V}$ has full rank.

Observe that components $x_1, \dots, x_n$ of vector $\mathbf{x}$ can be seen as its coordinate w.r.t. the canonical basis.

## Example – Coordinate Vector

Compute the coordinates $\mathbf{c}$ of $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

w.r.t. basis $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$

We have to solve equation $\mathbf{V}\mathbf{c} = \mathbf{x}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & -1 \\ 3 & 5 & 6 & 2 \end{array}\right)$$



## Example – Coordinate Vector

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & -1 \\ 3 & 5 & 6 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 3 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Back substitution yields $c_1 = 4$, $c_2 = -8$ and $c_3 = 5$.

The coordinate vector of $\mathbf{x}$ w.r.t. basis $B$ is thus

$$\mathbf{c}(\mathbf{x}) = \begin{pmatrix} 4 \\ -8 \\ 5 \end{pmatrix}$$

Alternatively we could compute $\mathbf{V}^{-1}$ and get as $\mathbf{c} = \mathbf{V}^{-1}\mathbf{x}$.



## Change of Basis

Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be the coordinate vectors of $\mathbf{x} \in \mathcal{V}$ w.r.t. bases $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$, resp.

Consequently $\quad \mathbf{c}_2(\mathbf{x}) = \mathbf{W}^{-1}\mathbf{x} = \mathbf{W}^{-1}\mathbf{V}\mathbf{c}_1(\mathbf{x})$ .

Such a transformation of a coordinate vector w.r.t. one basis into that of another one is called a **change of basis**.

Matrix

$$\boxed{\mathbf{U} = \mathbf{W}^{-1}\mathbf{V}}$$

is called the **transformation matrix** for this change from basis $\mathcal{B}_1$ to $\mathcal{B}_2$.



## Example – Change of Basis

Let

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \right\} \text{ and } B_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$$

two bases of $\mathbb{R}^3$.

Transformation matrix for the change of basis from $B_1$ to $B_2$:
$\mathbf{U} = \mathbf{W}^{-1} \cdot \mathbf{V}$.

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{pmatrix} \quad \Rightarrow \quad \mathbf{W}^{-1} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ 1 & 1 & 6 \end{pmatrix}$$



## Example – Change of Basis

Transformation matrix for the change of basis from $B_1$ to $B_2$:

$$\mathbf{U} = \mathbf{W}^{-1} \cdot \mathbf{V} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ 1 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -7 & 4 \\ -1 & 8 & 0 \\ 0 & -3 & -1 \end{pmatrix}$$

Let $\mathbf{c}_1 = (3, 2, 1)^\mathsf{T}$ be the coordinate vector of $\mathbf{x}$ w.r.t. basis $B_1$.
Then the coordinate vector $\mathbf{c}_2$ w.r.t. basis $B_2$ is given by

$$\mathbf{c}_2 = \mathbf{U}\mathbf{c}_1 = \begin{pmatrix} 2 & -7 & 4 \\ -1 & 8 & 0 \\ 0 & -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 13 \\ -7 \end{pmatrix}$$



## Linear Map

A map $\varphi$ from vector space $\mathcal{V}$ into $\mathcal{W}$

$$\varphi\colon \mathcal{V} \to \mathcal{W},\ \mathbf{x} \mapsto \mathbf{y} = \varphi(\mathbf{x})$$

is called **linear**, if for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$

**(i)** $\varphi(\mathbf{x} + \mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y})$
**(ii)** $\varphi(\alpha\,\mathbf{x}) = \alpha\,\varphi(\mathbf{x})$

We already have seen such a map: $\mathcal{V} \to \mathbb{R}^n,\ \mathbf{x} \mapsto \mathbf{c}(\mathbf{x})$



## Linear Map

Let $\mathbf{A}$ be an $m \times n$ matrix. Then map
$\varphi\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \varphi_\mathbf{A}(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$ is linear:

$$\varphi_\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot \mathbf{y} = \varphi_\mathbf{A}(\mathbf{x}) + \varphi_\mathbf{A}(\mathbf{y})$$
$$\varphi_\mathbf{A}(\alpha\,\mathbf{x}) = \mathbf{A} \cdot (\alpha\,\mathbf{x}) = \alpha\,(\mathbf{A} \cdot \mathbf{x}) = \alpha\,\varphi_\mathbf{A}(\mathbf{x})$$

Vice versa every linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an appropriate $m \times n$ matrix $\mathbf{A}_\varphi$: $\quad \varphi(\mathbf{x}) = \mathbf{A}_\varphi\,\mathbf{x}$.

Matrices represent all possible linear maps $\mathbb{R}^n \to \mathbb{R}^m$.

More generally they represent linear maps between any vector space once we have bases for these and do all computations with their coordinate vectors.

In this sense, *matrices "are" linear maps*.



## Geometric Interpretation of Linear Maps

We have the following "elementary" maps:

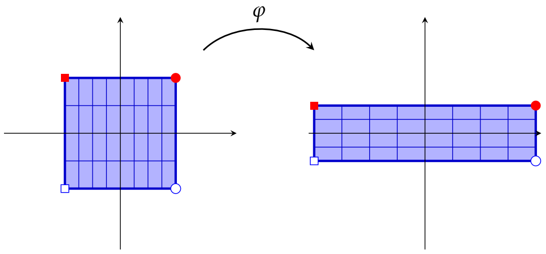
- *lengthening* / *shortening* in some direction
- *shear* in some direction
- *projection* into a subspace
- *rotation*
- *reflection* at a subspace

These maps can be combined into more complex ones.

## Lengthening / Shortening

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x}$

lengthens the $x$-coordinate by factor 2 and
shortens the $y$-coordinate by factor $\frac{1}{2}$.

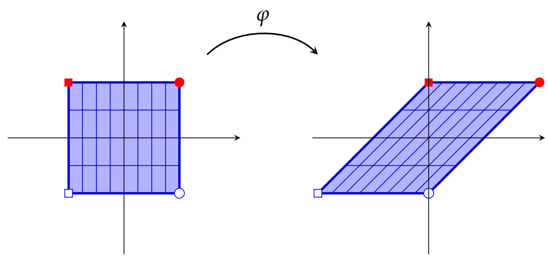





## Shear

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

shears the rectangle into the $x$-coordinate.

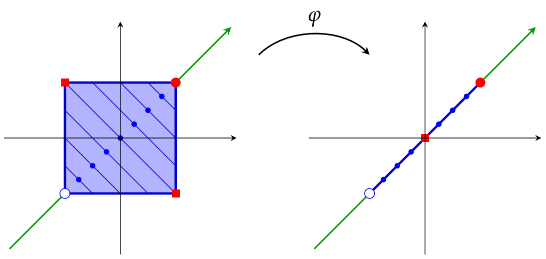





## Projection

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}$

projects a point $\mathbf{x}$ orthogonally into the subspace generated by vector $(1,1)^\mathsf{T}$, i.e., span $\left((1,1)^\mathsf{T}\right)$.

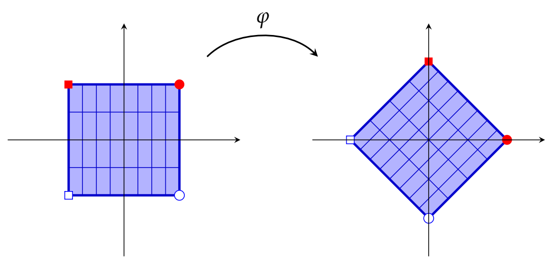





## Rotation

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \mathbf{x}$

rotates a point $\mathbf{x}$ clock-wise by 45° around the origin.

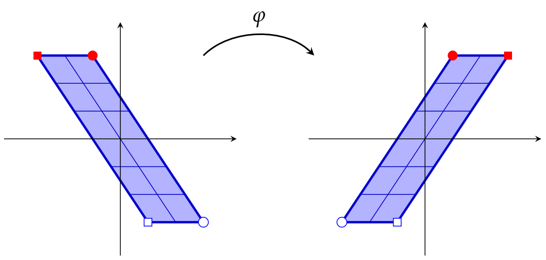





## Reflection

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

reflects a point $\mathbf{x}$ at the $y$-axis.






## Image and Kernel

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \varphi(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$ be a linear map.

The **image** of $\varphi$ is a subspace of $\mathbb{R}^m$.

$$\boxed{\operatorname{Im}(\varphi) = \{\varphi(\mathbf{v})\colon \mathbf{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m}$$

The **kernel** (or **null space**) of $\varphi$ is a subspace of $\mathbb{R}^n$.

$$\boxed{\operatorname{Ker}(\varphi) = \{\mathbf{v} \in \mathbb{R}^n\colon \varphi(\mathbf{v}) = 0\} \subseteq \mathbb{R}^n}$$

The kernel is the preimage of $0$.

*Image* $\operatorname{Im}(\mathbf{A})$ and *kernel* $\operatorname{Ker}(\mathbf{A})$ of a matrix $\mathbf{A}$ are the respective image and kernel of the corresponding linear map.



## Generating Set of the Image

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$, $\mathbf{x} \in \mathbb{R}^n$ an arbitrary vector, and $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

We can write $\mathbf{x}$ as a linear combination of the canonical basis:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \, \mathbf{e}_i$$

Recall that $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$.
So we can write $\varphi(\mathbf{x})$ as a linear combination of the columns of $\mathbf{A}$:

$$\varphi(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x} = \mathbf{A} \cdot \sum_{i=1}^{n} x_i \, \mathbf{e}_i = \sum_{i=1}^{n} x_i \, \mathbf{A}\mathbf{e}_i = \sum_{i=1}^{n} x_i \, \mathbf{a}_i$$

That is, the columns $\mathbf{a}_i$ of $\mathbf{A}$ span (generate) $\operatorname{Im}(\varphi)$.



## Basis of the Kernel

Let $\mathbf{A} = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

If $\mathbf{y}, \mathbf{z} \in \operatorname{Ker}(\varphi)$ and $\alpha, \beta \in \mathbb{R}$, then

$$\varphi(\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha\varphi(\mathbf{y}) + \beta\varphi(\mathbf{z}) = \alpha 0 + \beta 0 = 0$$

Thus $\operatorname{Ker}(\varphi)$ is closed under linear combination,
i.e., $\operatorname{Ker}(\varphi)$ is a subspace.

We obtain a basis of $\operatorname{Ker}(\varphi)$ by solving the homogeneous linear equation $\mathbf{A} \cdot \mathbf{x} = 0$ by means of Gaussian elimination.

## Dimension of Image and Kernel

Rank-nullity theorem:

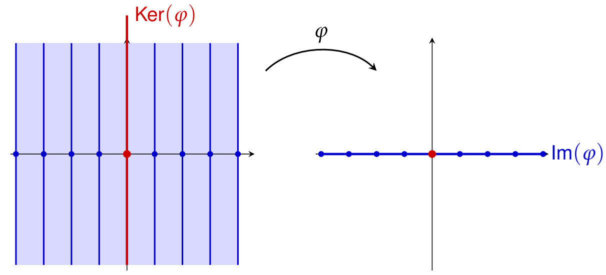
$$\boxed{\dim \mathcal{V} = \dim \mathsf{Im}(\varphi) + \dim \mathsf{Ker}(\varphi)}$$



## Example – Dimension of Image and Kern

Map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{x} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x}$

projects a point $\mathbf{x}$ orthogonally onto the $x$ axis.




$\dim \mathbb{R}^2 = 2$, $\dim \mathsf{Ker}(\varphi) = 1$ $\qquad \dim \mathsf{Im}(\varphi) = 1$



## Linear Map and Rank

The *rank* of matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ is (per definition) the dimension of $\mathsf{span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

Hence it is the dimension of the image of the corresponding linear map.

$$\boxed{\dim \mathsf{Im}(\varphi_{\mathbf{A}}) = \mathsf{rank}(\mathbf{A})}$$

The dimension of the solution set $\mathcal{L}$ of a homogeneous linear equation $\mathbf{A}\,\mathbf{x} = 0$ is then the kernel of this map.

$$\dim \mathcal{L} = \dim \mathsf{Ker}(\varphi_{\mathbf{A}}) = \dim \mathbb{R}^n - \dim \mathsf{Im}(\varphi_{\mathbf{A}}) = n - \mathsf{rank}(\mathbf{A})$$



## Matrix Multiplication

By *multiplying* two matrices $\mathbf{A}$ and $\mathbf{B}$ we obtain the matrix of a *compound* linear map:

$$(\varphi_{\mathbf{A}} \circ \varphi_{\mathbf{B}})(\mathbf{x}) = \varphi_{\mathbf{A}}(\varphi_{\mathbf{B}}(\mathbf{x})) = \mathbf{A}\,(\mathbf{B}\,\mathbf{x}) = (\mathbf{A} \cdot \mathbf{B})\,\mathbf{x}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\mathbf{B}"] \arrow[rr, bend left, "\mathbf{AB}"] & \mathbb{R}^m \arrow[r, "\mathbf{A}"] & \mathbb{R}^k \\
\mathbf{x} \arrow[r, mapsto] & \mathbf{B}\mathbf{x} \arrow[r, mapsto] & \mathbf{AB}\mathbf{x}
\end{tikzcd}$$

This point of view implies:

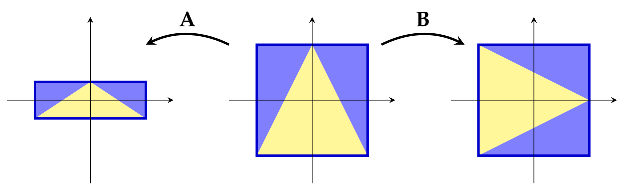
$$\boxed{\mathsf{rank}(\mathbf{A} \cdot \mathbf{B}) \le \min\{\mathsf{rank}(\mathbf{A}), \mathsf{rank}(\mathbf{B})\}}$$



## Non-Commutative Matrix Multiplication

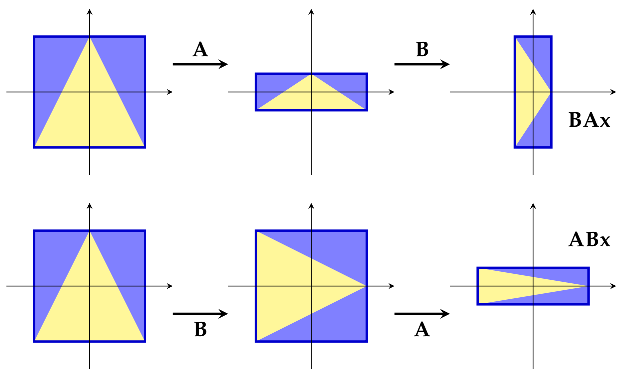
$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ represents a shortening of the $y$-coordinate.

$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a clock-wise rotation about 90°.






## Non-Commutative Matrix Multiplication






## Inverse Matrix

The *inverse matrix* $\mathbf{A}^{-1}$ of $\mathbf{A}$ exists if and only if map $\varphi_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\,\mathbf{x}$ is one-to-one and onto, i.e., if and only if

$$\varphi_{\mathbf{A}}(\mathbf{x}) = x_1\,\mathbf{a}_1 + \dots + x_n\,\mathbf{a}_n = 0 \quad \Leftrightarrow \quad \mathbf{x} = 0$$

i.e., if and only if$\mathbf{A}$ is *regular*.

From this point of view implies $(\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\mathbf{B}"] \arrow[rr, bend left, "\mathbf{AB}"] & \mathbb{R}^m \arrow[r, "\mathbf{A}"] & \mathbb{R}^k \\
\mathbf{x} \arrow[r, mapsto] & \mathbf{B}\mathbf{x} \arrow[r, mapsto] & \mathbf{AB}\mathbf{x} \\
\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{z} & \mathbf{A}^{-1}\mathbf{z} \arrow[l, mapsto] & \mathbf{z} \arrow[l, mapsto] \arrow[ll, bend left, "\mathbf{B}^{-1}\mathbf{A}^{-1}"]
\end{tikzcd}$$



## Similar Matrices

The basis of a vector space and thus the coordinates of a vector are not uniquely determined. Matrix $\mathbf{A}_\varphi$ of a linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ also depends on the chosen bases.
Let $\mathbf{A}$ be the matrix w.r.t. basis $B_1$.
Which matrix represents linear map $\varphi$ if we use basis $B_2$ instead?

$$\begin{tikzcd}
\text{basis } B_1 & \mathbf{U}\,\mathbf{x} \arrow[r, "\mathbf{A}"] & \mathbf{A}\,\mathbf{U}\,\mathbf{x} \arrow[d, "\mathbf{U}^{-1}"] \\
\text{basis } B_2 & \mathbf{x} \arrow[u, "\mathbf{U}"] \arrow[r, "\mathbf{C}"] & \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}\,\mathbf{x}
\end{tikzcd}$$

and thus $\quad \mathbf{C}\,\mathbf{x} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}\,\mathbf{x}$

Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{C}$ are called **similar**, if there exists a regular matrix $\mathbf{U}$ such that

$$\boxed{\mathbf{C} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}}$$

## Summary

- vector space and subspace
- linear independency and rank
- basis and dimension
- coordinate vector
- change of basis
- linear map
- image and kernel
- similar matrices



Chapter 5

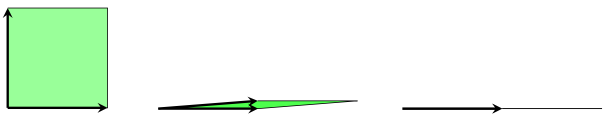
# Determinant



## What is a Determinant?

We want to *"compute"* whether $n$ vectors in $\mathbb{R}^n$ are linearly dependent and *measure* "how far" they are from being linearly dependent, resp.

**Idea:**
Two vectors in $\mathbb{R}^2$ span a parallelogram:



vectors are linearly *dependent* $\Leftrightarrow$ area is zero

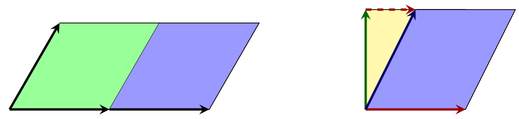
We use the *$n$-dimensional volume* of the created parallelepiped for our function that "measures" linear dependency.



## Properties of a Volume

We define our function indirectly by the properties of this volume.

- Multiplication of a vector by a scalar $\alpha$ yields the $\alpha$-fold volume.
- Adding some vector to another one does not change the volume.
- If two vectors coincide, then the volume is zero.
- The volume of a unit cube is one.





## Determinant

The **determinant** is a function which maps an $n \times n$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ into a real number $\det(\mathbf{A})$ with the following properties:

**(D1)** The determinant is **linear** in each column:

$$\det(\dots, \mathbf{a}_i + \mathbf{b}_i, \dots) = \det(\dots, \mathbf{a}_i, \dots) + \det(\dots, \mathbf{b}_i, \dots)$$
$$\det(\dots, \alpha\, \mathbf{a}_i, \dots) = \alpha \det(\dots, \mathbf{a}_i, \dots)$$

**(D2)** The determinant is zero, if two columns coincide:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_i, \dots) = 0$$

**(D3)** The determinant is **normalized**:

$$\det(\mathbf{I}) = 1$$

Notations: $\boxed{\det(\mathbf{A}) = |\mathbf{A}|}$



## Example – Properties

**(D1)**

$$\begin{vmatrix} 1 & 2+10 & 3 \\ 4 & 5+11 & 6 \\ 7 & 8+12 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 10 & 3 \\ 4 & 11 & 6 \\ 7 & 12 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3\cdot 2 & 3 \\ 4 & 3\cdot 5 & 6 \\ 7 & 3\cdot 8 & 9 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

**(D2)**

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 7 & 8 & 7 \end{vmatrix} = 0$$



## Determinant – Remarks

- Properties (D1)–(D3) define a function uniquely.
  (I.e., such a function does exist and two functions with these properties are identical.)
- The determinant as defined above can be negative.
  So it can be seen as "signed volume".
- We derive more properties of the determinant below.
- Take care about the notation:
  Do not mix up $|\mathbf{A}|$ with the absolute value of a number $|x|$.
- The determinant is a so called *normalized alternating multi-linear form.*



## Further Properties

**(D4)** The determinant is **alternating**:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_k, \dots) = -\det(\dots, \mathbf{a}_k, \dots, \mathbf{a}_i, \dots)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix}$$

## Further Properties

**(D5)** The determinant does not change if we add some multiple of a column to another column:

$$\det(\ldots,\mathbf{a}_i+\alpha\,\mathbf{a}_k,\ldots,\mathbf{a}_k,\ldots)=\det(\ldots,\mathbf{a}_i,\ldots,\mathbf{a}_k,\ldots)$$

$$\begin{vmatrix} 1 & 2+2\cdot 1 & 3 \\ 4 & 5+2\cdot 4 & 6 \\ 7 & 8+2\cdot 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$



## Further Properties

**(D6)** The determinant does not change if we *transpose* a matrix:

$$\boxed{\det(\mathbf{A}^\mathsf{T})=\det(\mathbf{A})}$$

Consequently,
all statements about columns hold analogously for rows.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$



## Further Properties

**(D7)** $\det(\mathbf{A})\neq 0$ $\Leftrightarrow$ columns (rows) of $\mathbf{A}$ are linearly independent
$\Leftrightarrow$ $\mathbf{A}$ ist regular
$\Leftrightarrow$ $\mathbf{A}$ ist invertible

**(D8)** The determinant of the product of two matrices is equal to the product of their determinants:

$$\boxed{\det(\mathbf{A}\cdot\mathbf{B})=\det(\mathbf{A})\cdot\det(\mathbf{B})}$$

**(D9)** The determinant if the inverse matrix is equal to the reciprocal of the determinant of the matrix:

$$\boxed{\det(\mathbf{A}^{-1})=\frac{1}{\det(\mathbf{A})}}$$



## Further Properties

**(10)** The determinant of a triangular matrix is the product of its diagonal elements:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\ 0 & a_{22} & a_{23} & \ldots & a_{2n} \\ 0 & 0 & a_{33} & \ldots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{nn} \end{vmatrix} = a_{11}\cdot a_{22}\cdot a_{33}\cdot\ldots\cdot a_{nn}$$

**(11)** The absolute value of the determinant $|\det(\mathbf{a}_1,\ldots,\mathbf{a}_n)|$ is the volume of the parallelepiped spanned by the column vectors $\mathbf{a}_1,\ldots,\mathbf{a}_n$.



## 2 × 2 Matrix

$$\boxed{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}\cdot a_{22}-a_{12}\cdot a_{21}}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4-2\cdot 3=-2$$



## 3 × 3 Matrix: Sarrus' Rule

$$\left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}\right|\begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix} = \begin{matrix} 1\cdot 5\cdot 9+2\cdot 6\cdot 7+3\cdot 4\cdot 8 \\ -7\cdot 5\cdot 3-8\cdot 6\cdot 1-9\cdot 4\cdot 2=0 \end{matrix}$$



## Source of Error

Determinants of $4\times 4$ matrices must be computed by means of transformation into a triangular matrix or by Laplace expansion.

There is no such thing like Sarrus' rule for $4\times 4$ matrices.



## Transform into Triangular Matrix

**(1)** Transform into upper triangular matrix similar to Gaussian elimination.

- Add a multiple of a row to another row. (D5)
- Multiply a row by some scalar $\alpha\neq 0$ *and* the determinant by the reciprocal $\frac{1}{\alpha}$. (D1)
- Exchange two rows *and* switch the *sign* of the determinant. (D4)

**(2)** Compute the determinant as the product of its diagonal elements. (Property D10)

## Example – Transform into Triangular Matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot (-3) \cdot 0 = 0$$

$$\begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -\frac{1}{2} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{vmatrix} = -\frac{1}{2} \cdot 1 \cdot 2 \cdot (-4) = 4$$



## Laplace Expansion

*Laplace expansion* along the $k$-th column and $i$-th row, resp.:

$$\boxed{\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik} = \sum_{k=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik}}$$

where $M_{ik}$ is the *determinant* of the $(n-1) \times (n-1)$ submatrix which we obtain by *deleting* the $i$-th row and the $k$-th column of $\mathbf{A}$. It is called a **minor** of $\mathbf{A}$.

We get the signs $(-1)^{i+k}$ by means of a chessboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$



## Expansion along the First Row

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3)$$

$$= 0$$



## Expansion along the Second Column

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}$$

$$= -2 \cdot (-6) + 5 \cdot (-12) - 8 \cdot (-6)$$

$$= 0$$



## Laplace and Leibzig Formula

Laplace expansion allows to compute the determinant *recursively*:

The deterimant of a $k \times k$ matrix is expanded into a sum of $k$ determinants of $(k-1) \times (k-1)$ matrices.

For an $n \times n$ matrices we can repeat this recursion step $n$ times and yield a summation of $n!$ products of $n$ numbers each:

$$\det(\mathbf{A}) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

where $\mathcal{S}_n$ is the permutation group of order $n$.

This formula is shown here just for completeness.
Its explanation is out of the scope of this course.



## Adjugate Matrix

In Laplace expansion the factors $\boxed{A_{ik} = (-1)^{i+k} M_{ik}}$ are called the **cofactors** of $a_{ik}$:

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot A_{ik}$$

The matrix formed by these cofactors is called the **cofactor matrix** $\mathbf{A}^*$.

Its transpose $\mathbf{A}^{*\mathsf{T}}$ is called the **adjugate** of $\mathbf{A}$.

$$\operatorname{adj}(\mathbf{A}) = \mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$



## Product $\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}}$

$$\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = |\mathbf{A}| \cdot \mathbf{I}$$



## Product $\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}}$

Product of the $k$-th row of $\mathbf{A}^{*\mathsf{T}}$ by the $j$-th column of $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$:

$$\left[\mathbf{A}^{*\mathsf{T}} \cdot \mathbf{A}\right]_{kj} = \sum_{i=1}^{n} A_{ik} \cdot a_{ij} = \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+k} M_{ik}$$

[expansion along $k$-th column] $\quad = \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{a}_j}_{k\text{-th column}}, \dots, \mathbf{a}_n)$

$$= \begin{cases} |\mathbf{A}| & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

$$= |\mathbf{A}| \delta_{kj}$$

Hence $\quad \mathbf{A}^{*\mathsf{T}} \cdot \mathbf{A} = |\mathbf{A}| \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathsf{T}}$

## Cramer's Rule for the Inverse Matrix

We get a formula for the inverse of Matrix $\mathbf{A}$:

$$\boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathsf{T}}}$$

This formula is not practical for inverting a matrix ...

... except for $2 \times 2$ matrices where it is very convenient:

$$\boxed{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$



## Example – Inverse Matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix}^{-1} = \frac{1}{4} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & 2 & -\frac{1}{2} \\ 0 & -2 & 1 \\ \frac{1}{4} & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$



## Cramer's Rule for Linear Equations

We want to solve linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

If $\mathbf{A}$ is regular (i.e., $|\mathbf{A}| \neq 0$), then we find

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{|\mathbf{A}|} \mathbf{A}^{*\mathsf{T}} \cdot \mathbf{b}$$

So we get for $x_k$

$$\begin{aligned} x_k &= \frac{1}{|\mathbf{A}|} \sum_{i=1}^{n} A_{ik} \cdot b_i = \frac{1}{|\mathbf{A}|} \sum_{i=1}^{n} b_i \cdot (-1)^{i+k} M_{ik} \\ &= \frac{1}{|\mathbf{A}|} \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{b}}_{k\text{-th column}}, \dots, \mathbf{a}_n) \end{aligned}$$



## Cramer's Rule for Linear Equations

Let $\mathbf{A}_k$ be the matrix where the $k$-th column of $\mathbf{A}$ is replaced by $\mathbf{b}$.

If $\mathbf{A}$ is an invertible matrix, then the solution of

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

is given by

$$\boxed{\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ \vdots \\ |\mathbf{A}_n| \end{pmatrix}}$$

This procedure does not work if $\mathbf{A}$ is not regular.



## Example – Cramer's Rule

Compute the solution of equation

$$\begin{pmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution:

$$\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ |\mathbf{A}_2| \\ |\mathbf{A}_3| \end{pmatrix} = \begin{pmatrix} 12 \\ -22 \\ 45 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 1$$

$$|\mathbf{A}_1| = \begin{vmatrix} 1 & 11 & 3 \\ 2 & 13 & 4 \\ 3 & 3 & 1 \end{vmatrix} = 12$$

$$|\mathbf{A}_2| = \begin{vmatrix} 9 & 1 & 3 \\ 9 & 2 & 4 \\ 2 & 3 & 1 \end{vmatrix} = -22$$

$$|\mathbf{A}_3| = \begin{vmatrix} 9 & 11 & 1 \\ 9 & 13 & 2 \\ 2 & 3 & 3 \end{vmatrix} = 45$$



## Summary

- definition of determinant
- properties
- relation between determinant and regularity
- volume of a parallelepiped
- compuation of the determinant (Sarrus' rule, transformation into triangular matrix)
- Laplace expansion
- Cramer's rule



Chapter 6

# Eigenvalues



## Closed Leontief Model

In a closed Leontief input-output-model consumption and production coincide, i.e.,

$$\mathbf{V} \cdot \mathbf{x} = \mathbf{x} = 1 \cdot \mathbf{x}$$

Is this possible for the given technology matrix $\mathbf{V}$?

This is a special case of a so called **eigenvalue problem**.

## Eigenvalue and Eigenvector

A vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, is called **eigenvector** of an $n \times n$ matrix $\mathbf{A}$ corresponding to **eigenvalue** $\lambda \in \mathbb{R}$, if

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}}$$

The eigenvalues of matrix $\mathbf{A}$ are all numbers $\lambda$ for which an eigenvector does exist.



## Example – Eigenvalue and Eigenvector

For a $3 \times 3$ diagonal matrix we find

$$\mathbf{A} \cdot \mathbf{e}_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \\ 0 \end{pmatrix} = a_{11} \cdot \mathbf{e}_1$$

Thus $\mathbf{e}_1$ is a eigenvector corresponding to eigenvalue $\lambda = a_{11}$.

Analogously we find for an $n \times n$ diagonal matrix

$$\mathbf{A} \cdot \mathbf{e}_i = a_{ii} \cdot \mathbf{e}_i$$

So the eigenvalue of a diagonal matrix are its diagonal elements with unit vectors $\mathbf{e}_i$ as the corresponding eigenvectors.



## Computation of Eigenvalues

In order to find eigenvectors of an $n \times n$ matrix $\mathbf{A}$ we have to solve equation

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} \quad \Leftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0\,.$$

If $(\mathbf{A} - \lambda\mathbf{I})$ is invertible then we get

$$\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})^{-1}0 = 0\,.$$

However, $\mathbf{x} = 0$ cannot be an eigenvector (by definition) and hence $\lambda$ cannot be an eigenvalue.

Thus $\lambda$ is an *eigenvalue* of $\mathbf{A}$ if and only if $(\mathbf{A} - \lambda\mathbf{I})$ is *not invertible*, i.e., if and only if

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0}$$



## Example – Eigenvalues

Compute the eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

We have to find all $\lambda \in \mathbb{R}$ where $|\mathbf{A} - \lambda\mathbf{I}|$ vanishes.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| =$$

$$\left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0.$$

The roots of this quadratic equation are

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3.$$

Thus matrix $\mathbf{A}$ has eigenvalues 2 and 3.



## Characteristic Polynomial

For an $n \times n$ matrix $\mathbf{A}$

$$\det(\mathbf{A} - \lambda\mathbf{I})$$

is a polynomial of degree $n$ in $\lambda$.
It is called the **characteristic polynomial** of matrix $\mathbf{A}$.

The eigenvalues are then the roots of the characteristic polynomial.

For that reason eigenvalues and eigenvectors are sometimes called the *characteristic roots* and *characteristic vectors*, resp., of $\mathbf{A}$.

The set of all eigenvalues of $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$.
*Spectral methods* make use of eigenvalues.

**Remark:**
It may happen that characteristic roots are complex ($\lambda \in \mathbb{C}$).
These are then called *complex eigenvalues*.



## Computation of Eigenvectors

Eigenvectors $\mathbf{x}$ corresponding to a *known* eigenvalue $\lambda_0$ can be computed by solving linear equation $(\mathbf{A} - \lambda_0\mathbf{I})\mathbf{x} = 0$.

Eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ corresponding to $\lambda_1 = 2$:

$$(\mathbf{A} - \lambda_1\mathbf{I})\mathbf{x} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gaussian elimination yields: $x_2 = \alpha$ and $x_1 = -2\alpha$

$$\mathbf{v}_1 = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{for an } \alpha \in \mathbb{R} \setminus \{0\}.$$



## Eigenspace

If $\mathbf{x}$ is an eigenvector corresponding to eigenvalue $\lambda$, then each multiple $\alpha\mathbf{x}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\alpha\mathbf{x}) = \alpha(\mathbf{A} \cdot \mathbf{x}) = \alpha\lambda \cdot \mathbf{x} = \lambda \cdot (\alpha\mathbf{x})$$

If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to the same eigenvalue $\lambda$, then $\mathbf{x} + \mathbf{y}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot \mathbf{y} = \lambda \cdot \mathbf{x} + \lambda \cdot \mathbf{y} = \lambda \cdot (\mathbf{x} + \mathbf{y})$$

The set of all eigenvectors corresponding to eigenvalue $\lambda$ (including zero vector 0) is thus a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** corresponding to $\lambda$.

Computer programs return *bases of eigenspaces*.
(Beware: Bases are not uniquely determined!)



## Example – Eigenspace

Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda_1 = 2$: $\quad \mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Eigenvector corresponding to eigenvalue $\lambda_2 = 3$: $\quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue $\lambda_i$ are all non-vanishing (i.e., non-zero) multiples of $\mathbf{v}_i$.

Computer programs often return normalized eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

## Example

Eigenvalues and Eigenvectors of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}$.

Create the characteristic polynomial and compute its roots:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 6 & 2-\lambda \end{vmatrix} = (2-\lambda)\cdot\lambda\cdot(\lambda-5) = 0$$

Eigenvalues:

$\lambda_1 = 2,\ \lambda_2 = 0,$ and $\lambda_3 = 5$ .



## Example

Eigenvector(s) corresponding to eigenvalue $\lambda_3 = 5$:

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = \begin{pmatrix} (2-5) & 0 & 1 \\ 0 & (3-5) & 1 \\ 0 & 6 & (2-5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Gaussian elimination yields

$$\left(\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $x_3 = \alpha$, $x_2 = \frac{1}{2}\alpha$, and $x_1 = \frac{1}{3}\alpha$ for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$.
Eigenvector $\mathbf{v}_3 = (2,3,6)^\mathsf{T}$.



## Example

Eigenvector corresponding to

- $\lambda_1 = 2$: $\quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\lambda_2 = 0$: $\quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix}$
- $\lambda_3 = 5$: $\quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue $\lambda_i$ are all non-vanishing multiples of $\mathbf{v}_i$.



## Properties of Eigenvalues

1. $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$ have the same eigenvalues.
2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$-matrices.
   Then $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ have the same eigenvalues.
3. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to $\lambda$,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue $\lambda^k$.
4. If $\mathbf{x}$ is an eigenvector of regular matrix $\mathbf{A}$ corresponding to $\lambda$,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue $\frac{1}{\lambda}$.



## Properties of Eigenvalues

5. The product of all eigenvalues $\lambda_i$ of an $n \times n$ matrix $\mathbf{A}$ is equal to the determinant of $\mathbf{A}$:

   $$\boxed{\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i}$$

   This implies:
   $\mathbf{A}$ is regular if and only if all its eigenvalues are non-zero.

6. The sum of all eigenvalues $\lambda_i$ of an $n \times n$ matrix $\mathbf{A}$ is equal to the sum of the diagonal elements of $\mathbf{A}$ (called the **trace** of $\mathbf{A}$).

   $$\boxed{\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i}$$



## Eigenvalues of Similar Matrices

Let $\mathbf{U}$ be a transformation matrix and $\mathbf{C} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}$.
If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue $\lambda$,
then $\mathbf{U}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{C}$ corresponding to $\lambda$:

$$\mathbf{C} \cdot (\mathbf{U}^{-1}\mathbf{x}) = (\mathbf{U}^{-1}\mathbf{A}\mathbf{U})\mathbf{U}^{-1}\mathbf{x} = \mathbf{U}^{-1}\mathbf{A}\mathbf{x} = \mathbf{U}^{-1}\lambda\mathbf{x} = \lambda \cdot (\mathbf{U}^{-1}\mathbf{x})$$

*Similar* matrices $\mathbf{A}$ and $\mathbf{C}$ have the same eigenvalues and (if we consider change of basis) the same eigenvectors.

We want to choose a basis such that the matrix that represents the given linear map becomes as simple as possible.
The simplest matrices are *diagonal matrices*.

Can we find a basis where the corresponding linear map is represented by a diagonal matrix?

Unfortunately not in the general case. But . . .



## Symmetric Matrix

An $n \times n$ matrix $\mathbf{A}$ is called **symmetric**, if $\boxed{\mathbf{A}^\mathsf{T} = \mathbf{A}}$.

For a *symmetric* matrix $\mathbf{A}$ we find:

- All $n$ eigenvalues are real.
- Eigenvectors $\mathbf{u}_i$ corresponding to distinct eigenvalues $\lambda_i$ are *orthogonal* (i.e., $\mathbf{u}_i^\mathsf{T} \cdot \mathbf{u}_j = 0$ if $i \neq j$).
- There exists an **orthonormal basis** $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ (i.e. the vectors $\mathbf{u}_i$ are normalized and mutually orthogonal) that consists of eigenvectors of $\mathbf{A}$,

Matrix $\mathbf{U} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ is then an **orthogonal matrix**:

$$\boxed{\mathbf{U}^\mathsf{T} \cdot \mathbf{U} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{U}^{-1} = \mathbf{U}^\mathsf{T}}$$



## Diagonalization

For the $i$-th unit vector $\mathbf{e}_i$ we find

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} \cdot \mathbf{e}_i = \mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{u}_i = \mathbf{U}^\mathsf{T}\,\lambda_i\,\mathbf{u}_i = \lambda_i\,\mathbf{U}^\mathsf{T}\,\mathbf{u}_i = \lambda_i \cdot \mathbf{e}_i$$

and thus

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{pmatrix}$$

Every symmetric matrix $\mathbf{A}$ becomes a diagonal matrix with the eigenvalues of $\mathbf{A}$ as its entries if we use the orthonormal basis of eigenvectors.

This procedure is called **diagonalization** of matrix $\mathbf{A}$.

## Example – Diagonalization

We want to diagonalize $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3$$

with respective normalized eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

With respect to basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ matrix $\mathbf{A}$ becomes diagonal matrix

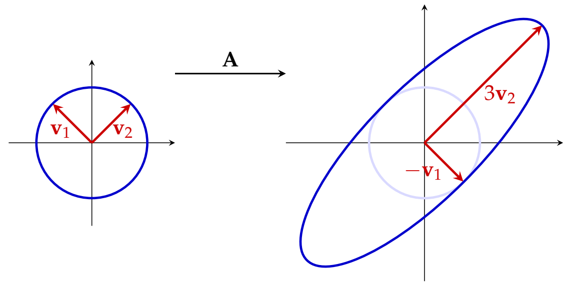
$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$



## A Geometric Interpretation I

Function $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ maps the unit circle in $\mathbb{R}^2$ into an ellipsis.

The two semi-axes of the ellipsis are given by $\lambda_1 \mathbf{v}_1$ and $\lambda_2 \mathbf{v}_2$, resp.






## Quadratic Form

Let $\mathbf{A}$ be a *symmetric matrix*. Then function

$$\boxed{q_{\mathbf{A}} \colon \mathbb{R}^n \to \mathbb{R},\ \mathbf{x} \mapsto q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{x}}$$

is called a **quadratic form**.

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then

$$q_{\mathbf{A}}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 2\,x_2^2 + 3\,x_3^2$$



## Example – Quadratic Form

In general we find for $n \times n$ matrix $\mathbf{A} = (a_{ij})$:

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\, x_i x_j$$

$$\begin{aligned} q_{\mathbf{A}}(\mathbf{x}) &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} x_1 + x_2 - 2x_3 \\ x_1 + 2x_2 + 3x_3 \\ -2x_1 + 3x_2 + x_3 \end{pmatrix} \\ &= x_1^2 + 2x_1x_2 - 4x_1x_3 + 2x_2^2 + 6x_2x_3 + x_3^2 \end{aligned}$$



## Definiteness

A quadratic form $q_{\mathbf{A}}$ is called

- **positive definite**, if for all $\mathbf{x} \neq 0$, $\quad q_{\mathbf{A}}(\mathbf{x}) > 0$.
- **positive semidefinite**, if for all $\mathbf{x}$, $\quad q_{\mathbf{A}}(\mathbf{x}) \geq 0$.
- **negative definite**, if for all $\mathbf{x} \neq 0$, $\quad q_{\mathbf{A}}(\mathbf{x}) < 0$.
- **negative semidefinite**, if for all $\mathbf{x}$, $\quad q_{\mathbf{A}}(\mathbf{x}) \leq 0$.
- **indefinite** in all other cases.

A matrix $\mathbf{A}$ is called *positive* (negative) *definite* (semidefinite), if the corresponding quadratic form is *positive* (negative) *definite* (semidefinite).



## Definiteness

Every symmetric matrix is *diagonalizable*. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be the orthonormal basis of eigenvectors of $\mathbf{A}$. Then for every $\mathbf{x}$:

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x})\mathbf{u}_i = \mathbf{U}\mathbf{c}(\mathbf{x})$$

$\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is the transformation matrix for the orthonormal basis, $\mathbf{c}$ the corresponding coefficient vector.

So if $\mathbf{D}$ is the diagonal matrix of eigenvalues $\lambda_i$ of $\mathbf{A}$ we find

$$q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{x} = (\mathbf{U}\mathbf{c})^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{U}\mathbf{c} = \mathbf{c}^\mathsf{T} \cdot \mathbf{U}^\mathsf{T}\mathbf{A}\mathbf{U} \cdot \mathbf{c} = \mathbf{c}^\mathsf{T} \cdot \mathbf{D} \cdot \mathbf{c}$$

and thus

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} c_i^2(\mathbf{x})\lambda_i$$



## Definiteness and Eigenvalues

Equation $q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} c_i^2(\mathbf{x})\lambda_i$ immediately implies:

Let $\lambda_i$ be the eigenvalues of symmetric matrix $\mathbf{A}$.
Then $\mathbf{A}$ (the quadratic form $q_{\mathbf{A}}$) is

- *positive definite*, if all $\lambda_i > 0$.
- *positive semidefinite*, if all $\lambda_i \geq 0$.
- *negative definite*, if all $\lambda_i < 0$.
- *negative semidefinite*, if all $\lambda_i \leq 0$.
- *indefinite*, if there exist $\lambda_i > 0$ and $\lambda_j < 0$.



## Example – Definiteness and Eigenvalues

- The eigenvalues of $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ are $\lambda_1 = 6$ and $\lambda_2 = 1$.
  Thus the matrix is positive definite.
- The eigenvalues of $\begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$ are
  $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 9$. The matrix is positive semidefinite.
- The eigenvalues of $\begin{pmatrix} 7 & -5 & 4 \\ -5 & 7 & 4 \\ 4 & 4 & -2 \end{pmatrix}$ are
  $\lambda_1 = -6$, $\lambda_2 = 6$ and $\lambda_3 = 12$. Thus the matrix is indefinite.

## Leading Principle Minors

The definiteness of a matrix can also be determined by means of minors.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$\boxed{A_k = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}}$$

is called the $k$-th **leading principle minor** of $\mathbf{A}$.



## Leading Principle Minors and Definiteness

A symmetric Matrix $\mathbf{A}$ is

- *positive definite*, if and only if all $A_k > 0$.
- *negative definite*, if and only if $(-1)^k A_k > 0$ for all $k$.
- *indefinite*, if $|\mathbf{A}| \neq 0$ and none of the two cases holds.

$(-1)^k A_k > 0$ means that

- $A_1, A_3, A_5, \ldots < 0$, and
- $A_2, A_4, A_6, \ldots > 0$.



## Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 2 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 \quad > 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are positive definite.



## Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 1 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = -28 \quad < 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are indefinite.



## Principle Minors

Unfortunately the condition for semidefiniteness is more tedious.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$\boxed{A_{i_1,\ldots,i_k} = \begin{vmatrix} a_{i_1,i_1} & \dots & a_{i_1,i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k,i_1} & \dots & a_{i_k,i_k} \end{vmatrix} \qquad 1 \leq i_1 < \ldots < i_k \leq n.}$$

is called a **principle minor** of order $k$ of $\mathbf{A}$.



## Principle Minors and Semidefiniteness

A symmetric matrix $\mathbf{A}$ is

- *positive semidefinite*, if and only if all $A_{i_1,\ldots,i_k} \geq 0$.
- *negative semidefinite*, if and only if $(-1)^k A_{i_1,\ldots,i_k} \geq 0$ for all $k$.
- *indefinite* in all other cases.

$(-1)^k A_{i_1,\ldots,i_k} \geq 0$ means that

- $A_{i_1,\ldots,i_k} \geq 0$, if $k$ is even, and
- $A_{i_1,\ldots,i_k} \leq 0$, if $k$ is odd.



## Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$$

The matrix is positive semidefinite.

(But not positive definite!)

principle minors of order 1:

$A_1 = 5 \quad \geq 0 \qquad A_2 = 2 \quad \geq 0$

$A_3 = 5 \quad \geq 0$

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} 5 & -1 \\ -1 & 2 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{1,3} = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{2,3} = \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 9 \quad \geq 0$$

principle minors of order 3:

$A_{1,2,3} = |\mathbf{A}| = 0 \quad \geq 0$



## Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} -5 & 1 & -4 \\ 1 & -2 & -1 \\ -4 & -1 & -5 \end{pmatrix}$$

The matrix is negative semidefinite.

(But not negative definite!)

principle minors of order 1:

$A_1 = -5 \quad \leq 0 \qquad A_2 = -2 \quad \leq 0$

$A_3 = -5 \quad \leq 0$

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{1,3} = \begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{2,3} = \begin{vmatrix} -2 & -1 \\ -1 & -5 \end{vmatrix} = 9 \quad \geq 0$$

principle minors of order 3:

$A_{1,2,3} = |\mathbf{A}| = 0 \quad \leq 0$

## Leading Principle Minors and Semidefiniteness

Obviously every positive definite matrix is also positive semidefinite (but not necessarily the other way round).

Matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is positive definite as all leading principle minors are positive (see above).

Therefore $\mathbf{A}$ is also positive semidefinite.

In this case there is no need to compute the non-leading principle minors.



## Recipe for Semidefiniteness

Recipe for finding semidefiniteness of matrix $\mathbf{A}$:

1. Compute all *leading principle minors*:
   - **If** the condition for positive definiteness holds, then
     $\mathbf{A}$ is *positive definite* and thus positive semidefinite.
   - Else **if** the condition for negative definiteness holds, then
     $\mathbf{A}$ is *negative definite* and thus negative semidefinite.
   - Else **if** $\det(\mathbf{A}) \neq 0$, then
     $\mathbf{A}$ is *indefinite*.
2. Else also compute all *non-leading principle minors*:
   - **If** the condition for positive semidefiniteness holds, then
     $\mathbf{A}$ is *positive semidefinite*.
   - Else **if** the condition for negative semidefiniteness holds, then
     $\mathbf{A}$ is *negative semidefinite*.
   - **Else**
     $\mathbf{A}$ is *indefinite*.

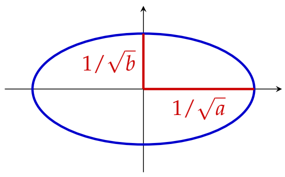


## Ellipse

Equation

$$ax^2 + by^2 = 1\,, \qquad a, b > 0$$

describes an *ellipse* in **canonical form**.




The semi-axes have length $\frac{1}{\sqrt{a}}$ and $\frac{1}{\sqrt{b}}$, resp.



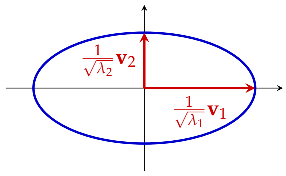
## A Geometric Interpretation II

Term $ax^2 + by^2$ is a quadratic form with matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

It has eigenvalues and *normalized* eigenvectors

$$\lambda_1 = a \text{ with } \mathbf{v}_1 = \mathbf{e}_1 \qquad \text{and} \qquad \lambda_2 = b \text{ with } \mathbf{v}_2 = \mathbf{e}_2\,.$$




These eigenvectors coincide with the semi-axes of the ellipse.

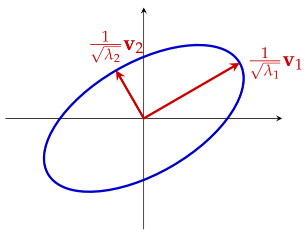


## A Geometric Interpretation II

Now let $\mathbf{A}$ be a symmetric $2 \times 2$ matrix with *positive* eigenvalues. Equation

$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 1$$

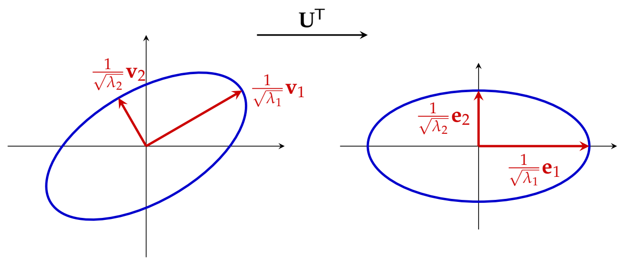
describes an *ellipse* where the semi-axes (*principle axes*) coincide with the *normalized* eigenvectors of $\mathbf{A}$.






## A Geometric Interpretation II

By a change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ using transformation $\mathbf{U} = (\mathbf{v}_1, \mathbf{v}_2)$ this ellipse is rotated into canonical form.




(That is, we diagonalize matrix $\mathbf{A}$.)



## An Application in Statistics

Suppose we have $n$ observations of $k$ metric attributes $X_1, \dots, X_k$ which we combine into a vector:

$$\mathbf{x}_i = (x_{i1}, \dots, x_{ik}) \in \mathbb{R}^k$$

The arithmetic mean then is given by

$$\bar{\mathbf{x}} = \frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i = (\bar{x}_1, \dots, \bar{x}_k)$$

The *total sum of squares* is a measure for the statistical dispersion

$$\mathrm{TSS} = \sum_{i=1}^{n} \|\mathbf{x}_i - \bar{\mathbf{x}}\|^2 = \sum_{j=1}^{k}\left(\sum_{i=1}^{n} |x_{ij} - \bar{x}_j|^2\right) = \sum_{j=1}^{k} \mathrm{TSS}_j$$

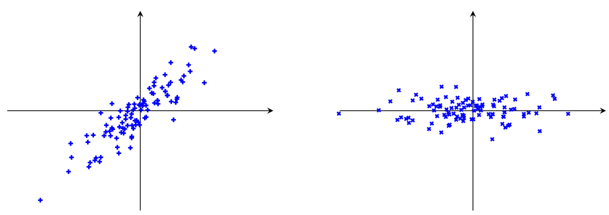
It can be computed component-wise.



## An Application in Statistics

A change of basis by means of an *orthogonal* matrix does not change TSS.

However, it changes the contributions of each of the components.



Can we find a basis such that a few components contribute much more to the TSS than the remaining ones?

## Principle Component Analysis (PCA)

**Assumptions:**

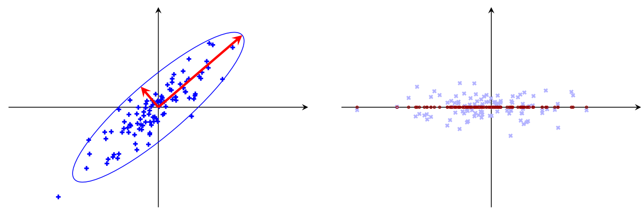
- The data are approximately *multinormal* distributed.

**Procedure:**

1. Compute the covariance matrix $\boldsymbol{\Sigma}$.
2. Compute all eigenvalues and normalized eigenvectors of $\boldsymbol{\Sigma}$.
3. Sort eigenvalues such that
$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \,.$$
4. Use corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ as new basis.
5. The contribution of the first $m$ components in this basis to TSS is
$$\frac{\sum_{j=1}^{m} \mathrm{TSS}j}{\sum_{j=1}^{k} \mathrm{TSS}j} \approx \frac{\sum_{j=1}^{m} \lambda_j}{\sum_{j=1}^{k} \lambda_j} \,.$$



## Principle Component Analysis (PCA)



By means of PCA it is possible to reduce the number of dimensions without reducing the TSS substantially.



## Summary

- eigenvalues and eigenvectors
- characteristic polynomial
- eigenspace
- properties of eigenvalues
- symmetric matrices and diagonalization
- quadratic forms
- definitness
- principle minors
- principle component analysis



Chapter 7

# Real Functions



## Real Function*

**Real functions** are maps where both *domain* and *codomain* are (unions of) intervals in $\mathbb{R}$.

Often only function terms are given but neither domain nor codomain. Then domain and codomain are implicitly given as following:

- *Domain* of the function is the largest *sensible* subset of the domain of the function terms (i.e., where the terms are defined).
- *Codomain* is the image (range) of the function
$$f(D) = \{y \mid y = f(x) \text{ for ein } x \in D_f\} \,.$$



## Implicit Domain*

Production function $f(x) = \sqrt{x}$ is an abbreviation for
$$f\colon [0,\infty) \to [0,\infty),\ x \mapsto f(x) = \sqrt{x}$$

(There are no negative amounts of goods.
Moreover, $\sqrt{x}$ is not real for $x < 0$.)

Its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is an abbreviation for
$$f'\colon (0,\infty) \to (0,\infty),\ x \mapsto f'(x) = \frac{1}{2\sqrt{x}}$$

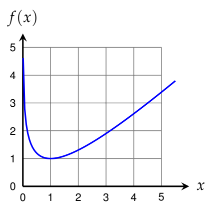
(Note the open interval $(0,\infty)$; $\frac{1}{2\sqrt{x}}$ is not defined for $x = 0$.)



## Graph of a Function*

Each tuple $(x, f(x))$ corresponds to a point in the $xy$-plane.
The set of all these points forms a curve called the **graph** of function $f$.
$$\mathcal{G}_f = \{(x,y) \mid x \in D_f,\ y = f(x)\}$$

Graphs can be used to visualize functions.
They allow to detect many properties of the given function.




$$f(x) = x - \ln(x)$$

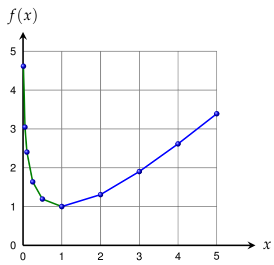


## How to Draw a Graph*

1. Get an idea about the possible shape of the graph. One should be able to sketch graphs of elementary functions by heart.
2. Find an appropriate range for the $x$-axis.
(It should show a characteristic detail of the graph.)
3. Create a table of function values and draw the corresponding points into the $xy$-plane.

   If known, use characteristic points like local extrema or inflection points.
4. Check if the curve can be constructed from the drawn points.
If not add adapted points to your table of function values.
5. Fit the curve of the graph through given points in a proper way.

## Example – How to Draw a Graph*




Graph of function

$$f(x) = x - \ln x$$

Table of values:

| $x$ | $f(x)$ |
|---|---|
| 0 | ERROR |
| 1 | 1 |
| 2 | 1.307 |
| 3 | 1.901 |
| 4 | 2.614 |
| 5 | 3.391 |
| 0.5 | 1.193 |
| 0.25 | 1.636 |
| 0.1 | 2.403 |
| 0.05 | 3.046 |



## Sources of Errors

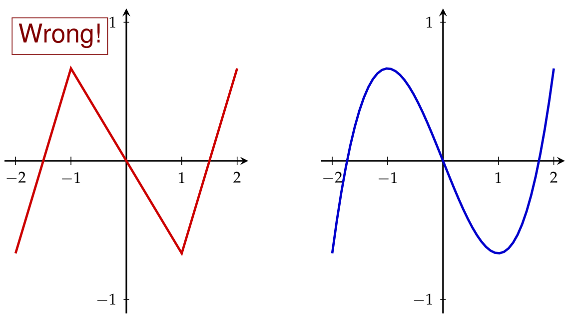
Most frequent errors when drawing function graphs:

- **Table of values is too small:**
  It is not possible to construct the curve from the computed function values.
- **Important points are ignored:**
  Ideally extrema and inflection points should be known and used.
- **Range for $x$ and $y$-axes not suitable:**
  The graph is tiny or important details vanish in the "noise" of handwritten lines (or pixel size in case of a computer program).



## Sources of Errors*

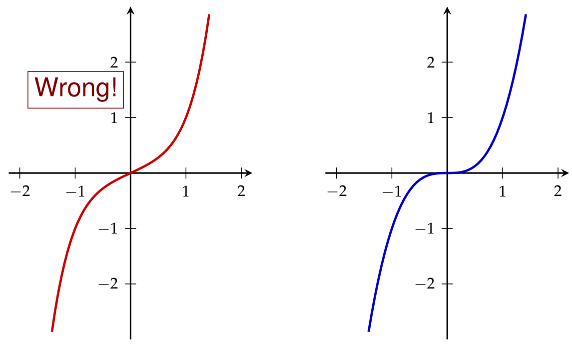
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:






## Sources of Errors*

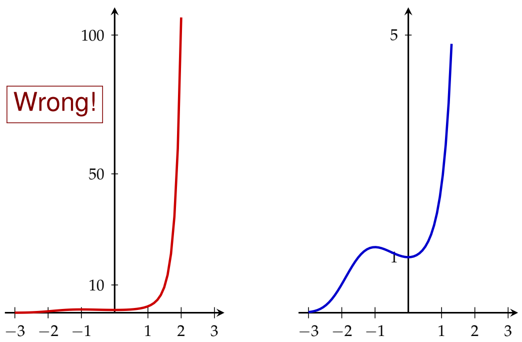
Graph of $f(x) = x^3$ has slope $0$ in $x = 0$:






## Sources of Errors*

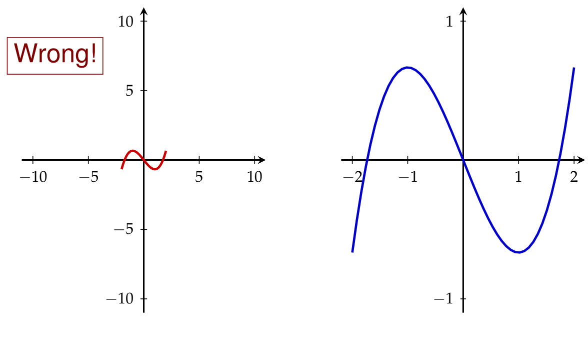
Function $f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$ has a local maximum in $x = -1$:






## Sources of Errors*

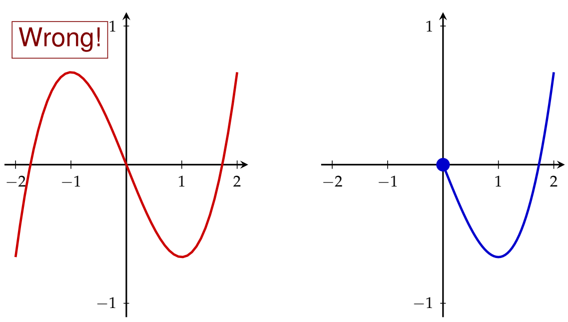
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:






## Sources of Errors*

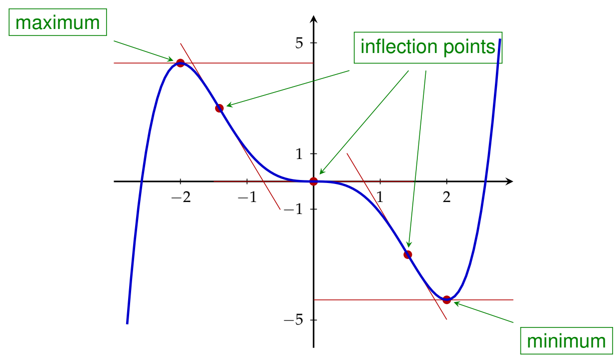
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[0, 2]$: (not in $[-2, 2]$!)






## Extrema and Inflection Points*

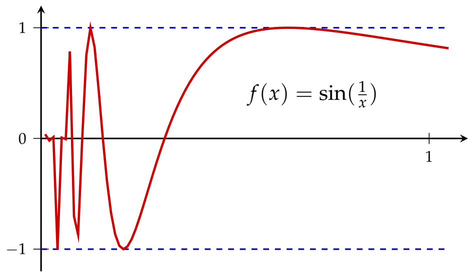
Graph of function $f(x) = \frac{1}{15}(3x^5 - 20x^3)$:



## Sources of Errors

It is *important* that one already has an *idea of the shape* of the function graph **before** drawing the curve.

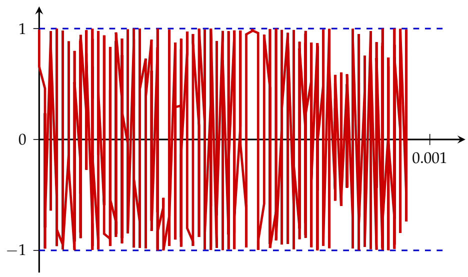
Even a graph drawn by means of a computer program can differ significantly from the true curve.






## Sources of Errors*

$$f(x) = \sin(\tfrac{1}{x})$$






## Sketch of a Function Graph*

Often a **sketch** of the graph is sufficient. Then the exact function values are not so important. Axes may not have scales.

*However*, it is important that the sketch clearly shows all characteristic details of the graph (like extrema or important function values).

Sketches can also be drawn like a caricature:
They stress prominent parts and properties of the function.

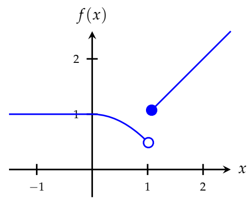


## Piece-wise Defined Functions*

The function term can be defined differently in subintervals of the domain.

At the boundary points of these subintervals we have to mark which points belong to the graph and which do not:

● (belongs) and ○ (does not belong).




$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1, \\ x, & \text{for } x \geq 1. \end{cases}$$



## Bijectivity*

Recall that each argument has exactly one image and that the number of preimages of an element in the codomain can vary.
Thus we can characterize maps by their possible number of preimages.

- A map $f$ is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Also recall that a function has an *inverse* if and only if it is one-to-one and onto (i.e., *bijective*).



## A Simple Horizontal Test*

How can we determine whether a real function is one-to-one or onto?
I.e., how many preimage may a $y \in W_f$ have?

**(1)** Draw the graph of the given function.

**(2)** Mark some $y \in W$ on the $y$-axis and draw a line parallel to the $x$-axis (*horizontal*) through this point.

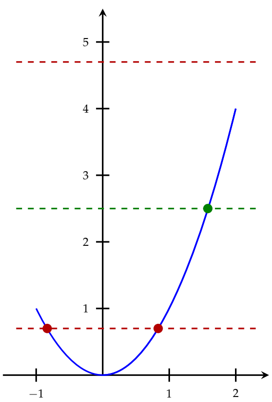
**(3)** The number of intersection points of horizontal line and graph coincides with the number of preimages of $y$.

**(4)** Repeat Steps (2) and (3) for a *representative* set of $y$-values.

**(5)** Interpretation: If all horizontal lines intersect the graph in
 **(a)** *at most one* point, then $f$ is **one-to-one**;
 **(b)** *at least one* point, then $f$ is **onto**;
 **(c)** *exactly one* point, then $f$ is **bijective**.



## Example – Horizontal Test*




$f\colon [-1, 2] \to \mathbb{R},\ x \mapsto x^2$

- is not one-to-one;
- is not onto.

$f\colon [0, 2] \to \mathbb{R},\ x \mapsto x^2$

- is one-to-one;
- is not onto.

$f\colon [0, 2] \to [0, 4],\ x \mapsto x^2$

- is one-to-one and onto.

**Beware!** *Domain* and *codomain* are part of the function!



## Function Composition*

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.

$$\boxed{g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))}$$

is called **composite function**.
(read: "$g$ composed with $f$", "$g$ circle $f$", or "$g$ after $f$")

Let $g\colon \mathbb{R} \to [0, \infty),\ x \mapsto g(x) = x^2,$
$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = 3x - 2.$

Then $(g \circ f)\colon \mathbb{R} \to [0, \infty),$
$x \mapsto (g \circ f)(x) = g(f(x)) = g(3x - 2) = (3x - 2)^2$

and $(f \circ g)\colon \mathbb{R} \to \mathbb{R},$
$x \mapsto (f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 2$

## Inverse Function*

If $f\colon D_f \to W_f$ is a **bijection**, then there exists a so called **inverse function**

$$\boxed{f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)}$$

with the property

$$\boxed{f^{-1} \circ f = \mathrm{id} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}}$$

We get the function term of the inverse by *interchanging* the roles of *argument* $x$ and *image* $y$.



## Example – Inverse Function*

We get the term for the inverse function by expressing $x$ as function of $y$

We need the inverse function of

$$y = f(x) = 2x - 1$$

By rearranging we obtain

$$y = 2x - 1 \quad \Leftrightarrow \quad y + 1 = 2x \quad \Leftrightarrow \quad \frac{1}{2}(y+1) = x$$

Thus the term of the inverse function is $f^{-1}(y) = \frac{1}{2}(y+1)$.

Arguments are usually denoted by $x$. So we write

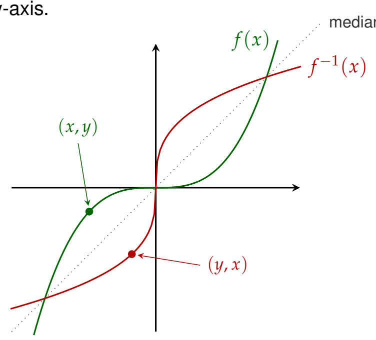
$$f^{-1}(x) = \frac{1}{2}(x+1)\ .$$

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.



## Geometric Interpretation*

Interchanging of $x$ and $y$ corresponds to reflection across the median between $x$ and $y$-axis.




(Graph of function $f(x) = x^3$ and its inverse.)

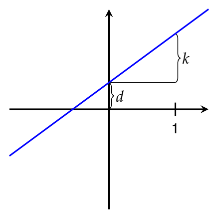


## Linear Function and Absolute Value*

- **Linear function**

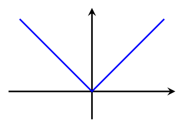
  $$\boxed{f(x) = k\,x + d}$$

  $k$ … **slope**
  $d$ … **intercept**




- **Absolute value** (or *modulus*)

  $$\boxed{f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}}$$

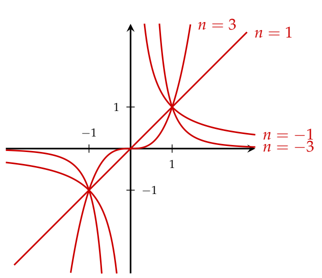


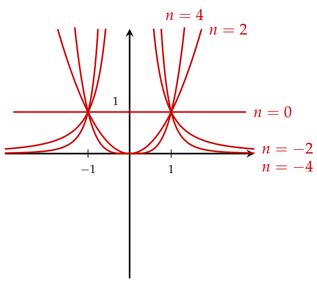


## Power Function*

**Power function** with *integer* exponents:

$$\boxed{f\colon x \mapsto x^n, \quad n \in \mathbb{Z}} \qquad D = \begin{cases} \mathbb{R} & \text{for } n \geq 0 \\ \mathbb{R} \setminus \{0\} & \text{for } n < 0 \end{cases}$$






## Calculation Rule for Powers and Roots*

| | | |
|---|---|---|
| $x^{-n} = \dfrac{1}{x^n}$ | $x^0 = 1$ | $(x \neq 0)$ |
| $x^{n+m} = x^n \cdot x^m$ | $x^{\frac{1}{m}} = \sqrt[m]{x}$ | $(x \geq 0)$ |
| $x^{n-m} = \dfrac{x^n}{x^m}$ | $x^{\frac{n}{m}} = \sqrt[m]{x^n}$ | $(x \geq 0)$ |
| $(x \cdot y)^n = x^n \cdot y^n$ | $x^{-\frac{n}{m}} = \dfrac{1}{\sqrt[m]{x^n}}$ | $(x \geq 0)$ |
| $(x^n)^m = x^{n \cdot m}$ | | |

**Important!**

$0^0$ is *not* defined!



## Sources of Errors

**Important!**

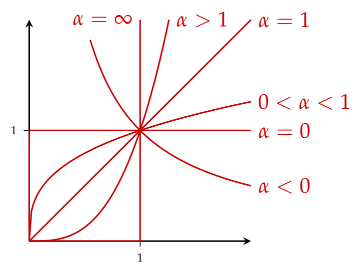
- $-x^2$ is **not** equal to $(-x)^2$ !
- $(x+y)^n$ is **not** equal to $x^n + y^n$ !
- $x^n + y^n$ can**not** be simplified (in general)!



## Power Function*

**Power function** with *real* exponents:

$$\boxed{f\colon x \mapsto x^\alpha \qquad \alpha \in \mathbb{R}} \qquad D = \begin{cases} [0, \infty) & \text{for } \alpha \geq 0 \\ (0, \infty) & \text{for } \alpha < 0 \end{cases}$$



## Polynomial and Rational Functions*

- **Polynomial** of degree $n$:

$$\boxed{f(x) = \sum_{k=0}^{n} a_k x^k}$$

$a_i \in \mathbb{R}$, for $i = 1, \dots, n$, $\quad a_n \neq 0$.

- **Rational Function:**

$$\boxed{D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}}$$

$p(x)$ and $q(x)$ are polynomials
$D = \mathbb{R} \setminus \{$roots of $q\}$



## Calculation Rule for Fractions and Rational Terms*

Let $b, c, e \neq 0$.

| | |
|---|---|
| $\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$ | *Reduce* |
| $\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$ | *Expand* |
| $\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$ | *Multiplying* |
| $\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e}$ | *Dividing* |
| $\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e}$ | *Compound fraction* |



## Calculation Rule for Fractions and Rational Terms*

Let $b, c \neq 0$.

| | |
|---|---|
| $\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$ | *Addition with common denominator* |
| $\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$ | *Addition* |

**Very important!** *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!



## Sources of Errors

**Very Important!** *Really!*

| | | |
|---|---|---|
| $\frac{a+c}{b+c}$ | is **not** equal to | $\frac{a}{b}$ |
| $\frac{x}{a} + \frac{y}{b}$ | is **not** equal to | $\frac{x+y}{a+b}$ |
| $\frac{a}{b+c}$ | is **not** equal to | $\frac{a}{b} + \frac{a}{c}$ |

$$\frac{x+2}{y+2} \neq \frac{x}{y} \qquad \frac{1}{2} + \frac{1}{3} \neq \frac{1}{5}.$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2} + \frac{1}{y^2}$$



## Exponential Function*

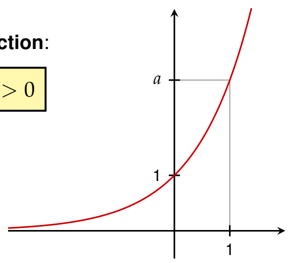
- **Exponential function**:

$$\boxed{\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \exp(x) = e^x}$$

$e = 2{,}7182818\dots$ Euler's number

- **Generalized exponential function**:

$$\boxed{\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto a^x \qquad a > 0}$$






## Logarithm Function*

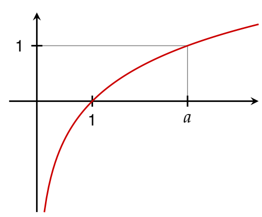
- **Logarithm**:

  Inverse function of *exponential function*.

$$\boxed{\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log(x) = \ln(x)}$$

- **Generalized Logarithm** to basis $a$:

$$\boxed{\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log_a(x)}$$






## Exponent and Logarithm*

A number $y$ is called the **logarithm** to basis $a$, if $a^y = x$.
The logarithm is the *exponent of a number to basis* $a$.
We write

$$y = \log_a(x) \quad \Leftrightarrow \quad x = a^y$$

Important logarithms:

- **natural logarithm** $\ln(x)$ with basis $e = 2.7182818\dots$
  (sometimes called *Euler's number*)
- **common logarithm** $\lg(x)$ with basis $10$
  (sometimes called decadic or decimal logarithm)



## Calculations with Exponent and Logarithm*

**Conversation formula:**

$$\boxed{a^x = e^{x \ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}}$$

**Important:**
Often one can see $\log(x)$ without a basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- In *mathematics*: *natural* logarithm
  financial mathematics, programs like R, *Mathematica*, Maxima, ...
- In *applied sciences*: *common* logarithm
  economics, pocket calculator, Excel, ...

## Calculation Rules for Exponent and Logarithm*

| | |
|---|---|
| $a^{x+y} = a^x \cdot a^y$ | $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$ |
| $a^{x-y} = \frac{a^x}{a^y}$ | $\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$ |
| $(a^x)^y = a^{x \cdot y}$ | $\log_a(x^\beta) = \beta \cdot \log_a(x)$ |
| $(a \cdot b)^x = a^x \cdot b^x$ | |
| $a^{\log_a(x)} = x$ | $\log_a(a^x) = x$ |
| $a^0 = 1$ | $\log_a(1) = 0$ |

$\log_a(x)$ has (as real-valued function) domain $x > 0$!



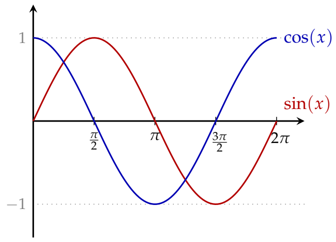
## Trigonometric Functions*

- **Sine**:

$$\boxed{\mathbb{R} \to [-1,1],\ x \mapsto \sin(x)}$$

- **Cosine**:

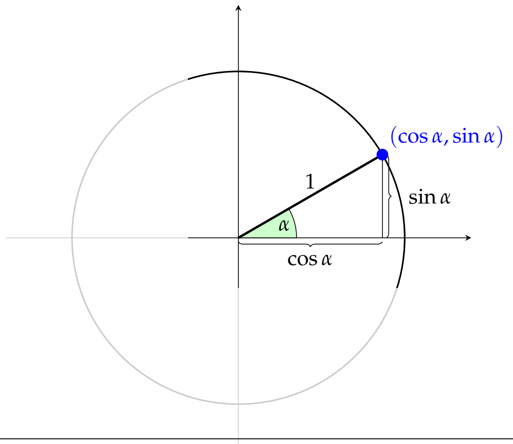
$$\boxed{\mathbb{R} \to [-1,1],\ x \mapsto \cos(x)}$$




**Beware!**
These functions use **radian** for their arguments, i.e., angles are measured by means of the length of arcs on the unit circle and not by degrees. A right angle then corresponds to $x = \pi/2$.



## Sine and Cosine*






## Sine and Cosine*

**Important formulas:**

*Periodic*: For all $k \in \mathbb{Z}$,

$$\boxed{\begin{aligned} \sin(x + 2k\pi) &= \sin(x) \\ \cos(x + 2k\pi) &= \cos(x) \end{aligned}}$$

Relation between sin and cos:

$$\boxed{\sin^2(x) + \cos^2(x) = 1}$$



## Multivariate Functions*

A **function of several variables** (or **multivariate function**) is a function with more than one argument which evaluates to a real number.

$$\boxed{f\colon \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)}$$

Arguments $x_i$ are the **variables** of function $f$.

$$f(x,y) = \exp(-x^2 - 2y^2)$$

is a bivariate function in variables $x$ and $y$.

$$p(x_1, x_2, x_3) = x_1^2 + x_1 x_2 - x_2^2 + 5x_1 x_3 - 2x_2 x_3$$

is a function in the three variables $x_1$, $x_2$, and $x_3$.



## Graphs of Bivariate Functions*

Bivariate functions (i.e., of *two* variables) can be visualized by its graph:

$$\boxed{\mathcal{G}_f = \{(x,y,z) \mid z = f(x,y) \text{ for } x,y \in \mathbb{R}\}}$$

It can be seen as the two dimensional *surface* of a three dimensional landscape.

The notion of *graph* exists analogously for functions of three or more variables.

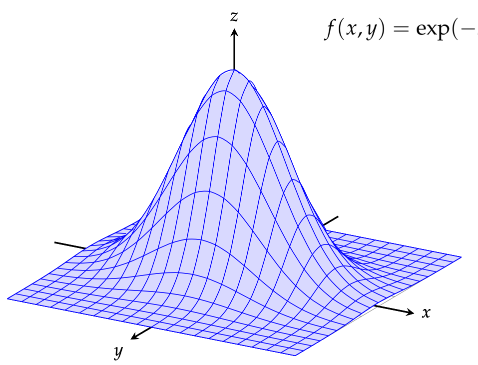
$$\mathcal{G}_f = \{(\mathbf{x}, y) \mid y = f(\mathbf{x}) \text{ for an } \mathbf{x} \in \mathbb{R}^n\}$$

However, it can hardly be used to visualize such functions.



## Graphs of Bivariate Functions

$$f(x,y) = \exp(-x^2 - 2y^2)$$






## Contour Lines of Bivariate Functions*

Let $c \in \mathbb{R}$ be fixed. Then the set of all points $(x,y)$ in the real plane with $f(x,y) = c$ is called **contour line** of function $f$.

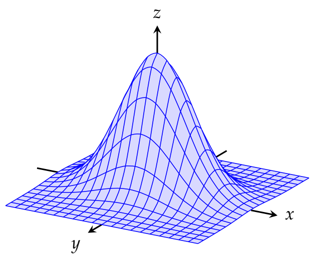
Obviously function $f$ is constant on each of its contour lines.

Other names:

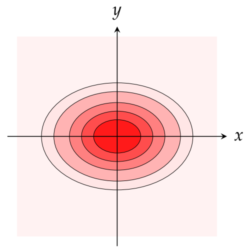
- *Indifference curve*
- *Isoquants*
- *Level set* (is a generalization of a contour line for functions of any number of variables.)

A collection of contour lines can be seen as a kind of “hiking map” for the “landscape” of the function.

## Contour Lines of Bivariate Functions*




graph




contour lines

$$f(x, y) = \exp(-x^2 - 2y^2)$$



## Indifference Curves*

Indifference curves are determined by an equation

$$F(x, y) = 0$$

We can (try to) draw such curves by expressing one of the variables as function of the other one
(i.e., solve the equation w.r.t. one of the two variables).

So we may get an univariate function. The graph of this function coincides with the indifference curve.

We then draw the graph of this univariate function by the method described above.



## Example – Cobb-Douglas-Function*

We want to draw indifference curve

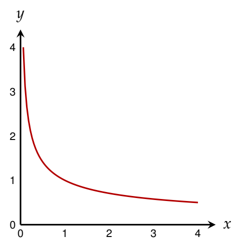
$$x^{\frac{1}{3}} y^{\frac{2}{3}} = 1 , \qquad x, y > 0.$$

Expressing $x$ by $y$ yields:

$$x = \frac{1}{y^2}$$

Alternatively we can express $y$ by $x$:

$$y = \frac{1}{\sqrt{x}}$$






## Example – CES-Function*

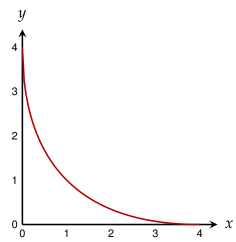
We want to draw indifference curve

$$\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 = 4 , \qquad x, y > 0.$$

Expressing $x$ by $y$ yields:

$$y = \left(2 - x^{\frac{1}{2}}\right)^2$$

(Take care about the domain of this curve!)





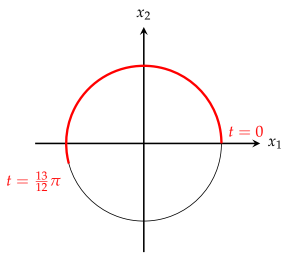

## Paths*

A function

$$\boxed{s\colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto s(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}}$$

is called a **path** in $\mathbb{R}^n$.
Variable $t$ is often interpreted as *time*.

$$[0, \infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$






## Vector-valued Functions*

Generalized vector-valued function:

$$\boxed{\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1, \dots, x_n) \\ \vdots \\ f_m(x_1, \dots, x_n) \end{pmatrix}}$$

- Univariate functions:
  $\mathbb{R} \to \mathbb{R},\ x \mapsto y = x^2$
- Multivariate functions:
  $\mathbb{R}^2 \to \mathbb{R},\ \mathbf{x} \mapsto y = x_1^2 + x_2^2$
- Paths:
  $[0, 1) \to \mathbb{R}^n,\ s \mapsto (s, s^2)^\mathsf{T}$
- Linear maps:
  $\mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x}$ $\qquad \mathbf{A} \dots\ m \times n$-Matrix



## Summary

- real functions
- implicit domain
- graph of a function
- sources of errors
- piece-wise defined functions
- one-to-one and onto
- function composition
- inverse function
- elementary functions
- multivariate functions
- paths
- vector-valued functions



Chapter 8

# Limits

## Sequences*

A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$\boxed{a\colon \mathbb{N} \to \mathbb{R},\ n \mapsto a_n}$$

Sequences are denoted by $(a_n)_{n=1}^{\infty}$ or just $(a_n)$ for short.

An alternative notation used in literature is $\langle a_n \rangle_{n=1}^{\infty}$.



## Sequences*

Sequences can be defined

- by **enumerating** of its terms,
- by a **formula**, or
- by **recursion**.
  Each term is determined by its predecessor(s).

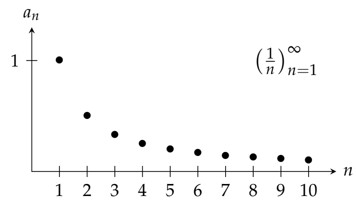
| | |
|---|---|
| Enumeration: | $(a_n) = (1, 3, 5, 7, 9, \dots)$ |
| Formula: | $(a_n) = (2n - 1)$ |
| Recursion: | $a_1 = 1,\ \ a_{n+1} = a_n + 2$ |



## Graphical Representation*

A sequence $(a_n)$ can by represented

**(1)** by drawing tuples $(n, a_n)$ in the plane, or




**(2)** by drawing points on the number line.




## Properties*

**Properties** of a sequence $(a_n)$:

| Property | Definition | |
|---|---|---|
| monotonically increasing | $a_{n+1} \geq a_n$ | for all $n \in \mathbb{N}$ |
| monotonically decreasing | $a_{n+1} \leq a_n$ | |
| alternating | $a_{n+1} \cdot a_n < 0$ | i.e. the sign changes |
| bounded | $\lvert a_n \rvert \leq M$ | for some $M \in \mathbb{R}$ |

Sequence $\left(\frac{1}{n}\right)$ is

- monotonically decreasing; and
- bounded, as for all $n \in \mathbb{N}$, $|a_n| = |1/n| \leq M = 1$;
  (we could also choose $M = 1000$)
- but *not* alternating.



## Limit of a Sequence*

Consider the following sequence of numbers

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\right)$$


The terms of this sequence *tend* to $0$ with increasing $n$.
We say that sequence $(a_n)$ **converges** to $0$.

We write

$$\boxed{(a_n) \to 0 \quad \text{or} \quad \lim_{n\to\infty} a_n = 0}$$

(read: "limit of $a_n$ for $n$ tends to $\infty$")



## Limit of a Sequence / Definition*

**Definition:**
A number $a \in \mathbb{R}$ is a **limit** of sequence $(a_n)$, if *for every interval* $(a - \varepsilon, a + \varepsilon)$ there *exists an* $N$ such that $a_n \in (a - \varepsilon, a + \varepsilon)$ for all $n \geq N$; i.e., all terms following $a_N$ are contained in this interval.

**Equivalent Definition:** A sequence $(a_n)$ converges to **limit** $a \in \mathbb{R}$ if *for every* $\varepsilon > 0$ there *exists an* $N$ such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

[Mathematicians like to use $\varepsilon$ for a very small positive number.]

A sequence that has a *limit* is called **convergent**.
It **converges** to its limit.

It can be shown that a limit of a sequence is *unique*ly defined *(if it exists)*.

A sequence *without* a limit is called **divergent**.



## Limit of a Sequence / Example*

Sequence

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\right)$$

has limit $a = 0$.

For example, if we set $\varepsilon = 0.3$, then all terms following $a_4$ are contained in interval $(a - \varepsilon, a + \varepsilon)$.

If we set $\varepsilon = \frac{1}{1\,000\,000}$, then all terms starting with the 1 000 001-st term are contained in the interval.

Thus

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0\,.$$



## Limit of a Sequence / Example*

Sequence $(a_n)_{n=1}^{\infty} = \left(\frac{1}{2^n}\right)_{n=1}^{\infty} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\right)$ converges to $0$:

$$\lim_{n\to\infty} a_n = 0$$

Sequence $(b_n)_{n=1}^{\infty} = \left(\frac{n-1}{n+1}\right)_{n=1}^{\infty} = \left(0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \dots\right)$ is convergent:

$$\lim_{n\to\infty} b_n = 1$$

Sequence $(c_n)_{n=1}^{\infty} = \left((-1)^n\right)_{n=1}^{\infty} = (-1, 1, -1, 1, -1, 1, \dots)$
is divergent.

Sequence $(d_n)_{n=1}^{\infty} = \left(2^n\right)_{n=1}^{\infty} = (2, 4, 8, 16, 32, \dots)$ is divergent,
but tends to $\infty$. By abuse of notation we write:

$$\lim_{n\to\infty} d_n = \infty$$

## Limits of Important Sequences*

$$\lim_{n\to\infty} n^a = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a = 0 \\ \infty & \text{for } a > 0 \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{for } |q| < 1 \\ 1 & \text{for } q = 1 \\ \infty & \text{for } q > 1 \\ \nexists & \text{for } q \le -1 \end{cases}$$

$$\lim_{n\to\infty} \frac{n^a}{q^n} = \begin{cases} 0 & \text{for } |q| > 1 \\ \infty & \text{for } 0 < q < 1 \\ \nexists & \text{for } -1 < q < 0 \end{cases} \qquad (|q| \notin \{0,1\})$$



## Rules for Limits*

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences with $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$, resp., and let $(c_n)_{n=1}^{\infty}$ be a bounded sequence. Then

(1) $\lim_{n\to\infty} (k \cdot a_n + d) = k \cdot a + d$

(2) $\lim_{n\to\infty} (a_n + b_n) = a + b$

(3) $\lim_{n\to\infty} (a_n \cdot b_n) = a \cdot b$

(4) $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ for $b \neq 0$

(5) $\lim_{n\to\infty} (a_n \cdot c_n) = 0$ provided $a = 0$

(6) $\lim_{n\to\infty} a_n^k = a^k$



## Example – Rules for Limits*

$$\lim_{n\to\infty} \left(2 + \frac{3}{n^2}\right) = 2 + 3 \underbrace{\lim_{n\to\infty} n^{-2}}_{=0} = 2 + 3 \cdot 0 = 2$$

$$\lim_{n\to\infty} (2^{-n} \cdot n^{-1}) = \lim_{n\to\infty} \frac{n^{-1}}{2^n} = 0$$

$$\lim_{n\to\infty} \frac{1 + \frac{1}{n}}{2 - \frac{3}{n^2}} = \frac{\lim_{n\to\infty} (1 + \frac{1}{n})}{\lim_{n\to\infty} (2 - \frac{3}{n^2})} = \frac{1}{2}$$

$$\lim_{n\to\infty} \underbrace{\sin(n)}_{\text{bounded}} \cdot \underbrace{\frac{1}{n^2}}_{\to 0} = 0$$



## Rules for Limits / Rational Terms*

**Important!**
When we apply these rules we have to take care that we never obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0 \cdot \infty$.

These expressions are **not defined**!

$$\lim_{n\to\infty} \frac{3n^2 + 1}{n^2 - 1} = \frac{\lim_{n\to\infty} 3n^2 + 1}{\lim_{n\to\infty} n^2 - 1} = \frac{\infty}{\infty} \quad \text{(not defined)}$$

**Trick: Reduce** the fraction by the *largest power* in its **denominator**.

$$\lim_{n\to\infty} \frac{3n^2 + 1}{n^2 - 1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{3 + n^{-2}}{1 - n^{-2}} = \frac{\lim_{n\to\infty} 3 + n^{-2}}{\lim_{n\to\infty} 1 - n^{-2}} = \frac{3}{1} = 3$$



## Euler's Number*

$$\boxed{\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7182818284590\ldots}$$

This limit is very important in many applications including finance (continuous compounding).

$$\begin{aligned} \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n &= \lim_{n\to\infty} \left(1 + \frac{1}{n/x}\right)^n \\ &= \lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^{mx} \qquad \left(m = \frac{n}{x}\right) \\ &= \left(\lim_{m\to\infty} \left(1 + \frac{1}{m}\right)^m\right)^x = e^x \end{aligned}$$



## Cauchy Sequence

How can we determine that a sequence converges if we have no clue about the limit?

A sequence $(a_n)_{n=1}^{\infty}$ converges if and only if *for every* $\varepsilon > 0$ there *exists an* $N$ such that $|a_n - a_m| < \varepsilon$ for all $n, m \ge N$.

A sequence with such a property is called a **Cauchy sequence**.



## Series*

The sum of the first $n$ terms of sequence $(a_i)_{i=1}^{\infty}$

$$s_n = \sum_{i=1}^{n} a_i$$

is called the $n$-th **partial sum** of the *sequence*.

The sequence $(s_n)_{n=1}^{\infty}$ of all partial sums is called the **series** of the sequence.

The series of sequence $(a_i) = (2i - 1)$ is

$$(s_n) = \left(\sum_{i=1}^{n} (2i - 1)\right) = (1, 4, 9, 16, 25, \ldots) = (n^2).$$



## Limit of a Series*

There are many cases where we have a summation over *infinitely* many terms, $\sum_{i=1}^{\infty} a_i$.

However, then the usual rules for addition (in particular associativity and commutativity) may not hold any more.
Applying them in such cases would result in contradicitions.

Thus an "infinite sum" is defined as the limit of a series:

$$\boxed{\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i}$$

For example, $\displaystyle\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{(\frac{1}{2})^n - 1}{\frac{1}{2} - 1} = 1.$

## Limit of a Function*

What happens with the value of a function $f$, if the argument $x$ tends to some value $x_0$ (which need not belong to the domain of $f$)?

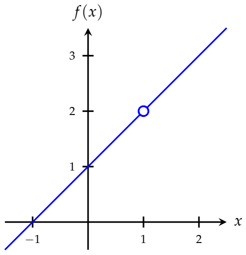
Function

$$f(x) = \frac{x^2-1}{x-1}$$

is not defined in $x = 1$.

By factorizing and reducing we get function

$$g(x) = x + 1 = \begin{cases} f(x), & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$






## Limit of a Function*

Suppose we approach argument $x_0 = 1$.

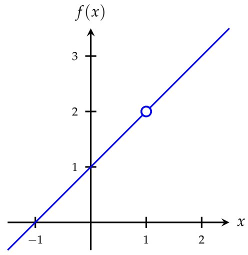
Then the value of function $f(x) = \dfrac{x^2-1}{x-1}$ tends to 2.

We say:

$f(x)$ **converges** to 2 when $x$ tends to 1

and write:

$$\boxed{\lim_{x \to 1} \frac{x^2-1}{x-1} = 2}$$






## Limit of a Function*

Formal **definition**:

If sequence $(f(x_n))_{n=1}^\infty$ of function values converges to number $y_0$ for every convergent sequence $(x_n)_{n=1}^\infty \to x_0$ of arguments, then $y_0$ is called the **limit** of $f$ as $x$ approaches $x_0$.

We write

$$\boxed{\lim_{x \to x_0} f(x) = y_0 \quad \text{or} \quad f(x) \to y_0 \text{ for } x \to x_0}$$

$x_0$ need not belong to the domain of $f$.

$y_0$ need not belong to the codomain of $f$.



## Rules for Limits*

Rules for limits of functions are analogous to rules for sequences.

Let $\lim_{x \to x_0} f(x) = a$ and $\lim_{x \to x_0} g(x) = b$.

$$\begin{array}{lll}
(1) & \lim_{x \to x_0} (c \cdot f(x) + d) = c \cdot a + d & \\
(2) & \lim_{x \to x_0} (f(x) + g(x)) = a + b & \\
(3) & \lim_{x \to x_0} (f(x) \cdot g(x)) = a \cdot b & \\
(4) & \lim_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b} & \text{for } b \neq 0 \\
(6) & \lim_{x \to x_0} (f(x))^k = a^k & \text{for } k \in \mathbb{N}
\end{array}$$



## How to Find Limits?*

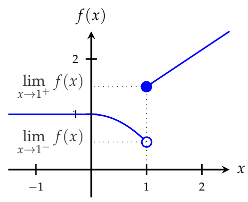
The following recipe is suitable for “simple” functions:

1. Draw the graph of the function.
2. Mark $x_0$ on the $x$-axis.
3. Follow the graph with your pencil until we reach $x_0$ starting from *right* of $x_0$.
4. The $y$-coordinate of your pencil in this point is then the so called **right-handed limit** of $f$ as $x$ approaches $x_0$ (from above):
   $\lim_{x \to x_0^+} f(x)$. (Other notations: $\lim_{x \downarrow x_0} f(x)$ or $\lim_{x \searrow x_0} f(x)$)
5. Analogously we get the **left-handed limit** of $f$ as $x$ approaches $x_0$ (from below): $\lim_{x \to x_0^-} f(x)$.
6. If both limits *coincide*, then the limit exists and we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$$



## Example – How to Find Limits?*




$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

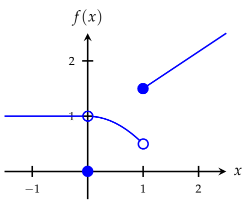
$0.5 = \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) = 1.5$

i.e., the limit of $f$ at $x_0 = 1$ does not exist.

The limits at other points, however, do exist, e.g. $\lim_{x \to 0} f(x) = 1$.



## Example – How to Find Limits?*




$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

The only difference is to above is the function value at $x_0 = 0$.

Nevertheless, the limit does exist:

$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x) \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 1 .$$

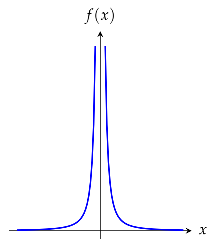


## Unbounded Function*

It may happen that $f(x)$ tends to $\infty$ (or $-\infty$) if $x$ tends to $x_0$.

We then write (by abuse of notation):

$$\boxed{\lim_{x \to x_0} f(x) = \infty}$$




$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

## Limit as $x \to \infty$*

By abuse of language we can define the *limit* analogously for $x_0 = \infty$ and $x_0 = -\infty$, resp.

Limit

$$\lim_{x\to\infty} f(x)$$

exists, if $f(x)$ converges whenever $x$ tends to infinity.

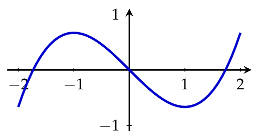
$$\lim_{x\to\infty} \frac{1}{x^2} = 0 \quad \text{and} \quad \lim_{x\to-\infty} \frac{1}{x^2} = 0$$



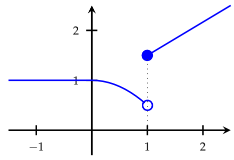
## Continuous Functions*

We may observe that we can draw the graph of a function *without removing the pencil from paper.* We call such functions **continuous**.

For some other functions we *have to remove* the pencil. At such points the function has a **jump discontinuity**.



continuous



jump discontinuity at $x = 1$



## Continuous Functions*

Formal **Definition:**

Function $f\colon D \to \mathbb{R}$ is called **continuous** *at* $x_0 \in D$, if

1. $\lim_{x\to x_0} f(x)$ exists, and
2. $\boxed{\lim_{x\to x_0} f(x) = f(x_0)}$.

The function is called **continuous** if it is continuous *at all* points of its domain.

Note that continuity is a *local* property of a function.



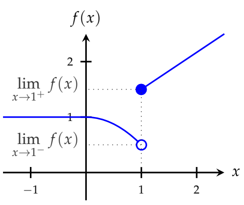
## Continuous Function and Limit*

Continuous functions have to important property that we can exchange function evaluation and the limiting process.

$$\boxed{\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x)}$$



## Discontinuous Function*




$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

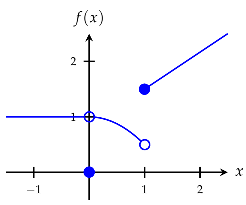
Not continuous in $x = 1$ as $\lim_{x\to 1} f(x)$ does not exist.

So $f$ is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{1\}$.
For example at $x = 0$, $\lim_{x\to 0} f(x)$ does exist and $\lim_{x\to 0} f(x) = 1 = f(0)$.



## Discontinuous Function*



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

Not continuous in all $x = 0$, either.
$\lim_{x\to 0} f(x) = 1$ does exist but $\lim_{x\to 0} f(x) \neq f(0)$.

So $f$ is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{0, 1\}$.

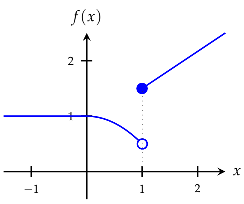


## Recipe for “Nice” Functions*

**(1)** Draw the graph of the given function.

**(2)** At all points of the *domain*, where we *have to remove* the pencil from paper the function is *not continuous*.

**(3)** At all other points of the domain (where we need not remove the pencil) the function is *continuous*.

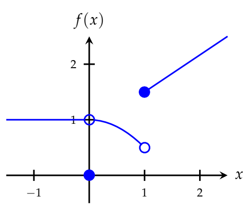


$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1, \\ \frac{x}{2} + 1, & \text{for } x \ge 1. \end{cases}$$

$f$ is continuous
*except* at point $x = 1$.



## Discontinuous Function*



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

Function $f$ is continuous *except* at points $x = 0$ and $x = 1$.

## Summary

- sequence
- limit of a sequence
- limit of a function
- convergent and divergent
- Euler's number
- rules for limits
- continuous functions



Chapter 9

# Derivatives

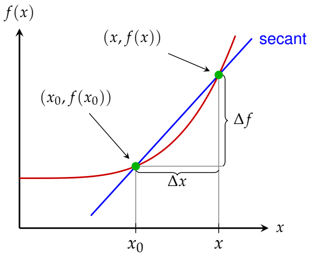


## Difference Quotient*

Let $f\colon \mathbb{R} \to \mathbb{R}$ be some function. Then the ratio

$$\boxed{\frac{\Delta f}{\Delta x} = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \frac{f(x) - f(x_0)}{x - x_0}}$$

is called **difference quotient**.






## Differential Quotient*

If the *limit*

$$\boxed{\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

exists, then function $f$ is called **differentiable** at $x_0$. This limit is then called **differential quotient** or **(first) derivative** of function $f$ at $x_0$.

We write

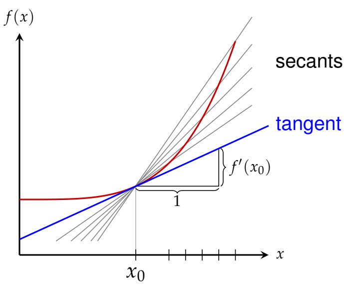
$$\boxed{f'(x_0) \qquad \text{or} \qquad \left.\frac{df}{dx}\right|_{x=x_0}}$$

Function $f$ is called *differentiable*, if it is differentiable at each point of its domain.



## Slope of Tangent*

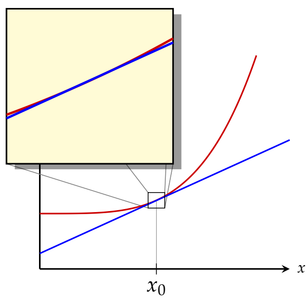
- The differential quotient gives the *slope of the tangent* to the graph of function $f(x)$ at $x_0$.






## Marginal Function*

- Instantaneous change of function $f$.
- "Marginal function" (as in *marginal utility*)





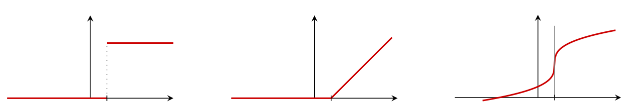

## Existence of Differential Quotient*

Function $f$ is differentiable at all points, where we can draw the tangent (with finite slope) uniquely to the graph.

Function $f$ is *not* differentiable at all points where this is not possible.

In particular these are

- jump discontinuities
- "kinks" in the graph of the function
- vertical tangents





## Computation of the Differential Quotient*

We can compute a differential quotient by determining the limit of the difference quotient.

Let $f(x) = x^2$. The we find for the first derivative

$$\begin{aligned}
f'(x_0) &= \lim_{h \to 0} \frac{(x_0 + h)^2 - x_0{}^2}{h} \\
&= \lim_{h \to 0} \frac{x_0{}^2 + 2\,x_0\,h + h^2 - x_0{}^2}{h} \\
&= \lim_{h \to 0} \frac{2\,x_0\,h + h^2}{h} = \lim_{h \to 0} (2\,x_0 + h) \\
&= 2\,x_0
\end{aligned}$$

## Derivative of a Function*

Function

$$\boxed{f' : D \to \mathbb{R},\ x \mapsto f'(x) = \left.\frac{df}{dx}\right|_x}$$

is called the **first derivative** of function $f$.
Its domain $D$ is the set of all points where the differential quotient (i.e., the limit of the difference quotient) exists.



## Derivatives of Elementary Functions*

| $f(x)$ | $f'(x)$ |
|---|---|
| $c$ | $0$ |
| $x^\alpha$ | $\alpha \cdot x^{\alpha-1}$ |
| $e^x$ | $e^x$ |
| $\ln(x)$ | $\frac{1}{x}$ |
| $\sin(x)$ | $\cos(x)$ |
| $\cos(x)$ | $-\sin(x)$ |



## Computation Rules for Derivatives*

- $(c \cdot f(x))' = c \cdot f'(x)$
- $(f(x) + g(x))' = f'(x) + g'(x)$ *Summation rule*
- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$ *Product rule*
- $(f(g(x)))' = f'(g(x)) \cdot g'(x)$ *Chain rule*
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$ *Quotient rule*



## Example – Computation Rules for Derivatives*

$$(3\,x^3 + 2\,x - 4)' = 3 \cdot 3 \cdot x^2 + 2 \cdot 1 - 0 = 9\,x^2 + 2$$

$$(e^x \cdot x^2)' = (e^x)' \cdot x^2 + e^x \cdot (x^2)' = e^x \cdot x^2 + e^x \cdot 2\,x$$

$$((3\,x^2 + 1)^2)' = 2\,(3\,x^2 + 1) \cdot 6\,x$$

$$(\sqrt{x})' = \left(x^{\frac{1}{2}}\right)' = \tfrac{1}{2} \cdot x^{-\frac{1}{2}} = \tfrac{1}{2\sqrt{x}}$$

$$(a^x)' = \left(e^{\ln(a)\cdot x}\right)' = e^{\ln(a)\cdot x} \cdot \ln(a) = a^x\, \ln(a)$$

$$\left(\frac{1 + x^2}{1 - x^3}\right)' = \frac{2x \cdot (1 - x^3) - (1 + x^2) \cdot 3x^2}{(1 - x^3)^2}$$



## Higher Order Derivatives*

We can compute derivatives of the derivative of a function.

Thus we obtain the

- **second derivative** $f''(x)$ of function $f$,
- **third derivative** $f'''(x)$, etc.,
- **$n$-th derivative** $f^{(n)}(x)$.

Other notations:

- $f''(x) = \frac{d^2 f}{dx^2}(x) = \left(\frac{d}{dx}\right)^2 f(x)$
- $f^{(n)}(x) = \frac{d^n f}{dx^n}(x) = \left(\frac{d}{dx}\right)^n f(x)$



## Example – Higher Order Derivatives*

The first five derivatives of function

$$f(x) = x^4 + 2x^2 + 5x - 3$$

are

$$\begin{aligned}
f'(x) &= (x^4 + 2x^2 + 5x - 3)' = 4x^3 + 4x + 5 \\
f''(x) &= (4x^3 + 4x + 5)' = 12x^2 + 4 \\
f'''(x) &= (12x^2 + 4)' = 24x \\
f^{\text{iv}}(x) &= (24x)' = 24 \\
f^{\text{v}}(x) &= 0
\end{aligned}$$



## Marginal Change*

We can estimate the derivative $f'(x_0)$ approximately by means of the difference quotient with *small* change $\Delta x$:

$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \approx \frac{\Delta f}{\Delta x}$$

Vice verse we can estimate the change $\Delta f$ of $f$ for *small* changes $\Delta x$ approximately by the first derivative of $f$:

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

**Beware:**

- $f'(x_0) \cdot \Delta x$ is a *linear function* in $\Delta x$.
- It is the *best possible* approximation of $f$ by a linear function *around* $x_0$.
- This approximation is useful only for *"small"* values of $\Delta x$.



## Differential*

The approximation

$$\Delta f = f(x_0 + \Delta x) - f(x_0) \approx f'(x_0) \cdot \Delta x$$

becomes exact if $\Delta x$ (and thus $\Delta f$) becomes *infinitesimally small*.
We then write $dx$ and $df$ instead of $\Delta x$ and $\Delta f$, resp.

$$\boxed{df = f'(x_0)\, dx}$$

Symbols $df$ and $dx$ are called the **differentials** of function $f$ and the independent variable $x$, resp.

## Differential*

Differential $df$ can be seen as a linear function in $dx$.
We can use it to compute $f$ approximately around $x_0$.

$$f(x_0 + dx) \approx f(x_0) + df$$

Let $f(x) = e^x$.

Differential of $f$ at point $x_0 = 1$:
$df = f'(1)\,dx = e^1\,dx$

Approximation of $f(1.1)$ by means of this differential:
$\Delta x = (x_0 + dx) - x_0 = 1.1 - 1 = 0.1$

$f(1.1) \approx f(1) + df = e + e \cdot 0.1 \approx 2.99$

Exact value: $f(1.1) = 3.004166\ldots$



## Elasticity*

The first derivative of a function gives *absolute* rate of change of $f$ at $x_0$.
Hence it depends on the scales used for argument and function values.

However, often *relative* rates of change are more appropriate.

We obtain *scale invariance* and *relative* rate of changes by

$$\frac{\text{change of function value relative to value of function}}{\text{change of argument relative to value of argument}}$$

and thus

$$\lim_{\Delta x \to 0} \frac{\frac{f(x+\Delta x)-f(x)}{f(x)}}{\frac{\Delta x}{x}} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} \cdot \frac{x}{f(x)} = f'(x) \cdot \frac{x}{f(x)}$$



## Elasticity*

Expression

$$\boxed{\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)}}$$

is called the **elasticity** of $f$ at point $x$.

Let $f(x) = 3\,e^{2x}$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{6\,e^{2x}}{3\,e^{2x}} = 2\,x$$

Let $f(x) = \beta\,x^\alpha$. Then

$$\varepsilon_f(x) = x \cdot \frac{f'(x)}{f(x)} = x \cdot \frac{\beta\,\alpha\,x^{\alpha-1}}{\beta\,x^\alpha} = \alpha$$



## Elasticity II*

The relative rate of change of $f$ can be expressed as

$$\ln(f(x))' = \frac{f'(x)}{f(x)}$$

What happens if we compute the derivative of $\ln(f(x))$ w.r.t. $\ln(x)$?

Let $\quad v = \ln(x) \quad \Leftrightarrow \quad x = e^v$

Derivation by means of the chain rule yields:

$$\frac{d(\ln(f(x)))}{d(\ln(x))} = \frac{d(\ln(f(e^v)))}{dv} = \frac{f'(e^v)}{f(e^v)}\,e^v = \frac{f'(x)}{f(x)}\,x = \varepsilon_f(x)$$

$$\boxed{\varepsilon_f(x) = \frac{d(\ln(f(x)))}{d(\ln(x))}}$$



## Elasticity II*

We can use the chain rule *formally* in the following way:

Let

- $u = \ln(y)$,
- $y = f(x)$,
- $x = e^v \quad \Leftrightarrow \quad v = \ln(x)$

Then we find

$$\frac{d(\ln f)}{d(\ln x)} = \frac{du}{dv} = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot \frac{dx}{dv} = \frac{1}{y} \cdot f'(x) \cdot e^v = \frac{f'(x)}{f(x)}\,x$$



## Elastic Functions*

A Function $f$ is called

- **elastic** in $x$, if $|\,\varepsilon_f(x)\,| > 1$
- **1-elastic** in $x$, if $|\,\varepsilon_f(x)\,| = 1$
- **inelastic** in $x$, if $|\,\varepsilon_f(x)\,| < 1$

For elastic functions we then have:
The value of the function changes *relatively* faster than the value of the argument.

Function $f(x) = 3\,e^{2x}$ is $\qquad [\,\varepsilon_f(x) = 2\,x\,]$

- 1-elastic, for $x = -\frac{1}{2}$ and $x = \frac{1}{2}$;
- inelastic, for $-\frac{1}{2} < x < \frac{1}{2}$;
- elastic, for $x < -\frac{1}{2}$ or $x > \frac{1}{2}$.



## Source of Errors

**Beware!**
Function $f$ is elastic if the **absolute value** of the *elasticity* is greater than 1.



## Elastic Demand*

Let $q(p)$ be an *elastic* demand function, where $p$ is the price.
We have: $p > 0$, $q > 0$, and $q' < 0$ ($q$ is decreasing). Hence

$$\varepsilon_q(p) = p \cdot \frac{q'(p)}{q(p)} < -1$$

What happens to the revenue (= price $\times$ selling)?

$$\begin{aligned} u'(p) &= (p \cdot q(p))' = 1 \cdot q(p) + p \cdot q'(p) \\ &= q(p) \cdot (1 + \underbrace{p \cdot \frac{q'(p)}{q(p)}}_{=\varepsilon_q < -1}) \\ &< 0 \end{aligned}$$

In other words, the revenue decreases if we raise prices.

## Partial Derivative*

We investigate the rate of change of function $f(x_1,\dots,x_n)$, when variable $x_i$ changes and the other variables remain fixed.
Limit

$$\boxed{\frac{\partial f}{\partial x_i} = \lim_{\Delta x_i \to 0} \frac{f(\dots,x_i+\Delta x_i,\dots) - f(\dots,x_i,\dots)}{\Delta x_i}}$$

is called the (first) **partial derivative** of $f$ w.r.t. $x_i$.

Other notations for partial derivative $\frac{\partial f}{\partial x_i}$:

- $f_{x_i}(\mathbf{x})$ (derivative w.r.t. variable $x_i$)
- $f_i(\mathbf{x})$ (derivative w.r.t. the $i$-th variable)
- $f_i'(\mathbf{x})$ ($i$-th component of the gradient)



## Computation of Partial Derivatives*

We obtain partial derivatives $\frac{\partial f}{\partial x_i}$ by applying the rules for *univariate* functions for variable $x_i$ while we treat *all other* variables *as constants*.

First partial derivatives of

$$f(x_1,x_2) = \sin(2\,x_1)\cdot\cos(x_2)$$

$$f_{x_1} = 2\cdot\cos(2\,x_1)\cdot \underbrace{\cos(x_2)}_{\text{treated as constant}}$$

$$f_{x_2} = \underbrace{\sin(2\,x_1)}_{\text{treated as constant}} \cdot(-\sin(x_2))$$



## Higher Order Partial Derivatives*

We can compute partial derivatives of partial derivatives analogously to their univariate counterparts and obtain
**higher order partial derivatives**:

$$\boxed{\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) \quad\text{and}\quad \frac{\partial^2 f}{\partial x_i^2}(\mathbf{x})}$$

Other notations for partial derivative $\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x})$:

- $f_{x_i x_k}(\mathbf{x})$ (derivative w.r.t. variables $x_i$ and $x_k$)
- $f_{ik}(\mathbf{x})$ (derivative w.r.t. the $i$-th and $k$-th variable)
- $f_{ik}''(\mathbf{x})$ (component of the Hessian matrix with index $ik$)



## Higher Order Partial Derivatives*

If all second order partial derivatives exists and are *continuous*, then the order of differentiation does not matter (Schwarz's theorem):

$$\boxed{\frac{\partial^2 f}{\partial x_k \partial x_i}(\mathbf{x}) = \frac{\partial^2 f}{\partial x_i \partial x_k}(\mathbf{x})}$$

Remark: Practically all differentiable functions in economic models have this property.



## Example – Higher Order Partial Derivatives*

Compute the first and second order partial derivatives of

$$f(x,y) = x^2 + 3\,x\,y$$

First order partial derivatives:

$$f_x = 2\,x + 3\,y \qquad f_y = 0 + 3\,x$$

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & f_{xy} &= 3 \\ f_{yx} &= 3 & f_{yy} &= 0 \end{aligned}$$



## Gradient

We collect all *first order partial derivatives* into a (row) vector which is called the **gradient** at point $\mathbf{x}$.

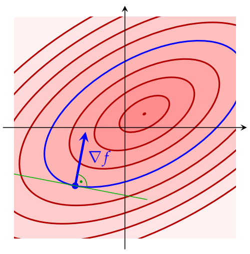
$$\boxed{\nabla f(\mathbf{x}) = (f_{x_1}(\mathbf{x}),\dots,f_{x_n}(\mathbf{x}))}$$

- read: "gradient of $f$" or "nabla $f$".
- Other notation: $f'(\mathbf{x})$
- Alternatively the gradient can also be a column vector.
- The gradient is the analog of the first derivative of univariate functions.



## Properties of the Gradient

- The gradient of $f$ always points in the direction of *steepest ascent*.
- Its length is equal to the slope at this point.
- The gradient is *normal* (i.e. in right angle) to the corresponding *contour line* (level set).






## Example – Gradient

Compute the gradient of

$$f(x,y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (3,2)$.

$$f_x = 2\,x + 3\,y$$
$$f_y = 0 + 3\,x$$

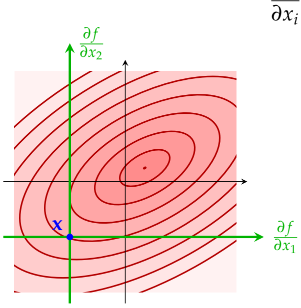
$$\nabla f(\mathbf{x}) = (2x+3y,\, 3x)$$

$$\nabla f(3,2) = (12, 9)$$

## Directional Derivative

We obtain partial derivative $\frac{\partial f}{\partial x_i}$ by differentiating the univariate function $g(t) = f(x_1, \dots, x_i + t, \dots, x_n) = f(\mathbf{x} + t \cdot \mathbf{h})$ with $\mathbf{h} = \mathbf{e}_i$ at point $t = 0$:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = \left.\frac{dg}{dt}\right|_{t=0} = \left.\frac{d}{dt} f(\mathbf{x} + t \cdot \mathbf{h})\right|_{t=0}$$

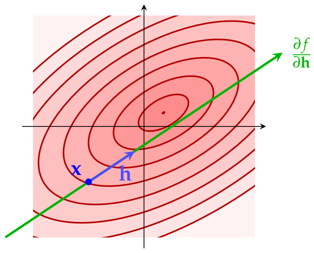





## Directional Derivative

**Generalization:**
We obtain the **directional derivative** $\frac{\partial f}{\partial \mathbf{h}}$ along $\mathbf{h}$ with length 1 by differentiating the univariate function $g(t) = f(\mathbf{x} + t \cdot \mathbf{h})$ at point $t = 0$:

$$\frac{\partial f}{\partial \mathbf{h}}(\mathbf{x}) = \left.\frac{dg}{dt}\right|_{t=0} = \left.\frac{d}{dt} f(\mathbf{x} + t \cdot \mathbf{h})\right|_{t=0}$$




The directional derivative describes the change of $f$, if we move $\mathbf{x}$ in direction $\mathbf{h}$.



## Directional Derivative

We have (for $\|\mathbf{h}\| = 1$):

$$\frac{\partial f}{\partial \mathbf{h}}(\mathbf{x}) = f_{x_1}(\mathbf{x}) \cdot h_1 + \dots + f_{x_n}(\mathbf{x}) \cdot h_n = \nabla f(\mathbf{x}) \cdot \mathbf{h}$$

If $\mathbf{h}$ does not have norm 1, we first have to normalize first:

$$\boxed{\frac{\partial f}{\partial \mathbf{h}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \frac{\mathbf{h}}{\|\mathbf{h}\|}}$$



## Example – Directional Derivative

Compute the directional derivative of

$$f(x_1, x_2) = x_1^2 + 3\,x_1\,x_2$$

along $\mathbf{h} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$ at $\mathbf{x} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

Norm of $\mathbf{h}$:

$$\|\mathbf{h}\| = \sqrt{\mathbf{h}^\top \mathbf{h}} = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Directional derivative:

$$\frac{\partial f}{\partial \mathbf{h}}(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \frac{\mathbf{h}}{\|\mathbf{h}\|} = \frac{1}{\sqrt{5}}(12, 9) \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\frac{6}{\sqrt{5}}$$



## Total Differential

We want to approximate a function $f$ by some linear function such that the approximation error is as small as possible:

$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) \approx f_{x_1}(\mathbf{x})\,h_1 + \dots + f_{x_n}(\mathbf{x})\,h_n = \nabla f(\mathbf{x}) \cdot \mathbf{h}$$

The approximation becomes exact if $\mathbf{h}$ (and thus $\Delta f$) becomes *infinitesimally small*.

The *linear function*

$$\boxed{df = f_{x_1}(\mathbf{x})\,dx_1 + \dots + f_{x_n}(\mathbf{x})\,dx_n = \sum_{i=1}^{n} f_{x_i}\,dx_i = \nabla f(\mathbf{x}) \cdot d\mathbf{x}}$$

is called the **total Differential** of $f$ at $\mathbf{x}$.



## Example – Total Differential

Compute the total differential of

$$f(x_1, x_2) = x_1^2 + 3\,x_1\,x_2$$

at $\mathbf{x} = (3, 2)$.

$$df = f_{x_1}(3,2)\,dx_1 + f_{x_2}(3,2)\,dx_2 = 12\,dx_1 + 9\,dx_2$$

Approximation of $f(3.1, 1.8)$ by means of the total differential:

$$\begin{aligned} f(3.1, 1.8) &\approx f(3;2) + df \\ &= 27 + 12 \cdot 0.1 + 9 \cdot (-0.2) = 26.40 \end{aligned}$$

Exact value: $f(3.1, 1.8) = 26.35$

$$\mathbf{h} = (\mathbf{x} + \mathbf{h}) - \mathbf{x} = \begin{pmatrix} 3.1 \\ 1.8 \end{pmatrix} - \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0.1 \\ -0.2 \end{pmatrix}$$



## Hessian Matrix

Let $f(\mathbf{x}) = f(x_1, \dots, x_n)$ be two times differentiable. Then matrix

$$\boxed{\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \dots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \dots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \dots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}}$$

is called the **Hessian matrix** of $f$ at $\mathbf{x}$.

- The Hessian matrix is symmetric, i.e., $f_{x_ix_k}(\mathbf{x}) = f_{x_kx_i}(\mathbf{x})$.
- Other notation: $f''(\mathbf{x})$
- The Hessian matrix is the analog of the second derivative of univariate functions.



## Example – Hessian Matrix

Compute the Hessian matrix of

$$f(x, y) = x^2 + 3\,x\,y$$

at point $\mathbf{x} = (1, 2)$.

Second order partial derivatives:

$$\begin{aligned} f_{xx} &= 2 & f_{xy} &= 3 \\ f_{yx} &= 3 & f_{yy} &= 0 \end{aligned}$$

Hessian matrix:

$$\mathbf{H}_f(x, y) = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{yx}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 3 & 0 \end{pmatrix} = \mathbf{H}_f(1, 2)$$

## Differentiability

**Theorem:**
A function $f\colon \mathbb{R} \to \mathbb{R}$ is **differentiable** at $x_0$ if and only if there exists a linear map $\ell$ which approximates $f$ in $x_0$ in an optimal way:

$$\lim_{h\to 0} \frac{|(f(x_0+h) - f(x_0)) - \ell(h)|}{|h|} = 0$$

Obviously $\ell(h) = f'(x_0) \cdot h$.

**Definition:**
A function $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ is **differentiable** at $\mathbf{x}_0$ if there exists a linear map $\ell$ which approximates $\mathbf{f}$ in $\mathbf{x}_0$ in an optimal way:

$$\lim_{\mathbf{h}\to 0} \frac{\|(\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0)) - \ell(\mathbf{h})\|}{\|\mathbf{h}\|} = 0$$

Function $\ell(\mathbf{h}) = \mathbf{J} \cdot \mathbf{h}$ is called the *total derivative* of $\mathbf{f}$.



## Jacobian Matrix

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$

The $m \times n$ matrix

$$D\mathbf{f}(\mathbf{x}_0) = \mathbf{f}'(\mathbf{x}_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

is called the **Jacobian matrix** of $\mathbf{f}$ at point $\mathbf{x}_0$.

It is the generalization of *derivatives* (and gradients) for vector-valued functions.



## Jacobian Matrix

For $f\colon \mathbb{R}^n \to \mathbb{R}$ the Jacobian matrix is the gradient of $f$:

$$Df(\mathbf{x}_0) = \nabla f(\mathbf{x}_0)$$

For vector-valued functions the Jacobian matrix can be written as

$$D\mathbf{f}(\mathbf{x}_0) = \begin{pmatrix} \nabla f_1(\mathbf{x}_0) \\ \vdots \\ \nabla f_m(\mathbf{x}_0) \end{pmatrix}$$



## Example – Jacobian Matrix

- $f(\mathbf{x}) = f(x_1,x_2) = \exp(-x_1^2 - x_2^2)$
  $$\begin{aligned} Df(\mathbf{x}) &= \left(\tfrac{\partial f}{\partial x_1}, \tfrac{\partial f}{\partial x_2}\right) = \nabla f(\mathbf{x}) \\ &= \left(-2\,x_1 \exp(-x_1^2 - x_2^2), -2\,x_2 \exp(-x_1^2 - x_2^2)\right) \end{aligned}$$
- $\mathbf{f}(\mathbf{x}) = \mathbf{f}(x_1,x_2) = \begin{pmatrix} f_1(x_1,x_2) \\ f_2(x_1,x_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 \\ x_1^2 - x_2^2 \end{pmatrix}$
  $$D\mathbf{f}(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2\,x_1 & 2\,x_2 \\ 2\,x_1 & -2\,x_2 \end{pmatrix}$$
- $\mathbf{s}(t) = \begin{pmatrix} s_1(t) \\ s_2(t) \end{pmatrix} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$
  $$D\mathbf{s}(t) = \begin{pmatrix} \frac{ds_1}{dt} \\ \frac{ds_2}{dt} \end{pmatrix} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix}$$



## Chain Rule

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{g}\colon \mathbb{R}^m \to \mathbb{R}^k$. Then

$$\boxed{(\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x})}$$

$$\mathbf{f}(x,y) = \begin{pmatrix} e^x \\ e^y \end{pmatrix} \qquad \mathbf{g}(x,y) = \begin{pmatrix} x^2 + y^2 \\ x^2 - y^2 \end{pmatrix}$$

$$\mathbf{f}'(x,y) = \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \qquad \mathbf{g}'(x,y) = \begin{pmatrix} 2\,x & 2\,y \\ 2\,x & -2\,y \end{pmatrix}$$

$$\begin{aligned} (\mathbf{g} \circ \mathbf{f})'(\mathbf{x}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) \cdot \mathbf{f}'(\mathbf{x}) &= \begin{pmatrix} 2\,e^x & 2\,e^y \\ 2\,e^x & -2\,e^y \end{pmatrix} \cdot \begin{pmatrix} e^x & 0 \\ 0 & e^y \end{pmatrix} \\ &= \begin{pmatrix} 2\,e^{2x} & 2\,e^{2y} \\ 2\,e^{2x} & -2\,e^{2y} \end{pmatrix} \end{aligned}$$



## Example – Directional Derivative

We can derive the formula for the directional derivative of $f\colon \mathbb{R}^n \to \mathbb{R}$ along $\mathbf{h}$ (with $\|\mathbf{h}\| = 1$) at $\mathbf{x}_0$ by means of the chain rule:

Let $\mathbf{s}(t)$ be a path through $\mathbf{x}_0$ along $\mathbf{h}$, i.e.,

$$\mathbf{s}\colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto \mathbf{x}_0 + t\mathbf{h}\,.$$

Then

$$\begin{aligned} &f'(\mathbf{s}(0)) = f'(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \\ &\mathbf{s}'(0) = \mathbf{h} \end{aligned}$$

and hence

$$\frac{\partial f}{\partial \mathbf{h}} = (f \circ \mathbf{s})'(0) = f'(\mathbf{s}(0)) \cdot \mathbf{s}'(0) = \nabla f(\mathbf{x}_0) \cdot \mathbf{h}\,.$$



## Example – Indirect Dependency

Let $f(x_1,x_2,t)$ where $x_1(t)$ and $x_2(t)$ also depend on $t$.
What is the total derivative of $f$ w.r.t. $t$?

Chain rule:
Let $\mathbf{x}\colon \mathbb{R} \to \mathbb{R}^3,\ t \mapsto \begin{pmatrix} x_1(t) \\ x_2(t) \\ t \end{pmatrix}$

$$\begin{aligned} \frac{df}{dt} &= (f \circ \mathbf{x})'(t) = f'(\mathbf{x}(t)) \cdot \mathbf{x}'(t) \\ &= \nabla f(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} = (f_{x_1}(\mathbf{x}(t)), f_{x_2}(\mathbf{x}(t)), f_t(\mathbf{x}(t)) \cdot \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ 1 \end{pmatrix} \\ &= f_{x_1}(\mathbf{x}(t)) \cdot x_1'(t) + f_{x_2}(\mathbf{x}(t)) \cdot x_2'(t) + f_t(\mathbf{x}(t)) \\ &= f_{x_1}(x_1,x_2,t) \cdot x_1'(t) + f_{x_2}(x_1,x_2,t) \cdot x_2'(t) + f_t(x_1,x_2,t) \end{aligned}$$



## L'Hôpital's Rule

Suppose we want to compute

$$\lim_{x\to x_0} \frac{f(x)}{g(x)}$$

and find

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0 \qquad \text{(or} = \pm\infty)$$

However, expressions like $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are not defined.

(You must not reduce the fraction by 0 or $\infty$!)

## L'Hôpital's Rule

If $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0$ (or $= \infty$ or $= -\infty$), then

$$\boxed{\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)}}$$

Assumption: $f$ and $g$ are differentiable in $x_0$.

This formula is called **l'Hôpital's rule** (also spelled as *l'Hospital's rule*).



## Example – L'Hôpital's Rule

$$\lim_{x\to 2} \frac{x^3 - 7x + 6}{x^2 - x - 2} = \lim_{x\to 2} \frac{3x^2 - 7}{2x - 1} = \frac{5}{3}$$

$$\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$$

$$\lim_{x\to 0} \frac{x - \ln(1+x)}{x^2} = \lim_{x\to 0} \frac{1 - (1+x)^{-1}}{2x} = \lim_{x\to 0} \frac{(1+x)^{-2}}{2} = \frac{1}{2}$$



## Example – L'Hôpital's Rule

L'Hôpital's rule can be applied iteratively:

$$\lim_{x\to 0} \frac{e^x - x - 1}{x^2} = \lim_{x\to 0} \frac{e^x - 1}{2x} = \lim_{x\to 0} \frac{e^x}{2} = \frac{1}{2}$$



## Summary

- difference quotient and differential quotient
- differential quotient and derivative
- derivatives of elementary functions
- differentiation rules
- higher order derivatives
- total differential
- elasticity
- partial derivatives
- gradient and Hessian matrix
- Jacobian matrix and chain rule
- l'Hôpital's rule



Chapter 10

# Inverse and Implicit Functions



## Inverse Function

Let $\mathbf{f}\colon D_f \subseteq \mathbb{R}^n \to W_f \subseteq \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x})$. A Function

$$\boxed{\mathbf{f}^{-1}\colon W_f \to D_f,\ \mathbf{y} \mapsto \mathbf{x} = \mathbf{f}^{-1}(\mathbf{y})}$$

is called **inverse function** of $\mathbf{f}$, if

$$\boxed{\mathbf{f}^{-1} \circ \mathbf{f} = \mathbf{f} \circ \mathbf{f}^{-1} = \mathbf{id}}$$

where $\mathbf{id}$ denotes the **identity function**, $\mathbf{id}(\mathbf{x}) = \mathbf{x}$:

$$\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = \mathbf{f}^{-1}(\mathbf{y}) = \mathbf{x} \quad \text{and} \quad \mathbf{f}(\mathbf{f}^{-1}(\mathbf{y})) = \mathbf{f}(\mathbf{x}) = \mathbf{y}$$

$\mathbf{f}^{-1}$ exists if and only if $\mathbf{f}$ is bijective.

We then obtain $\mathbf{f}^{-1}(\mathbf{y})$ as the *unique* solution $\mathbf{x}$ of equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$.



## Linear Function

Let $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = a\,x + b$.

$$y = a\,x + b \quad \Leftrightarrow \quad a\,x = y - b \quad \Leftrightarrow \quad x = \tfrac{1}{a}\,y - \tfrac{b}{a}$$

That is,

$$f^{-1}(y) = a^{-1}\,y - a^{-1}\,b$$

**Provided** that $\quad a \neq 0 \qquad [\,a = f'(x)\,]$

Observe:

$$(f^{-1})'(y) = a^{-1} = \frac{1}{a} = \frac{1}{f'(x)}$$



## Linear Function

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$ for some $m \times n$ matrix $\mathbf{A}$.

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\,\mathbf{y} - \mathbf{A}^{-1}\,\mathbf{b}$$

That is,

$$\mathbf{f}^{-1}(\mathbf{y}) = \mathbf{A}^{-1}\,\mathbf{y} - \mathbf{A}^{-1}\,\mathbf{b}\,.$$

**Provided** that $\mathbf{A}$ is invertible, $\qquad [\,\mathbf{A} = D\mathbf{f}(\mathbf{x})\,]$
(and thus: $n = m$)

Observe:

$$(\mathbf{f}^{-1})'(\mathbf{y}) = \mathbf{A}^{-1} = (\mathbf{f}'(\mathbf{x}))^{-1}$$

## Locally Invertible Function

Function

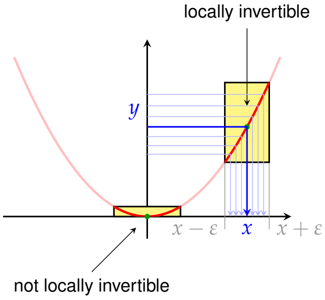
$$f: \mathbb{R} \to [0,\infty),\ x \mapsto f(x) = x^2$$

is not bijective. Thus
$f^{-1}$ does *not* exist *globally*.

For some $x_0$ there exists an *open* interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ where $y = f(x)$ can be solved w.r.t. $x$.

We say:
$f$ is **locally invertible** around $x_0$.

For other $x_0$ such an interval does not exist (even if it is very short).






## Existence and Derivative

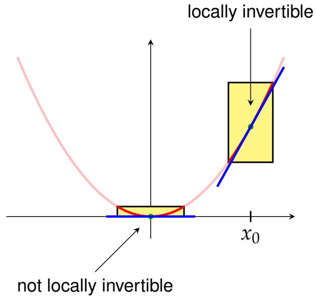
1. For which $\mathbf{x}_0$ is $\mathbf{f}$ locally invertible?
2. What is the derivative of $\mathbf{f}^{-1}$ at $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$.

**Idea:**
Replace $\mathbf{f}$ by its differential:

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0) + \mathbf{f}'(\mathbf{x}_0) \cdot \mathbf{h}$$

**Hence:**

1. $\mathbf{f}'(\mathbf{x}_0)$ must be invertible.
2. $(\mathbf{f}^{-1})'(\mathbf{y}_0) = (\mathbf{f}'(\mathbf{x}_0))^{-1}$






## Inverse Function Theorem

Let $f: D_f \subseteq \mathbb{R} \to \mathbb{R}$ be a function and $x_0$ some point with $f'(x_0) \neq 0$.

Then there exist open intervals $U$ around $x_0$ and $V$ around $y_0 = f(x_0)$ such that $f: U \to V$ is one-to-one and onto,
i.e., the inverse function $f^{-1}: \mathcal{V} \to \mathcal{U}$ exists.

Moreover, we find for its derivative:

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$



## Example – Inverse Function Theorem

Let $f: \mathbb{R} \to \mathbb{R},\ x \mapsto y = f(x) = x^2$ and $x_0 = 3$, $y_0 = f(x_0) = 9$.

As $f'(x_0) = 6 \neq 0$, $f$ is locally invertible around $x_0 = 3$ and

$$(f^{-1})'(9) = \frac{1}{f'(3)} = \frac{1}{6}$$

For $x_0 = 0$ we *cannot apply* this theorem as $f'(0) = 0$.
(The inverse function theorem provides a *sufficient* condition.)



## Inverse Function Theorem II

Let $\mathbf{f}: \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{x}_0$ and $x_0$ some point with $|\mathbf{f}'(\mathbf{x}_0)| \neq 0$.

Then there exist open hyper-rectangles $U$ around $\mathbf{x}_0$ and $V$ around $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$ such that $\mathbf{f}: U \to V$ is one-to-one and onto,
i.e., the inverse function $\mathbf{f}^{-1}: \mathcal{V} \to \mathcal{U}$ exists.

Moreover, we find for its derivative:

$$\boxed{(\mathbf{f}^{-1})'(\mathbf{y}_0) = (\mathbf{f}'(\mathbf{x}_0))^{-1}}$$

The **Jacobian determinant** $|\mathbf{f}'(\mathbf{x}_0)|$ is also denoted by

$$\boxed{\frac{\partial(f_1,\dots,f_n)}{\partial(x_1,\dots,x_n)} = |\mathbf{f}'(\mathbf{x}_0)|}$$



## Example – Inverse Function Theorem

Let $\mathbf{f}: \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1 x_2 \end{pmatrix}$ and $\mathbf{x}_0 = (1,1)^\mathsf{T}$.

Then $\quad \mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{pmatrix} \quad$ and

$$\frac{\partial(f_1,f_2)}{\partial(x_1,x_2)} = \begin{vmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{vmatrix} = 2x_1^2 + 2x_2^2 \neq 0 \qquad \text{for all } \mathbf{x} \neq 0.$$

That is, $\mathbf{f}$ is locally invertible around all $\mathbf{x}_0 \neq 0$.
In particular for $\mathbf{x}_0 = (1,1)^\mathsf{T}$ we find

$$(\mathbf{f}^{-1})'(\mathbf{f}(1,1)) = (\mathbf{f}'(1,1))^{-1} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{2}{4} \\ -\frac{1}{4} & \frac{2}{4} \end{pmatrix}$$

However, $\mathbf{f}$ is not bijective: $\quad \mathbf{f}(1,1) = \mathbf{f}(-1,-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$



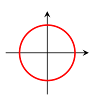
## Explicit and Implicit Function

The relation between two variables $x$ and $y$ can be described by an

**explicit** function:

$$y = f(x)$$

Example:
$y = x^2$
does not exist



**implicit** function:

$$F(x,y) = 0$$

Example:
$y - x^2 = 0$
$x^2 + y^2 - 1 = 0$

**Questions:**

- When can an implicit function be represented (**locally**) by an explicit function?
- What is the derivative of $y$ w.r.t. variable $x$?



## Case: Linear Function

For a linear function

$$F(x,y) = ax + by$$

both questions can be easily answered:

$$ax + by = 0 \quad \Rightarrow \quad y = -\frac{a}{b}x \qquad \textbf{(if } F_y = b \neq 0\textbf{)}$$

$$\frac{dy}{dx} = -\frac{a}{b} = -\frac{F_x}{F_y}$$

## Case: General Function

Let $F(x,y)$ be a function and $(x_0,y_0)$ some point with $F(x_0,y_0)=0$.

If $F$ is not linear, then we can compute the derivative $\frac{dy}{dx}$ in $x_0$ by replacing $F$ *locally* by its total differential

$$dF = F_x\,dx + F_y\,dy = d0 = 0$$

and yield[1]

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

[1] The given “computation” is not correct but yields the correct result. Note that the differential quotient is not the quotient of differentials.



## Example – Implicit Derivative

Compute the implicit derivative $\frac{dy}{dx}$ of

$$F(x,y) = x^2 + y^2 - 1 = 0\,.$$

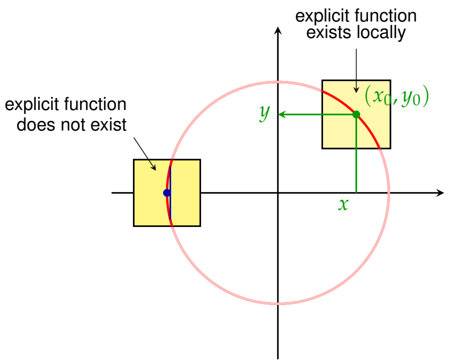
$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

We also can compute the derivative of $x$ w.r.t. variable $y$:

$$\frac{dx}{dy} = -\frac{F_y}{F_x} = -\frac{2y}{2x} = -\frac{y}{x}$$



## Local Existence of an Explicit Function




$y = f(x)$ exists locally, if $F_y \neq 0$.



## Implicit Function Theorem

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ and let $(x_0,y_0)$ be some point with

$$F(x_0,y_0) = 0 \qquad \text{and} \qquad F_y(x_0,y_0) \neq 0\,.$$

Then there exists a rectangle $R$ around $(x_0,y_0)$ such that

- $F(x,y)=0$ has a unique solution $y=f(x)$ in $R$, and
- $\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$



## Example – Implicit Function Theorem

Let $F(x,y) = x^2+y^2-8$ and $(x_0,y_0) = (2,2)$.

As $F(x_0,y_0)=0$ and $F_y(x_0,y_0) = 2\,y_0 = 4 \neq 0$,
variable $y$ can be represented locally as a function of variable $x$ and

$$\frac{dy}{dx}(x_0) = -\frac{F_x(x_0,y_0)}{F_y(x_0,y_0)} = -\frac{2x_0}{2y_0} = -1\,.$$



## Implicit Function Theorem II

Let $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $(\mathbf{x},y) \mapsto F(\mathbf{x},y) = F(x_1,\dots,x_n,y)$, and let $(\mathbf{x}_0,y_0)$ be some point with

$$F(\mathbf{x}_0,y_0) = 0 \qquad \text{and} \qquad F_y(\mathbf{x}_0,y_0) \neq 0\,.$$

Then there exists a hyper-rectangle $R$ around $(\mathbf{x}_0,y_0)$ such that

- $F(\mathbf{x},y)=0$ has a unique solution $y=f(\mathbf{x})$ in $R$, where $f\colon \mathbb{R}^n \to \mathbb{R}$, and
- $\boxed{\frac{\partial y}{\partial x_i} = -\frac{F_{x_i}}{F_y}}$

The independent variable $y$ can be any of the variables of $F$ and need not be in the last position.



## Example – Implicit Function Theorem

Compute $\frac{\partial x_2}{\partial x_3}$ of the implicit function

$$F(x_1,x_2,x_3,x_4) = x_1^2 + x_2\,x_3 + x_3^2 - x_3\,x_4 - 1 = 0$$

at point $(x_1,x_2,x_3,x_4) = (1,0,1,1)$.

As $F(1,0,1,1)=0$ and $F_{x_2}(1,0,1,1) = 1 \neq 0$ we can represent $x_2$ locally as a function of $(x_1,x_3,x_4)$: $\quad x_2 = f(x_1,x_3,x_4)$.

The partial derivative w.r.t. $x_3$ is given by

$$\frac{\partial x_2}{\partial x_3} = -\frac{F_{x_3}}{F_{x_2}} = -\frac{x_2 + 2\,x_3 - x_4}{x_3} = -1$$

At $(1,1,1,1)$ and $(1,1,0,1)$ the implicit function theorem cannot be applied for independent variable $x_2$:
$F(1,1,1,1) \neq 0$ and $F_{x_2}(1,1,0,1) = 0$.



## Jacobian Matrix

Let

$$\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} F_1(x_1,\dots,x_n,y_1,\dots,y_m) \\ \vdots \\ F_m(x_1,\dots,x_n,y_1,\dots,y_m) \end{pmatrix} = 0$$

then matrix

$$\frac{\partial \mathbf{F}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$$

is called the **Jacobian matrix** of $\mathbf{F}(\mathbf{x},\mathbf{y})$ w.r.t. $\mathbf{y}$.

Analogous: $\frac{\partial \mathbf{F}(\mathbf{x},\mathbf{y})}{\partial \mathbf{x}}$

## Implicit Function Theorem III

Let $\mathbf{F}\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$,

$$(\mathbf{x},\mathbf{y}) \mapsto \mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} F_1(x_1,\dots,x_n,y_1,\dots,y_m) \\ \vdots \\ F_m(x_1,\dots,x_n,y_1,\dots,y_m) \end{pmatrix}$$

and let $(\mathbf{x}_0,\mathbf{y}_0)$ be a point with

$$\mathbf{F}(\mathbf{x}_0,\mathbf{y}_0) = 0 \quad \text{and} \quad \left|\frac{\partial \mathbf{F}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}\right| \neq 0 \quad \text{for } (\mathbf{x},\mathbf{y}) = (\mathbf{x}_0,\mathbf{y}_0).$$

Then there exists a hyper-rectangle $R$ around $(\mathbf{x}_0,\mathbf{y}_0)$ such that

- $\mathbf{F}(\mathbf{x},\mathbf{y}) = 0$ has a unique solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ in $R$, where $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$, and
- $\boxed{\dfrac{\partial \mathbf{y}}{\partial \mathbf{x}} = -\left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{y}}\right)^{-1} \cdot \left(\dfrac{\partial \mathbf{F}}{\partial \mathbf{x}}\right)}$



## Example – Implicit Function Theorem

Let $\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} F_1(x_1,x_2,y_1,y_2) \\ F_2(x_1,x_2,y_1,y_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - y_1^2 - y_2^2 + 3 \\ x_1^3 + x_2^3 + y_1^3 + y_2^3 - 11 \end{pmatrix}$

and $(\mathbf{x}_0,\mathbf{y}_0) = (1,1,1,2)$.

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \\ 3x_1^2 & 3x_2^2 \end{pmatrix} \qquad \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(1,1,1,2) = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} -2y_1 & -2y_2 \\ 3y_1^2 & 3y_2^2 \end{pmatrix} \qquad \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(1,1,1,2) = \begin{pmatrix} -2 & -4 \\ 3 & 12 \end{pmatrix}$$

As $\mathbf{F}(1,1,1,2) = 0$ and $\left|\frac{\partial \mathbf{F}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}\right| = -12 \neq 0$ we can apply the implicit function theorem and we find

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}\right)^{-1} \cdot \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right) = -\frac{1}{-12}\begin{pmatrix} 12 & 4 \\ -3 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$$



## Summary

- local existence of an inverse function
- derivative of an inverse Function
- inverse function theorem
- explicit and implicit function
- explicit representation of an implicit function
- derivative of an implicit function
- implicit function theorem



Chapter 11

# Taylor Series



## First-Order Approximation

We want to approximate function $f$ by some *simple* function.

Best possible approximation by a **linear function**:

$$\boxed{f(x) \doteq f(x_0) + f'(x_0)\,(x - x_0)}$$

$\doteq$ means "*first-order approximation*".

If we use this approximation, we calculate the value of the tangent at $x$ instead of $f$.



## Polynomials

We get a better approximation (i.e., with smaller approximation error) if we use a **polynomial** $P_n(x) = \sum_{k=0}^{n} a_k x^k$ of higher degree.

**Ansatz:**

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + R_n(x)$$

**Remainder term** $R_n(x)$ gives the approximation error when we replace function $f$ by approximation $P_n(x)$.

**Idea:**
Choose coefficients $a_i$ such that the derivatives of $f$ and $P_n$ coincide at $x_0 = 0$ up to order $n$.



## Derivatives

$$\begin{aligned}
f(x) &= a_0 + a_1 x + \cdots + a_n x^n &&= P_n(x) \\
&\Rightarrow f(0) = a_0 \\
f'(x) &= a_1 + 2\cdot a_2 x + \cdots + n \cdot a_n x^{n-1} &&= P_n'(x) \\
&\Rightarrow f'(0) = a_1 \\
f''(x) &= 2\cdot a_2 + 3\cdot 2\cdot a_3 x + \cdots + n\cdot(n-1)\cdot a_n x^{n-2} &&= P_n''(x) \\
&\Rightarrow f''(0) = 2\,a_2 \\
f'''(x) &= 3\cdot 2\cdot a_3 + \cdots + n\cdot(n-1)\cdot(n-2)\cdot a_n x^{n-3} &&= P_n'''(x) \\
&\Rightarrow f'''(0) = 3!\,a_3 \\
&\vdots \\
f^{(n)}(x) &= n\cdot(n-1)\cdot(n-2)\cdot\ldots\cdot 1\cdot a_n &&= P_n^{(n)}(x) \\
&\Rightarrow f^{(n)}(0) = n!\,a_n
\end{aligned}$$



## MacLaurin Polynomial

Thus we find for the coefficients of the polynomial

$$a_k = \frac{f^{(k)}(0)}{k!}$$

$f^{(k)}(x_0)$ denotes the $k$-th derivatives of $f$ at $x_0$, $\quad f^{(0)}(x_0) = f(x_0)$.

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k + R_n(x)}$$

This polynomial is called the **MacLaurin polynomial** of degree $n$ of $f$:

$$f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n$$

## Taylor Polynomial

This idea can be generalized to arbitrary exansion points $x_0$.
We then get the **Taylor polynomial** of degree $n$ of $f$ around point $x_0$:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k + R_n(x)}$$



## Taylor Series

The (infinite) series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k$$

is called the **Taylor series** of $f$ around $x_0$.

If $\lim_{n\to\infty} R_n(x) = 0$, then the Taylor series converges to $f(x)$.

We then say that we **expand** $f$ into a *Taylor series* around **expansion point** $x_0$.



## Example – Exponential Function

Taylor series expansion of $f(x) = e^x$ around $x_0 = 0$:

$$f(x) = f(0) + f'(0)\,x + \tfrac{f''(0)}{2!}\,x^2 + \tfrac{f'''(0)}{3!}\,x^3 + \cdots + \tfrac{f^{(n)}(0)}{n!}\,x^n + R_n(x)$$

$$\begin{aligned}
f(x) &= e^x &&\Rightarrow& f(0) &= 1\\
f'(x) &= e^x &&\Rightarrow& f'(0) &= 1\\
f''(x) &= e^x &&\Rightarrow& f''(0) &= 1\\
f'''(x) &= e^x &&\Rightarrow& f'''(0) &= 1\\
&\vdots\\
f^{(n)}(x) &= e^x &&\Rightarrow& f^{(n)}(0) &= 1
\end{aligned}$$

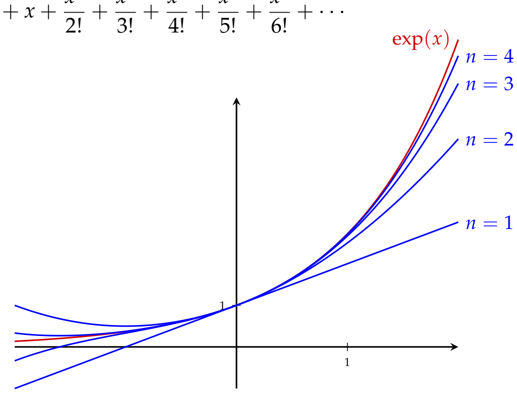
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

The Taylor series converges for all $x \in \mathbb{R}$.



## Example – Exponential Function

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$






## Example – Logarithm

Taylor series expansion of $f(x) = \ln(1+x)$ around $x_0 = 0$:

$$f(x) = f(0) + f'(0)\,x + \tfrac{f''(0)}{2!}\,x^2 + \tfrac{f'''(0)}{3!}\,x^3 + \cdots + \tfrac{f^{(n)}(0)}{n!}\,x^n + R_n(x)$$

$$\begin{aligned}
f(x) &= \ln(1+x) &&\Rightarrow& f(0) &= 0\\
f'(x) &= (1+x)^{-1} &&\Rightarrow& f'(0) &= 1\\
f''(x) &= -1\cdot(1+x)^{-2} &&\Rightarrow& f''(0) &= -1\\
f'''(x) &= 2\cdot 1\cdot(1+x)^{-3} &&\Rightarrow& f'''(0) &= 2!\\
&\vdots\\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!(1+x)^{-n+1} &&\Rightarrow& f^{(n)}(0) &= (-1)^{n-1}(n-1)!
\end{aligned}$$

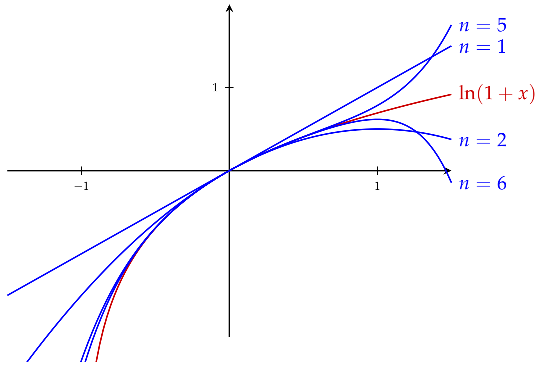
$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

The Taylor series converges for all $x \in (-1, 1)$.



## Example – Logarithm

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$





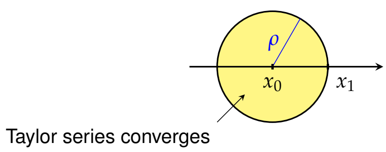

## Radius of Convergence

Some Taylor series do not converge for all $x \in \mathbb{R}$.
For example: $\ln(1+x)$

At least the following holds:

If a Taylor series converges for some $x_1$ with $|x_1 - x_0| = \rho$,
then it also converges for all $x$ with $|x - x_0| < \rho$.

The maximal value for $\rho$ is called the **radius of convergence** of the Taylor series.



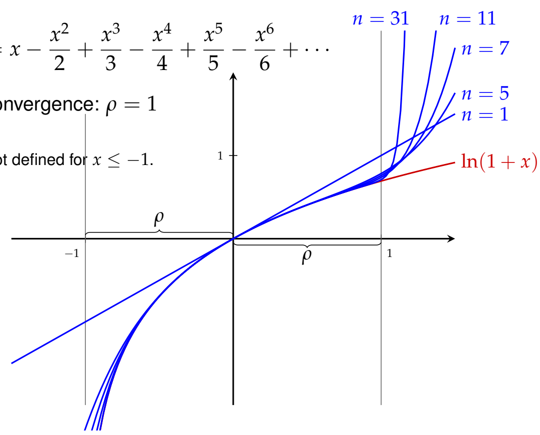



## Example – Radius of Convergence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

Radius of convergence: $\rho = 1$

Indication:
$\ln(1+x)$ is not defined for $x \leq -1$.



## Approximation Error

The remainder term indicates the error of the approximation by a Taylor polynomial.
It can be estimated by means of **Lagrange's form of the remainder**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-x_0)^{n+1}$$

for some point $\xi \in (x, x_0)$.

If the Taylor series converges we have

$$\boxed{R_n(x) = \mathcal{O}\left((x-x_0)^{n+1}\right) \quad \text{for} \quad x \to x_0}$$

We say: the remainder is of big O of $x^{n+1}$ as $x$ tends to $x_0$.

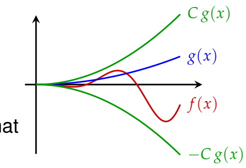


## Big O Notation

Let $f(x)$ and $g(x)$ be two functions.

We write

$$\boxed{f(x) = \mathcal{O}(g(x)) \quad \text{as} \quad x \to x_0}$$




if there exist reals numbers $C$ and $\varepsilon$ such that

$$|f(x)| < C \cdot |g(x)|$$

for all $x$ with $|x - x_0| < \varepsilon$.

We say that $f(x)$ is of big O of $g(x)$ as $x$ tends to $x_0$.

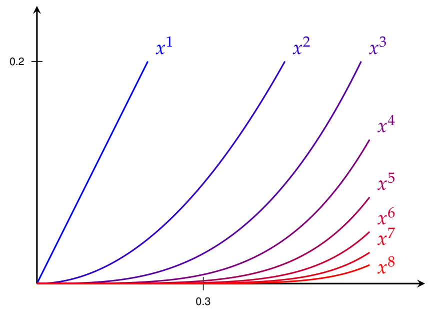
Symbol $\mathcal{O}(\cdot)$ belongs to the family of *Bachmann-Landau* notations.

Some books use notation $\quad f(x) \in \mathcal{O}(g(x)) \quad \text{as} \quad x \to x_0$



## Impact of Order of Powers

The higher the order of a monomial is,
the smaller is its contribution to the summation.






## Important Taylor Series

| $f(x)$ | | MacLaurin Series | $\rho$ |
|---|---|---|---|
| $\exp(x)$ | $=$ | $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$ | $\infty$ |
| $\ln(1+x)$ | $=$ | $x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$ | $1$ |
| $\sin(x)$ | $=$ | $x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ | $\infty$ |
| $\cos(x)$ | $=$ | $1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ | $\infty$ |
| $\frac{1}{1-x}$ | $=$ | $1 + x + x^2 + x^3 + x^4 + \cdots$ | $1$ |



## Calculations with Taylor Series

Taylor series can be conveniently

- added (term by term)
- differentiated (term by term)
- integrated (term by term)
- multiplied
- divided
- substituted

Therefore Taylor series are also used for the *Definition* of some function.

For example:

$$\exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$



## Example – Derivative

We get the first derivative of $\exp(x)$ by computing the derivative of its Taylor series:

$$\begin{aligned}
(\exp(x))' &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)' \\
&= 0 + 1 + \frac{2\,x}{2!} + \frac{3\,x^2}{3!} + \frac{4\,x^3}{4!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= \exp(x)
\end{aligned}$$



## Example – Product

We get the MacLaurin series of $f(x) = x^2 \cdot e^x$ by multiplying the MacLaurin series of $x^2$ with the MacLaurin series of $\exp(x)$:

$$\begin{aligned}
x^2 \cdot e^x &= x^2 \cdot \left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \cdots\right) \\
&= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \cdots
\end{aligned}$$



## Example – Substitution

We get the MacLaurin series of $f(x) = \exp(-x^2)$ by substituting of $-x^2$ into the MacLaurin series of $\exp(x)$:

$$\begin{aligned}
e^u &= 1 + u && + \tfrac{u^2}{2!} && + \tfrac{u^3}{3!} && + \tfrac{u^4}{4!} && + \cdots \\
e^{-x^2} &= 1 + (-x^2) && + \tfrac{(-x^2)^2}{2!} && + \tfrac{(-x^2)^3}{3!} && + \tfrac{(-x^2)^4}{4!} && + \cdots \\
&= 1 - x^2 && + \tfrac{x^4}{2!} && - \tfrac{x^6}{3!} && + \tfrac{x^8}{4!} && - \cdots
\end{aligned}$$

## Polynomials

The concept of Taylor series can be generalized to multivariate functions.

A polynomial of degree $n$ in *two* variables has the form

$$\begin{aligned} P_n(x_1, x_2) = \quad & a_0 \\ & + a_{10}\, x_1 + a_{11}\, x_2 \\ & + a_{20}\, x_1^2 + a_{21}\, x_1\, x_2 + a_{22}\, x_2^2 \\ & + a_{30}\, x_1^3 + a_{31}\, x_1^2\, x_2 + a_{32}\, x_1\, x_2^2 + a_{33}\, x_2^3 \\ & \quad \vdots \\ & + a_{n0}\, x_1^n + a_{n1}\, x_1^{n-1}\, x_2 + a_{n2}\, x_1^{n-2}\, x_2^2 + \cdots + a_{nn}\, x_2^n \end{aligned}$$

We choose coefficients $a_{kj}$ such that all its partial derivatives in expansion point $\mathbf{x}_0 = 0$ up to order $n$ coincides with the respective derivatives of $f$.



## Taylor Polynomial of Degree 2

We obtain the coefficients as

$$a_{kj} = \frac{1}{k!} \binom{k}{j} \frac{\partial^k f(0)}{(\partial x_1)^{k-j}\, (\partial x_2)^j} \qquad k \in \mathbb{N},\ j = 0, \cdots, k$$

In particular we find for the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$\begin{aligned} f(\mathbf{x}) = \quad & f(0) \\ & + f_{x_1}(0)\, x_1 + f_{x_2}(0)\, x_2 \\ & + \tfrac{1}{2} f_{x_1x_1}(0)\, x_1^2 + f_{x_1x_2}(0)\, x_1\, x_2 + \tfrac{1}{2} f_{x_2x_2}(0)\, x_2^2 + \cdots \end{aligned}$$



## Taylor Polynomial of Degree 2

Observe that the linear term can be written by means of the gradient:

$$f_{x_1}(0)\, x_1 + f_{x_2}(0)\, x_2 = \nabla f(0) \cdot \mathbf{x}$$

The quadratic term can be written by means of the Hessian matrix:

$$f_{x_1x_1}(0)\, x_1^2 + 2\, f_{x_1x_2}(0)\, x_1\, x_2 + f_{x_2x_2}(0)\, x_2^2 = \mathbf{x}^\mathsf{T} \cdot \mathbf{H}_f(0) \cdot \mathbf{x}$$

So we find for the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$\boxed{f(\mathbf{x}) = f(0) + \nabla f(0) \cdot \mathbf{x} + \tfrac{1}{2} \mathbf{x}^\mathsf{T} \cdot \mathbf{H}_f(0) \cdot \mathbf{x} + \mathcal{O}(\|\mathbf{x}\|^3)}$$

or in different notation

$$\boxed{f(\mathbf{x}) = f(0) + f'(0)\mathbf{x} + \tfrac{1}{2} \mathbf{x}^\mathsf{T} f''(0)\mathbf{x} + \mathcal{O}(\|\mathbf{x}\|^3)}$$



## Example – Bivariate Function

Compute the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$f(x,y) = e^{x^2-y^2} + x$$

$$\begin{aligned} f(x,y) &= e^{x^2-y^2} + x & \Rightarrow\ & f(0,0) = 1 \\ f_x(x,y) &= 2x\, e^{x^2-y^2} + 1 & \Rightarrow\ & f_x(0,0) = 1 \\ f_y(x,y) &= -2y\, e^{x^2-y^2} & \Rightarrow\ & f_y(0,0) = 0 \\ f_{xx}(x,y) &= 2\, e^{x^2-y^2} + 4x^2\, e^{x^2-y^2} & \Rightarrow\ & f_{xx}(0,0) = 2 \\ f_{xy}(x,y) &= -4xy\, e^{x^2-y^2} & \Rightarrow\ & f_{xy}(0,0) = 0 \\ f_{yy}(x,y) &= -2\, e^{x^2-y^2} + 4y^2\, e^{x^2-y^2} & \Rightarrow\ & f_{yy}(0,0) = -2 \end{aligned}$$

gradient:

$$\nabla f(0) = (1, 0)$$

Hessian matrix:

$$\mathbf{H}_f(0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$



## Example – Bivariate Function

Thus we find for the Taylor polynomial

$$\begin{aligned} f(x,y) &\approx f(0) + \nabla f(0) \cdot \mathbf{x} + \tfrac{1}{2} \mathbf{x}^\mathsf{T} \cdot \mathbf{H}_f(0) \cdot \mathbf{x} \\ &= 1 + (1,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \tfrac{1}{2}\, (x,y) \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\ &= 1 + x + x^2 - y^2 \end{aligned}$$



## Summary

- MacLaurin and Taylor polynomial
- Taylor series expansion
- radius of convergence
- calculations with Taylor series
- Taylor series of multivariate functions



Chapter 12

# Integration



## Antiderivative

A function $F(x)$ is called an **antiderivative** (or *primitive*) of function $f(x)$, if

$$\boxed{F'(x) = f(x)}$$

**Computation:**

Guess and verify

**Example:** We want the antiderivative of $f(x) = \ln(x)$.

Guess: $F(x) = x\,(\ln(x) - 1)$

Verify: $F'(x) = (x\,(\ln(x) - 1)' =$
$= 1 \cdot (\ln(x) - 1) + x \cdot \frac{1}{x} = \ln(x)$

But also: $F(x) = x\,(\ln(x) - 1) + 5$

## Antiderivative

The antiderivative is denoted by symbol

$$\boxed{\int f(x)\,dx + c}$$

and is also called the **indefinite integral** of function $f$. Number $c$ is called **integration constant**.

Unfortunately, there are no *"recipes"* for computing antiderivatives (but tools one can try and which may help).

There are functions where antiderivatives cannot be expressed by means of elementary functions.

E.g., the antiderivative of $\exp(-\frac{1}{2}x^2)$.



## Basic Integrals

Integrals of some elementary functions:

| $f(x)$ | $\int f(x)\,dx$ |
|---|---|
| $0$ | $c$ |
| $x^a$ | $\frac{1}{a+1} \cdot x^{a+1} + c$ |
| $e^x$ | $e^x + c$ |
| $\frac{1}{x}$ | $\ln\lvert x\rvert + c$ |
| $\cos(x)$ | $\sin(x) + c$ |
| $\sin(x)$ | $-\cos(x) + c$ |

(The table is created by swapping the columns in the list of derivatives.)



## Integration Rules

- **Summation rule**

$$\boxed{\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx}$$

- **Integration by parts**

$$\boxed{\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx}$$

- **Integration by substitution**

$$\boxed{\begin{array}{l}\displaystyle\int f(g(x)) \cdot g'(x)\,dx = \int f(z)\,dz \\ \text{with } z = g(x) \text{ and } dz = g'(x)\,dx\end{array}}$$



## Example – Summation Rule

Antiderivative of $f(x) = 4\,x^3 - x^2 + 3\,x - 5$.

$$\begin{aligned}\int f(x)\,dx &= \int 4\,x^3 - x^2 + 3\,x - 5\,dx \\ &= 4\int x^3\,dx - \int x^2\,dx + 3\int x\,dx - 5\int dx \\ &= 4\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3\frac{1}{2}x^2 - 5x + c \\ &= x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + c\end{aligned}$$



## Example – Integration by Parts

Antiderivative of $f(x) = x \cdot e^x$.

$$\int \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'}\,dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g}\,dx = x \cdot e^x - e^x + c$$

$$\begin{aligned}f &= x &&\Rightarrow & f' &= 1 \\ g' &= e^x &&\Rightarrow & g &= e^x\end{aligned}$$



## Example – Integration by Parts

Antiderivative of $f(x) = x^2\,\cos(x)$.

$$\int \underbrace{x^2}_{f} \cdot \underbrace{\cos(x)}_{g'}\,dx = \underbrace{x^2}_{f} \cdot \underbrace{\sin(x)}_{g} - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_{g}\,dx$$

Integration by parts of the second terms yields:

$$\begin{aligned}\int \underbrace{2x}_{f} \cdot \underbrace{\sin(x)}_{g'}\,dx &= \underbrace{2x}_{f} \cdot \underbrace{(-\cos(x))}_{g} - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_{g}\,dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c\end{aligned}$$

Thus the antiderivative of $f$ is given by

$$\int x^2\,\cos(x)\,dx = x^2\,\sin(x) + 2x\,\cos(x) - 2\,\sin(x) + c$$



## Example – Integration by Substitution

Antiderivative of $f(x) = 2x \cdot e^{x^2}$.

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)}\,dx = \int \exp(z)\,dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad \Rightarrow \quad dz = g'(x)\,dx = 2x\,dx$$



## Integration Rules – Derivation

Integration by parts follows from the product rule for derivatives:

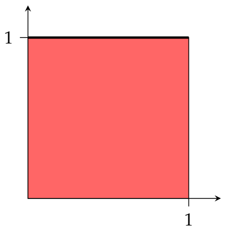
$$\begin{aligned}f(x) \cdot g(x) &= \int (f(x) \cdot g(x))'\,dx = \int (f'(x)\,g(x) + f(x)\,g'(x))\,dx \\ &= \int f'(x)\,g(x)\,dx + \int f(x)\,g'(x)\,dx\end{aligned}$$

Integration by substitution follows from the chain rule:
Let $F$ be an antiderivative of $f$ and let $z = g(x)$. Then

$$\begin{aligned}\int f(z)\,dz = F(z) = F(g(x)) &= \int (F(g(x)))'\,dx \\ &= \int F'(g(x))\,g'(x)\,dx = \int f(g(x))\,g'(x)\,dx\end{aligned}$$

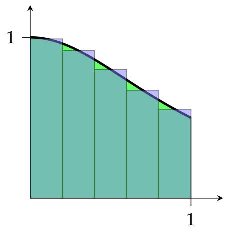
## Area

Compute the areas of the given regions.




$f(x) = 1$

Area: $A = 1$

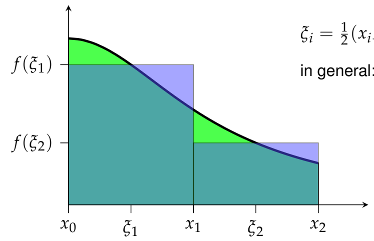



$f(x) = \frac{1}{1+x^2}$

Approximation by step function



## Riemann Sum



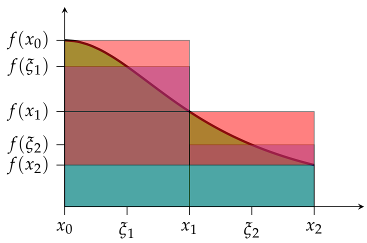

$\xi_i = \frac{1}{2}(x_{i-1} + x_i)$

in general: $\xi_i \in (x_i, x_{i-1})$

$$A = \int_a^b f(x)\,dx \approx \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$$



## Approximation Error




$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1}) \right| \leq (f_{\max} - f_{\min})\,(b-a)\,\frac{1}{n} \to 0$$

**Assumption:** Function monotone; $x_0, x_1, \dots, x_n$ equidistant



## Riemann Integral

If all sequences of **Riemann sums**

$$\boxed{I_n = \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})}$$

converge, then their (uniquely determined) limit is called the **Riemann integral** of $f$ and is denoted by $\int_a^b f(x)\,dx$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})}$$

Almost all functions in economics have a Riemann integral.



## Riemann Integral – Properties

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx \qquad \text{if } f(x) \leq g(x) \text{ for all } x \in [a,b]$$



## Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a *continuous* function $f(x)$, then we find

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason $\int f(x)\,dx$ is called an *indefinite integral* of $f$; and $\int_a^b f(x)\,dx$ is called a **definite integral** of $f$.



## Example – Fundamental Theorem

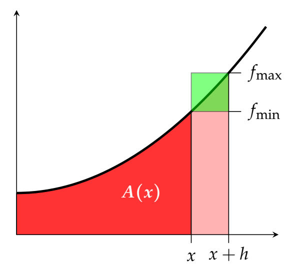
Compute the integral of $f(x) = x^2$ over interval $[0,1]$.

$$\int_0^1 x^2\,dx = \tfrac{1}{3}\,x^3\Big|_0^1 = \tfrac{1}{3} \cdot 1^3 - \tfrac{1}{3} \cdot 0^3 = \frac{1}{3}$$



## Fundamental Theorem / Proof Idea

Let $A(x)$ be the area between the graph of a continuous function $f$ and the $x$-axis from $0$ to $x$.




$$f_{\min} \cdot h \leq A(x+h) - A(x) \leq f_{\max} \cdot h$$

$$f_{\min} \leq \frac{A(x+h) - A(x)}{h} \leq f_{\max}$$

Limit for $h \to 0$: $(\lim_{h\to 0} f_{\min} = f(x))$

$$f(x) \leq \underbrace{\lim_{h\to 0} \frac{A(x+h) - A(x)}{h}}_{=A'(x)} \leq f(x)$$

$$A'(x) = f(x)$$

i.e. $A(x)$ is an antiderivative of $f(x)$.

## Integration Rules / (Definite Integrals)

- **Summation rule**

$$\boxed{\int_a^b \alpha f(x) + \beta g(x)\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx}$$

- **Integration by parts**

$$\boxed{\int_a^b f \cdot g'\,dx = f \cdot g\Big|_a^b - \int_a^b f' \cdot g\,dx}$$

- **Integration by Substitution**

$$\boxed{\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(z)\,dz}$$

with $z = g(x)$ and $dz = g'(x)\,dx$



## Example – Integration by Parts

Compute the definite integral $\int_0^2 x \cdot e^x\,dx$.

$$\int_0^2 \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'}\,dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g}\Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g}\,dx$$

$$= x \cdot e^x\Big|_0^2 - e^x\Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0)$$

$$= e^2 + 1$$

Note: we also could use our indefinite integral from above,

$$\int_0^2 x \cdot e^x\,dx = (x \cdot e^x - e^x)\Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$



## Example – Integration by Substitution

Compute the definite integral $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx$.

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\,dx = \int_1^{\ln(10)} \frac{1}{z}\,dz =$$

$$z = \ln(x) \quad \Rightarrow \quad dz = \frac{1}{x}\,dx$$

$$= \ln(z)\Big|_1^{\ln(10)} =$$

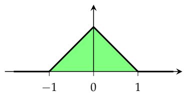
$$= \ln(\ln(10)) - \ln(1) \approx 0.834$$



## Example – Subdomains

Compute $\int_{-2}^{2} f(x)\,dx$ for function

$$f(x) = \begin{cases} 1 + x, & \text{for } -1 \le x < 0, \\ 1 - x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$




We have

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx$$

$$= \int_{-2}^{-1} 0\,dx + \int_{-1}^{0} (1 + x)\,dx + \int_{0}^{1} (1 - x)\,dx + \int_{1}^{2} 0\,dx$$

$$= (x + \tfrac{1}{2}x^2)\Big|_{-1}^{0} + (x - \tfrac{1}{2}x^2)\Big|_{0}^{1}$$

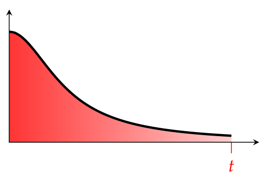
$$= \tfrac{1}{2} + \tfrac{1}{2} = 1$$

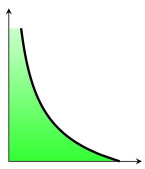


## Improper Integral

An **improper integral** is an integral where

- the domain of integration is unbounded, or
- the integrand is unbounded.



$$\boxed{\int_0^\infty f(x)\,dx = \lim_{t \to \infty} \int_0^t f(x)\,dx}$$

$$\boxed{\int_0^1 f(x)\,dx = \lim_{t \to 0} \int_t^1 f(x)\,dx}$$



## Example – Improper Integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t \to 0} \int_t^1 x^{-\frac{1}{2}}\,dx = \lim_{t \to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t \to 0} (2 - 2\sqrt{t}) = 2$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t \to \infty} \int_1^t x^{-2}\,dx = \lim_{t \to \infty} -\frac{1}{x}\Big|_1^t = \lim_{t \to \infty} -\frac{1}{t} - (-1) = 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx = \lim_{t \to \infty} \ln(x)\Big|_1^t = \lim_{t \to \infty} \ln(t) - \ln(1) = \infty$$

The improper integral does not exist.



## Two Limits

In probability theory we often have integrals where both boundaries are infinite.
For example, the expectation of random variable $X$ with density $f$ is defined as

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

In such a case we have to separate the domain of integration:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \lim_{t \to -\infty} \int_t^0 x \cdot f(x)\,dx + \lim_{s \to \infty} \int_0^s x \cdot f(x)\,dx$$

**Beware!**
If we yield $\infty - \infty$, then the result is **not** $\infty - \infty = 0$!



## Leibniz Integration Rule

Let $f(x,t)$ be *continously differentiable* function
(i.e., all partial derivatives exist and are continuous) and let
$F(x) = \int_{a(x)}^{b(x)} f(x,t)\,dt$.

Then

$$\boxed{F'(x) = f(x, b(x))\, b'(x) - f(x, a(x))\, a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt}$$

If $a(x) = a$ and $b(x) = b$ are constant, then

$$\boxed{\frac{d}{dx}\left(\int_a^b f(x,t)\,dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\,dt}$$

## Example – Leibniz Integration Rule

Let $F(x) = \int_x^{2x} t\,x^2\,dt$ for $x > 0$. Compute $F'(x)$.

We set $f(x,t) = t\,x^2$, $a(x) = x$ and $b(x) = 2x$
and apply Leibniz's integration rule:

$$\begin{aligned}
F'(x) &= f(x,b)\cdot b' - f(x,a)\cdot a' + \int_a^b f_x(x,t)\,dt \\
&= (2x)\,x^2\cdot 2 - (x)\,x^2\cdot 1 + \int_x^{2x} 2x\,t\,dt \\
&= 4x^3 - x^3 + \left(2x\,\tfrac{1}{2}t^2\right)\Big|_x^{2x} \\
&= 4x^3 - x^3 + (4x^3 - x^3) \\
&= 6x^3
\end{aligned}$$

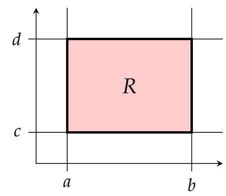


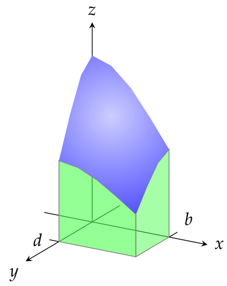
## Volume

Let $f(x,y)$ be a function with domain

$$R = [a,b]\times[c,d] = \{(x,y)\in\mathbb{R}^2 \colon a \le x \le b,\ c \le y \le d\}$$

What is the Volumen $V$ below the graph of $f$?

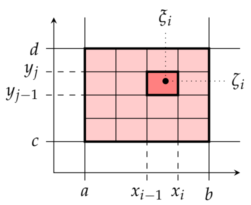


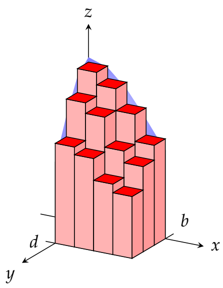






## Riemann Sums

- Partition $R$ into smaller rectangles $R_{ij} = [x_{i-1}, x_i]\times[y_{j-1}, y_j]$
- Estimate $V \approx \sum_{i=1}^{n}\sum_{j=1}^{k} f(\xi_i, \zeta_j)\,(x_i - x_{i-1})\,(y_j - y_{j-1})$





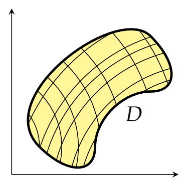

## Riemann Integral

If these **Riemann Sums** converge for partitions of $R$ with increasing number of rectangles, then the limit is called the
**Riemann Integral** of $f$ over $R$:

$$\boxed{\iint_R f(x,y)\,dx\,dy = \lim_{n,k\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{k} f(\xi_i,\zeta_j)\,(x_i - x_{i-1})\,(y_j - y_{j-1})}$$

The Riemann integral is defined analogously for arbitrary domains $D$.

$$\iint_D f(x,y)\,dx\,dy$$






## Fubini's Theorem

Let $f\colon R = [a,b]\times[c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ a *continuous* function. Then

$$\boxed{\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy\,.\end{aligned}}$$

Fubini's theorem provides a recipe for computating double integrals stepwise:

1. Treat $x$ like a constant and compute the inner integral $\int_c^d f(x,y)\,dy$ w.r.t. variabel $y$.
2. Integrate the result from Step 1 w.r.t. $x$.

We also may change the order of integration.



## Example – Fubini's Theorem

Compute $\int_{-1}^{1}\int_0^1 (1 - x - y^2 + xy^2)\,dx\,dy$.

We have to integrate twice:

$$\begin{aligned}
&\int_{-1}^{1}\int_0^1 (1 - x - y^2 + xy^2)\,dx\,dy \\
&= \int_{-1}^{1}\left( x - \frac{1}{2}x^2 - xy^2 + \frac{1}{2}x^2y^2\Big|_0^1 \right) dy \\
&= \int_{-1}^{1}\left(\frac{1}{2} - \frac{1}{2}y^2\right) dy = \frac{1}{2}y - \frac{1}{6}y^3\Big|_{-1}^{1} \\
&= \frac{1}{2} - \frac{1}{6} - \left(-\frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3}
\end{aligned}$$



## Bounds of Integration

**Beware!**
The integration variables and the corresponding integration limits have to be read from **inside** to **outside**.

If we change the order of integration, then we also have to exchange the integration limits:

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = \int_c^d\int_a^b f(x,y)\,dx\,dy$$

This may be more obvious if we add (redundant) parenthesis:

$$\int_a^b\left(\int_c^d f(x,y)\,dy\right)dx = \int_c^d\left(\int_a^b f(x,y)\,dx\right)dy\,.$$



## Example – Fubini's Theorem

Integration in reversed order:

$$\begin{aligned}
&\int_{-1}^{1}\int_0^1 (1 - x - y^2 + xy^2)\,dx\,dy \\
&= \int_0^1\left(\int_{-1}^{1} (1 - x - y^2 + xy^2)\,dy\right) dx \\
&= \int_0^1\left( y - xy - \frac{1}{3}y^3 + \frac{1}{3}xy^3\Big|_{-1}^{1}\right) dx \\
&= \int_0^1\left(1 - x - \frac{1}{3} + \frac{1}{3}x - \left(-1 + x + \frac{1}{3} - \frac{1}{3}x\right)\right) dx \\
&= \int_0^1\left(\frac{4}{3} - \frac{4}{3}x\right) dx = \frac{4}{3}x - \frac{4}{6}x^2\Big|_0^1 = \frac{2}{3}
\end{aligned}$$

## Fubini's Theorem – Interpretation

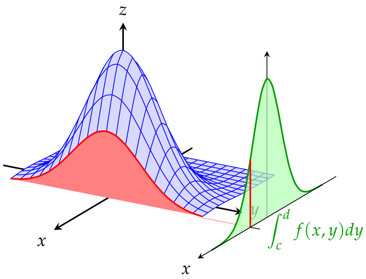
$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \int_a^b A(x)dx$$

If we fix $x$, then

$$A(x) = \int_c^d f(x,y)\,dy$$

is the area below curve

$$g(y) = f(x,y).$$






## Summary

- antiderivate
- Riemann sum and Riemann integral
- indefinite and definite integral
- Fundamental Theorem of Calculus
- integration rules
- Leibniz integration rule
- improper integral
- double integral
- Fubini's theorem



Chapter 13

# Convex and Concave



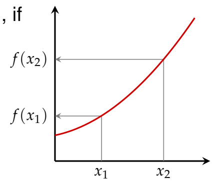
## Monotone Functions*

Function $f$ is called **monotonically increasing**, if

$$\boxed{x_1 \leq x_2 \;\Rightarrow\; f(x_1) \leq f(x_2)}$$

It is called *strictly monotonically increasing*, if

$$\boxed{x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)}$$

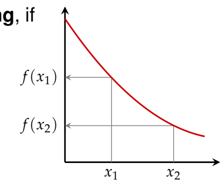



Function $f$ is called **monotonically decreasing**, if

$$\boxed{x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)}$$

It is called *strictly monotonically decreasing*, if

$$\boxed{x_1 < x_2 \;\Leftrightarrow\; f(x_1) > f(x_2)}$$






## Monotone Functions*

For differentiable functions we have

| | | | |
|---|---|---|---|
| $f$ monotonically increasing | $\Leftrightarrow$ | $f'(x) \geq 0$ | for all $x \in D_f$ |
| $f$ monotonically decreasing | $\Leftrightarrow$ | $f'(x) \leq 0$ | for all $x \in D_f$ |

| | | | |
|---|---|---|---|
| $f$ strictly monotonically increasing | $\Leftarrow$ | $f'(x) > 0$ | for all $x \in D_f$ |
| $f$ strictly monotonically decreasing | $\Leftarrow$ | $f'(x) < 0$ | for all $x \in D_f$ |

Function $f\colon (0,\infty),\ x \mapsto \ln(x)$ is strictly monotonically increasing, as

$$f'(x) = (\ln(x))' = \frac{1}{x} > 0 \quad \text{for all } x > 0$$



## Locally Monotone Functions*

A function $f$ can be monotonically increasing in some interval and decreasing in some other interval.

For *continuously* differentiable functions (i.e., when $f'(x)$ is continuous) we can use the following procedure:

1. Compute first derivative $f'(x)$.
2. Determine all roots of $f'(x)$.
3. We thus obtain intervals where $f'(x)$ does not change sign.
4. Select appropriate points $x_i$ in each interval and determine the sign of $f'(x_i)$.



## Example – Locally Monotone Functions*

In which region is function $f(x) = 2x^3 - 12x^2 + 18x - 1$ monotonically increasing?

We have to solve inequality $f'(x) \geq 0$:

1. $f'(x) = 6x^2 - 24x + 18$
2. Roots: $x^2 - 4x + 3 = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 3$
3. Obtain 3 intervals: $(-\infty, 1]$, $[1,3]$, and $[3,\infty)$
4. Sign of $f'(x)$ at appropriate points in each interval: $f'(0) = 3 > 0$, $f'(2) = -1 < 0$, and $f'(4) = 3 > 0$.
5. $f'(x)$ cannot change sign in each interval: $f'(x) \geq 0$ in $(-\infty, 1]$ and $[3,\infty)$.

Function $f(x)$ is monotonically increasing in $(-\infty, 1]$ and in $[3,\infty)$.



## Monotone and Inverse Function

If $f$ is *strictly monotonically increasing*, then

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)$$

immediately implies

$$x_1 \neq x_2 \;\Leftrightarrow\; f(x_1) \neq f(x_2)$$

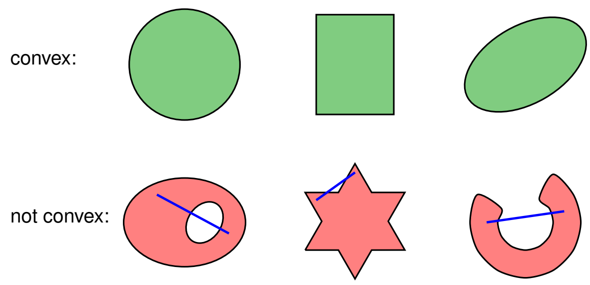
That is, $f$ is *one-to-one*.

So if $f$ is onto and strictly monotonically increasing (or decreasing), then $f$ is **invertible**.

## Convex Set

A set $D \subseteq \mathbb{R}^n$ is called **convex**, if for any two points $\mathbf{x}, \mathbf{y} \in D$ the straight line segment between these points also belongs to $D$, i.e.,

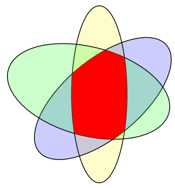
$$(1-h)\,\mathbf{x} + h\,\mathbf{y} \in D \qquad \text{for all } h \in [0,1], \text{ and } \mathbf{x}, \mathbf{y} \in D\,.$$






## Intersection of Convex Sets

Let $S_1, \ldots, S_k$ be convex subsets of $\mathbb{R}^n$. Then their *intersection* $S_1 \cap \ldots \cap S_k$ is also convex.



The union of convex sets need not be convex.



## Example – Half-Space

Let $\mathbf{p} \in \mathbb{R}^n$ and $m \in \mathbb{R}$ be fixed, $\mathbf{p} \neq 0$. Then

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{p}^\mathsf{T} \cdot \mathbf{x} = m\}$$

is a so called **hyper-plane** which partitions the $\mathbb{R}^n$ into two **half-spaces**

$$H_+ = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \geq m\}\,,$$
$$H_- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m\}\,.$$

Sets $H$, $H_+$ and $H_-$ are convex.

Let $\mathbf{x}$ be a vector of goods, $\mathbf{p}$ the vector of prices and $m$ the budget. Then the budget set is convex.

$$\{\mathbf{x} \in \mathbb{R}^n : \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m, \mathbf{x} \geq 0\}$$
$$= \{\mathbf{x} : \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m\} \cap \{\mathbf{x} : x_1 \geq 0\} \cap \ldots \cap \{\mathbf{x} : x_n \geq 0\}$$



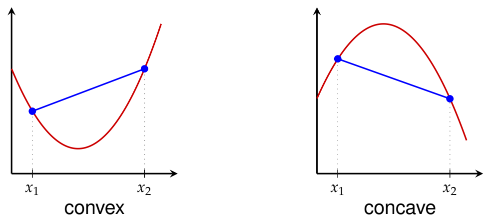
## Convex and Concave Functions

Function $f$ is called **convex** in domain $D \subseteq \mathbb{R}^n$, if $D$ is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \leq (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ and all $h \in [0,1]$. It is called **concave**, if

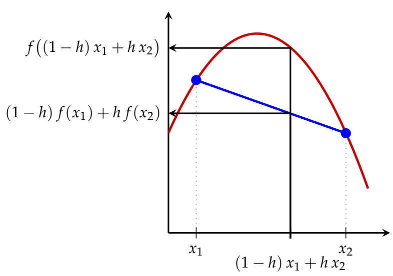
$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \geq (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$






## Concave Function*

$$f\big((1-h)\,x_1 + h\,x_2\big) \;\geq\; (1-h)\,f(x_1) + h\,f(x_2)$$




Secant is below the graph of function $f$.



## Strictly Convex and Concave Functions

Function $f$ is **strictly convex** in domain $D \subseteq \mathbb{R}^n$, if $D$ is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) < (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ and all $h \in (0,1)$.

Function $f$ is **strictly concave** in domain $D \subseteq \mathbb{R}^n$, if $D$ is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) > (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ and all $h \in (0,1)$.



## Example – Linear Function

Let $\mathbf{a} \in \mathbb{R}^n$ be fixed.
Then $f(\mathbf{x}) = \mathbf{a}^\mathsf{T} \cdot \mathbf{x}$ is a linear map and we find:

$$\begin{aligned} f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) &= \mathbf{a}^\mathsf{T} \cdot ((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \\ &= (1-h)\,\mathbf{a}^\mathsf{T} \cdot \mathbf{x}_1 + h\,\mathbf{a}^\mathsf{T} \cdot \mathbf{x}_2 \\ &= (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2) \end{aligned}$$

That is, every *linear function* is both *concave and convex.*

However, a linear function is neither strictly concave nor strictly convex, as the inequality is never strict.



## Example – Quadratic Univariate Function

Function $f(x) = x^2$ is *strictly convex*:

$$\begin{aligned} & f((1-h)\,x + h\,y) - \big[(1-h)\,f(x) + h\,f(y)\big] \\ &\quad = ((1-h)\,x + h\,y)^2 - \big[(1-h)\,x^2 + h\,y^2\big] \\ &\quad = (1-h)^2\,x^2 + 2(1-h)h\,xy + h^2\,y^2 - (1-h)\,x^2 - h\,y^2 \\ &\quad = -h(1-h)\,x^2 + 2(1-h)h\,xy - h(1-h)\,y^2 \\ &\quad = -h(1-h)\,(x-y)^2 \\ &\quad < 0 \quad \text{for } x \neq y \text{ and } 0 < h < 1. \end{aligned}$$

Thus

$$f((1-h)\,x + h\,y) < (1-h)\,f(x) + h\,f(y)$$

for all $x \neq y$ and $0 < h < 1$,
i.e., $f(x) = x^2$ is strictly convex, as claimed.

## Properties

- If $f(\mathbf{x})$ is (strictly) *convex*, then $-f(\mathbf{x})$ is (strictly) *concave* (and vice versa).
- If $f_1(\mathbf{x}), \dots, f_k(\mathbf{x})$ are *convex* (concave) functions and $\alpha_1, \dots, \alpha_k > 0$, then

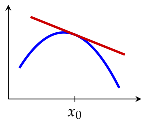
  $$g(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \dots + \alpha_k f_k(\mathbf{x})$$

  is also *convex* (concave).
- If (at least) one of the functions $f_i(x)$ is *strictly convex* (strictly concave), then $g(x)$ is strictly convex (strictly concave).



## Properties

For a differentiable functions the following holds:




- Function $f$ is **concave** if and only if

  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) \le \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}$$

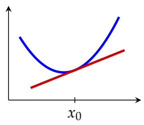
  i.e., the function graph is always below the tangent.
- Function $f$ is **strictly concave** if and only if

  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) < \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \quad \text{for all } \mathbf{x} \neq \mathbf{x}_0}$$

- Function $f$ is **convex** if and only if

  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) \ge \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}$$

  (Analogous for strictly convex functions.)

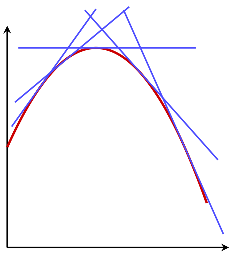





## Univariate Functions*

For two times differentiable functions we have

| | | | |
|---|---|---|---|
| $f$ convex | $\Leftrightarrow$ | $f''(x) \ge 0$ | for all $x \in D_f$ |
| $f$ concave | $\Leftrightarrow$ | $f''(x) \le 0$ | for all $x \in D_f$ |



Derivative $f'(x)$ is monotonically decreasing,

thus $f''(x) \le 0$.



## Univariate Functions*

For two times differentiable functions we have

| | | | |
|---|---|---|---|
| $f$ strictly convex | $\Leftarrow$ | $f''(x) > 0$ | for all $x \in D_f$ |
| $f$ strictly concave | $\Leftarrow$ | $f''(x) < 0$ | for all $x \in D_f$ |

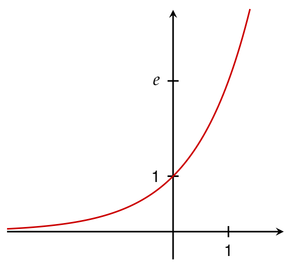


## Example – Convex Function*

Exponential function:

$$f(x) = e^x$$
$$f'(x) = e^x$$
$$f''(x) = e^x > 0 \quad \text{for all } x \in \mathbb{R}$$

$\exp(x)$ is (strictly) convex.



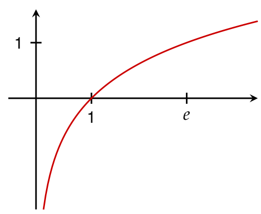



## Example – Concave Function*

Logarithm function: $(x > 0)$

$$f(x) = \ln(x)$$
$$f'(x) = \frac{1}{x}$$
$$f''(x) = -\frac{1}{x^2} < 0 \quad \text{for all } x > 0$$

$\ln(x)$ is (strictly) concave.






## Locally Convex Functions*

A function $f$ can be convex in some interval and concave in some other interval.

For two times *continuously* differentiable functions (i.e., when $f''(x)$ is continuous) we can use the following procedure:

1. Compute second derivative $f''(x)$.
2. Determine all roots of $f''(x)$.
3. We thus obtain intervals where $f''(x)$ does not change sign.
4. Select appropriate points $x_i$ in each interval and determine the sign of $f''(x_i)$.



## Locally Concave Function*

In which region is $f(x) = 2x^3 - 12x^2 + 18x - 1$ concave?

We have to solve inequality $f''(x) \le 0$.

1. $f''(x) = 12x - 24$
2. Roots: $12x - 24 = 0 \quad \Rightarrow \quad x = 2$
3. Obtain 2 intervals: $(-\infty, 2]$ and $[2, \infty)$
4. Sign of $f''(x)$ at appropriate points in each interval: $f''(0) = -24 < 0$ and $f''(4) = 24 > 0$.
5. $f''(x)$ cannot change sign in each interval: $f''(x) \le 0$ in $(-\infty, 2]$

Function $f(x)$ is concave in $(-\infty, 2]$.

## Univariate Restrictions

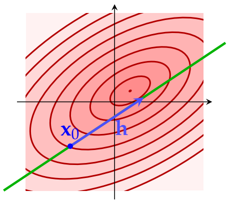
Notice, that by definition a (multivariate) function is convex if and only if every restriction of its domain to a straight line results in a convex univariate function. That is:

Function $f\colon D \subset \mathbb{R}^n \to \mathbb{R}$ is convex

if and only if

$g(t) = f(\mathbf{x}_0 + t \cdot \mathbf{h})$ is convex

for all $\mathbf{x}_0 \in D$ and
all non-zero $\mathbf{h} \in \mathbb{R}^n$.






## Quadratic Form

Let $\mathbf{A}$ be a symmetric matrix
and $q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^{\mathsf{T}}\mathbf{A}\mathbf{x}$ be the corresponding quadratic form.

Matrix $\mathbf{A}$ can be diagonalized, i.e., if we use an orthonormal basis of its eigenvectors, then $\mathbf{A}$ becomes a diagonal matrix with the eigenvalues of $\mathbf{A}$ as its elements:

$$q_{\mathbf{A}}(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \dots + \lambda_n x_n^2 \,.$$

- It is convex if all eigenvalues $\lambda_i \geq 0$
  as it is the sum of convex functions.
- It is concave if all $\lambda_i \leq 0$
  as it is the negative of a convex function.
- It is neither convex nor concave if we have eigenvalues with $\lambda_i > 0$ and $\lambda_i < 0$.



## Quadratic Form

We find for a quadratic form $q_{\mathbf{A}}$:

- *strictly convex* $\Leftrightarrow$ *positive definite*
- *convex* $\Leftrightarrow$ *positive semidefinite*
- *strictly concave* $\Leftrightarrow$ *negative definite*
- *concave* $\Leftrightarrow$ *negative semidefinite*
- neither $\Leftrightarrow$ *indefinite*

We can determine the definiteness of $\mathbf{A}$ by means of

- the eigenvalues of $\mathbf{A}$, or
- the (leading) principle minors of $\mathbf{A}$.



## Example – Quadratic Form

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Leading principle minors:

$$A_1 = 2 \quad > 0$$

$$A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 \quad > 0$$

$\mathbf{A}$ is thus positive definite.
Quadratic form $q_{\mathbf{A}}$ is *strictly convex*.



## Example – Quadratic Form

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -2 \end{pmatrix}$. Principle Minors:

$$A_1 = -1 \qquad A_2 = -4 \qquad A_3 = -2$$

$$A_{1,2} = \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} = 4 \qquad A_{1,3} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 \qquad A_{2,3} = \begin{vmatrix} -4 & 2 \\ 2 & -2 \end{vmatrix} = 4$$

$$A_{1,2,3} = \begin{vmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$A_i \quad \leq 0$
$A_{i,j} \quad \geq 0$
$A_{1,2,3} \quad \leq 0$

$\mathbf{A}$ is thus negative semidefinite.
Quadratic form $q_{\mathbf{A}}$ is *concave* (but not strictly concave).



## Concavity of Differentiable Functions

Le $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ with Taylor series expansion

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{h} + \tfrac{1}{2}\mathbf{h}^{\mathsf{T}} \cdot \mathbf{H}_f(\mathbf{x}_0) \cdot \mathbf{h} + \mathcal{O}(\|\mathbf{h}\|^3)$$

*Hessian matrix* $\mathbf{H}_f(\mathbf{x}_0)$ determines the concavity or convexity of $f$ around expansion point $\mathbf{x}_0$.

- $\mathbf{H}_f(\mathbf{x}_0)$ *positive definite* $\Rightarrow$ $f$ *strictly convex* around $\mathbf{x}_0$
- $\mathbf{H}_f(\mathbf{x}_0)$ *negative definite* $\Rightarrow$ $f$ *strictly concave* around $\mathbf{x}_0$

- $\mathbf{H}_f(\mathbf{x})$ *positive semidefinite* for all $\mathbf{x} \in D$ $\Leftrightarrow$ $f$ *convex* in $D$
- $\mathbf{H}_f(\mathbf{x})$ *negative semidefinite* for all $\mathbf{x} \in D$ $\Leftrightarrow$ $f$ *concave* in $D$



## Recipe – Strictly Convex

1. Compute Hessian matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \cdots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \cdots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

2. Compute all *leading principle minors* $H_i$.

3. 
   - $f$ *strictly convex* $\Leftrightarrow$ all $H_k > 0$ for (almost) **all** $\mathbf{x} \in D$
   - $f$ *strictly concave* $\Leftrightarrow$ all $(-1)^k H_k > 0$ for (almost) **all** $\mathbf{x} \in D$

   [ $(-1)^k H_k > 0$ implies: $H_1, H_3, \ldots < 0$ and $H_2, H_4, \ldots > 0$ ]

4. Otherwise $f$ is *neither* **strictly** convex *nor* strictly concave.



## Recipe – Convex

1. Compute Hessian matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \cdots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \cdots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

2. Compute all *principle minors* $H_{i_1,\ldots,i_k}$.
   (Only required if $\det(\mathbf{H}_f) = 0$, see below)

3. 
   - $f$ *convex* $\Leftrightarrow$ all $H_{i_1,\ldots,i_k} \geq 0$ for **all** $\mathbf{x} \in D$.
   - $f$ *concave* $\Leftrightarrow$ all $(-1)^k H_{i_1,\ldots,i_k} \geq 0$ for **all** $\mathbf{x} \in D$.

4. Otherwise $f$ is *neither* convex *nor* concave.

## Recipe – Convex II

Computation of *all* principle minors can be avoided if $\det(\mathbf{H}_f) \neq 0$. Then a function is either strictly convex/concave (and thus convex/concave) or neither convex nor concave.

In particular we have the following recipe:

1. Compute Hessian matrix $\mathbf{H}_f(\mathbf{x})$.
2. Compute all *leading principle minors* $H_i$.
3. Check if $\det(\mathbf{H}_f) \neq 0$.
4. Check for strict convexity or concavity.
5. If $\det(\mathbf{H}_f) \neq 0$ and $f$ is neither strictly convex nor concave, then $f$ is neither convex nor concave, either.



## Example – Strict Convexity

Is function $f$ (strictly) concave or convex?

$$f(x,y) = x^4 + x^2 - 2\,x\,y + y^2$$

1. Hessian matrix: $\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} 12\,x^2+2 & -2 \\ -2 & 2 \end{pmatrix}$
2. Leading principle minors:
   $H_1 = 12\,x^2 + 2 \qquad > 0$
   $H_2 = |\mathbf{H}_f(\mathbf{x})| = 24\,x^2 \qquad > 0 \qquad$ for all $x \neq 0$.
3. All leading principle minors $> 0$ for almost all $\mathbf{x}$
   $\Rightarrow$ $f$ is *strictly convex*. (and thus convex, too)



## Example – Cobb-Douglas Function

Let $f(x,y) = x^\alpha y^\beta$ with $\alpha,\beta \geq 0$ and $\alpha+\beta \leq 1$,
and $D = \{(x,y)\colon x,y \geq 0\}$.

Hessian matrix at $\mathbf{x}$:

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} \alpha(\alpha-1)\,x^{\alpha-2}y^\beta & \alpha\beta\,x^{\alpha-1}y^{\beta-1} \\ \alpha\beta\,x^{\alpha-1}y^{\beta-1} & \beta(\beta-1)\,x^\alpha y^{\beta-2} \end{pmatrix}$$

Principle Minors:

$$H_1 = \underbrace{\alpha}_{\geq 0}\ \underbrace{(\alpha-1)}_{\leq 0}\ \underbrace{x^{\alpha-2}y^\beta}_{\geq 0} \qquad \leq 0$$

$$H_2 = \underbrace{\beta}_{\geq 0}\ \underbrace{(\beta-1)}_{\leq 0}\ \underbrace{x^{\alpha}y^{\beta-2}}_{\geq 0} \qquad \leq 0$$



## Example – Cobb-Douglas Function

$$\begin{aligned} H_{1,2} &= |\mathbf{H}_f(\mathbf{x})| \\ &= \alpha(\alpha-1)\,x^{\alpha-2}y^\beta \cdot \beta(\beta-1)\,x^\alpha y^{\beta-2} - (\alpha\beta\,x^{\alpha-1}y^{\beta-1})^2 \\ &= \alpha(\alpha-1)\,\beta(\beta-1)\,x^{2\alpha-2}y^{2\beta-2} - \alpha^2\beta^2\,x^{2\alpha-2}y^{2\beta-2} \\ &= \alpha\beta[(\alpha-1)(\beta-1) - \alpha\beta]x^{2\alpha-2}y^{2\beta-2} \\ &= \underbrace{\alpha\beta}_{\geq 0}\ \underbrace{(1-\alpha-\beta)}_{\geq 0}\ \underbrace{x^{2\alpha-2}y^{2\beta-2}}_{\geq 0} \qquad \geq 0 \end{aligned}$$

$H_1 \leq 0$ and $H_2 \leq 0$, and $H_{1,2} \geq 0$ **for all** $(x,y) \in D$.
$f(x,y)$ thus is *concave* in $D$.

For $0 < \alpha,\beta < 1$ and $\alpha+\beta < 1$ we even find:
$H_1 = H_1 < 0$ and $H_2 = |\mathbf{H}_f(\mathbf{x})| > 0$ **for** almost **all** $(x,y) \in D$.
$f(x,y)$ is then *strictly concave*.

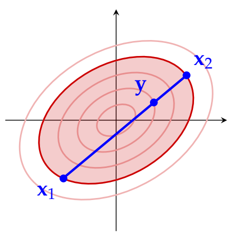


## Lower Level Sets of Convex Functions

Assume that $f$ is *convex*.
Then the **lower level sets** of $f$

$$\{\mathbf{x} \in D_f\colon f(\mathbf{x}) \leq c\}$$

are *convex*.




Let $\mathbf{x}_1, \mathbf{x}_2 \in \{\mathbf{x} \in D_f\colon f(\mathbf{x}) \leq c\}$,
i.e., $f(\mathbf{x}_1), f(\mathbf{x}_2) \leq c$.

Then for $\mathbf{y} = (1-h)\mathbf{x}_1 + h\mathbf{x}_2$
where $h \in [0,1]$ we find

$$\begin{aligned} f(\mathbf{y}) &= f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \\ &\leq (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2) \\ &\leq (1-h)c + hc = c \end{aligned}$$

That is, $\mathbf{y} \in \{\mathbf{x} \in D_f\colon f(\mathbf{x}) \leq c\}$, too.

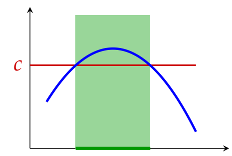


## Upper Level Sets of Concave Functions

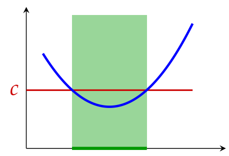
Assume that $f$ is *concave*.
Then the **upper level sets** of $f$

$$\{\mathbf{x} \in D_f\colon f(\mathbf{x}) \geq c\}$$

are *convex*.




upper level set




lower level set



## Extremum and Monotone Transformation

Let $T\colon \mathbb{R} \to \mathbb{R}$ be a *strictly monotonically increasing* function.
If $\mathbf{x}^*$ is a *maximum* of $f$, then $\mathbf{x}^*$ is also a maximum of $T \circ f$.

As $\mathbf{x}^*$ is a *maximum* of $f$, we have

$f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x}$.

As $T$ is strictly monotonically increasing,we have

$T(x_1) > T(x_2)$ falls $x_1 > x_2$.

Thus we find

$(T \circ f)(\mathbf{x}^*) = T(f(\mathbf{x}^*)) \geq T(f(\mathbf{x})) = (T \circ f)(\mathbf{x})$ for all $\mathbf{x}$,

i.e., $\mathbf{x}^*$ is a maximum of $T \circ f$.

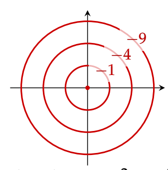
As $T$ is one-to-one we also get the converse statement:
If $\mathbf{x}^*$ is a *maximum* of $T \circ f$, then it also is a maximum of $f$.



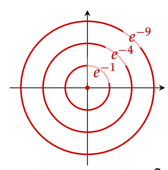
## Extremum and Monotone Transformation

A strictly monotonically increasing Transformation $T$ preserves the extrema of $f$.

Transformation $T$ also preserves the level sets of $f$:




$f(x,y) = -x^2 - y^2$




$T(f(x,y)) = \exp(-x^2 - y^2)$

## Quasi-Convex and Quasi-Concave

Function $f$ is called **quasi-convex** in $D \subseteq \mathbb{R}^n$, if $D$ is *convex* and every *lower level set* $\{\mathbf{x} \in D_f : f(\mathbf{x}) \leq c\}$ is *convex*.

Function $f$ is called **quasi-concave** in $D \subseteq \mathbb{R}^n$, if $D$ is *convex* and every *upper level set* $\{\mathbf{x} \in D_f : f(\mathbf{x}) \geq c\}$ is *convex*.



## Convex and Quasi-Convex

Every *concave* (convex) function also is *quasi-concave* (and quasi-convex, resp.).

However, a quasi-concave function need not be concave.

Let $T$ be a strictly monotonically increasing function.
If function $f(\mathbf{x})$ is *concave* (convex), then $T \circ f$ is *quasi-concave* (and quasi-convex, resp.).

Function $g(x, y) = e^{-x^2-y^2}$ is quasi-concave, as $f(x, y) = -x^2 - y^2$ is concave and $T(x) = e^x$ is strictly monotonically increasing.
However, $g = T \circ f$ is not concave.



## A Weaker Condition

The notion of *quasi-convex* is **weaker** than that of *convex* in the sense that every convex function also is quasi-convex but not vice versa.
There are much more quasi-convex functions than convex ones.

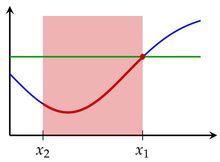
The importance of such a weaker notions is based on the observation that a couple of propositions still hold if “convex” is replaced by “quasi-convex”.

In this way we get a generalization of a theorem, where a *stronger* condition is replaced by a *weaker* one.

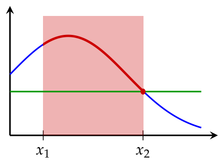


## Quasi-Convex and Quasi-Concave II

- Function $f$ is *quasi-convex* if and only if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \leq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$ and $h \in [0, 1]$.
- Function $f$ is *quasi-concave* if and only if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \geq \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$ and $h \in [0, 1]$.



quasi-convex



quasi-concave



## Strictly Quasi-Convex and Quasi-Concave

- Function $f$ is called **strictly quasi-convex** if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) < \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$, with $\mathbf{x}_1 \neq \mathbf{x}_2$, and $h \in (0, 1)$.
- Function $f$ is called **strictly quasi-concave** if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) > \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$, with $\mathbf{x}_1 \neq \mathbf{x}_2$, and $h \in (0, 1)$.



## Quasi-convex and Quasi-Concave III

For a differentiable function $f$ we find:

- Function $f$ is *quasi-convex* if and only if
$$\boxed{f(\mathbf{x}) \leq f(\mathbf{x}_0) \quad \Rightarrow \quad \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \leq 0}$$
- Function $f$ is *quasi-concave* if and only if
$$\boxed{f(\mathbf{x}) \geq f(\mathbf{x}_0) \quad \Rightarrow \quad \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0}$$



## Summary

- monotone function
- convex set
- convex and concave function
- convexity and definiteness of quadratic form
- minors of Hessian matrix
- quasi-convex and quasi-concave function



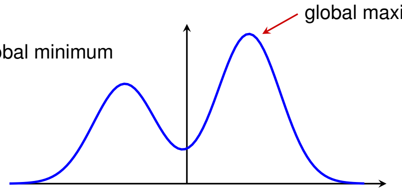
Chapter 14

# Extrema

## Global Extremum (Optimum)

A point $x^*$ is called **global maximum** (*absolute maximum*) of $f$, if for all $x \in D_f$,
$$f(x^*) \geq f(x)\ .$$

A point $x^*$ is called **global minimum** (*absolute minimum*) of $f$, if for all $x \in D_f$,
$$f(x^*) \leq f(x)\ .$$

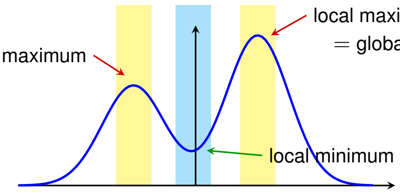





## Local Extremum (Optimum)

A point $x_0$ is called **local maximum** (*relative maximum*) of $f$, if for all $x$ in some *neighborhood* of $x_0$,
$$f(x_0) \geq f(x)\ .$$

A point $x_0$ is called **local minimum** (*relative minimum*) of $f$, if for all $x$ in some neighborhood of $x_0$,
$$f(x_0) \leq f(x)\ .$$

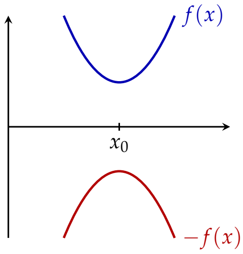





## Minima and Maxima

**Notice!**
Every minimization problem can be transformed into a maximization problem (and vice versa).

Point $x_0$ is a minimum of $f(x)$, if and only if $x_0$ is a maximum of $-f(x)$.






## Critical Point

At a (local) maximum or minimum the first derivative of the function must vanish (i.e., must be equal to 0).

A point $x_0$ is called a **critical point** (or *stationary point*) of function $f$, if
$$f'(x_0) = 0$$

*Necessary condition* for differentiable functions:

Each extremum of $f$ is a critical point of $f$.



## Global Extremum

*Sufficient condition*:

Let $x_0$ be a critical point of $f$.
If $f$ is **concave**, then $x_0$ is a **global maximum** of $f$.
If $f$ is *convex*, then $x_0$ is a *global minimum* of $f$.

If $f$ is **strictly** concave (or convex), then the extremum is *unique*.

This condition immediately follows from the properties of (strictly) concave functions. Indeed, we have for all $\mathbf{x} \neq \mathbf{x}_0$,
$$f(\mathbf{x}) - f(\mathbf{x}_0) \leq \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) = 0 \cdot (\mathbf{x} - \mathbf{x}_0) = 0$$
and thus
$$f(\mathbf{x}_0) \geq f(\mathbf{x})\ .$$



## Example – Global Extremum / Univariate*

Let $f(x) = e^x - 2x$.

Function $f$ is strictly convex:
$$f'(x) = e^x - 2$$
$$f''(x) = e^x \quad > 0 \quad \textbf{for all } x \in \mathbb{R}$$

Critical point:
$$f'(x) = e^x - 2 = 0 \quad \Rightarrow \quad x_0 = \ln 2$$

$x_0 = \ln 2$ is the (unique) global minimum of $f$.



## Example – Global Extremum / Multivariate

Let $f\colon D = [0,\infty)^2 \to \mathbb{R},\ f(x,y) = 4\,x^{\frac{1}{4}}y^{\frac{1}{4}} - x - y$

Hessian matrix at $\mathbf{x}$:
$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} -\frac{3}{4}\,x^{-\frac{7}{4}}y^{\frac{1}{4}} & \frac{1}{4}\,x^{-\frac{3}{4}}y^{-\frac{3}{4}} \\ \frac{1}{4}\,x^{-\frac{3}{4}}y^{-\frac{3}{4}} & -\frac{3}{4}\,x^{\frac{1}{4}}y^{-\frac{7}{4}} \end{pmatrix}$$

Leading principle minors:
$$H_1 = -\tfrac{3}{4}\,x^{-\frac{7}{4}}y^{\frac{1}{4}} \quad < 0$$
$$H_2 = \tfrac{1}{2}\,x^{-\frac{3}{2}}y^{-\frac{3}{2}} \quad > 0$$
$f$ is strictly concave in $D$.

critical point: $\quad \nabla f = (x^{-\frac{3}{4}}y^{\frac{1}{4}} - 1, x^{\frac{1}{4}}y^{-\frac{3}{4}} - 1) = 0$
$$\begin{aligned} f_x &= x^{-\frac{3}{4}}y^{\frac{1}{4}} - 1 = 0 \\ f_y &= x^{\frac{1}{4}}y^{-\frac{3}{4}} - 1 = 0 \end{aligned} \quad \Rightarrow \quad \mathbf{x}_0 = (1,1)$$

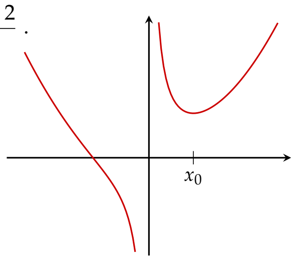
$\mathbf{x}_0$ is the global maximum of $f$.



## Sources of Errors

Find all global minima of $f(x) = \dfrac{x^3 + 2}{3x}$.

1. $f'(x) = \frac{2(x^3-1)}{3x^2}$,
   $f''(x) = \frac{2x^3+4}{3x^3}$.
2. critical point at $x_0 = 1$.
3. $f''(1) = 2 > 0$
   $\Rightarrow$ global minimum **???**

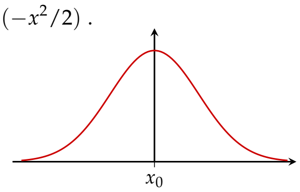



*However*, looking *just* at $f''(\mathbf{1})$ is not sufficient as we are looking for *global* minima!

**Beware!** We have to look at $f''(x)$ *at all* $x \in D_f$.
However, $f''(-1) = -\frac{2}{3} < 0$.
Moreover, domain $D = \mathbb{R} \setminus \{0\}$ is not an interval.
So $f$ is not convex and we cannot apply our theorem.

## Sources of Errors

Find all global maxima of $f(x) = \exp(-x^2/2)$ .




1. $f'(x) = x\exp(-x^2)$,
   $f''(x) = (x^2-1)\exp(-x^2)$.
2. critical point at $x_0 = 0$.
3. However,
   $f''(0) = -1 < 0$ but $f''(2) = 2e^{-2} > 0$.
   So $f$ is not concave and thus there cannot be a global maximum.
   **Really ???**

**Beware!** We are checking a *sufficient* condition.
Since an assumption does not hold ($f$ is not concave),
we simply **cannot apply** the theorem.
We *cannot* conclude that $f$ does not have a global maximum.



## Global Extrema in $[a,b]$*

Extrema of $f(x)$ in **closed** interval $[a,b]$.

**Procedure** for differentiable functions:

**(1)** Compute $f'(x)$.

**(2)** Find all stationary points $x_i$ (i.e., $f'(x_i) = 0$).

**(3)** Evaluate $f(x)$ for all *candidates*:
- all stationary points $x_i$,
- boundary points $a$ and $b$.

**(4)** Largest of these values is **global maximum**,
smallest of these values is **global minimum**.

It is *not* necessary to compute $f''(x_i)$.



## Global Extrema in $[a,b]$*

Find all *global* extrema of function

$$f\colon [0{,}5;8{,}5] \to \mathbb{R},\ x \mapsto \frac{1}{12}x^3 - x^2 + 3x + 1$$

**(1)** $f'(x) = \frac{1}{4}x^2 - 2x + 3$.

**(2)** $\frac{1}{4}x^2 - 2x + 3 = 0$ has roots $x_1 = 2$ and $x_2 = 6$.

**(3)**
$f(0.5) = 2.260$
$f(2) = 3.667$
$f(6) = 1.000 \quad \Rightarrow$ global minimum
$f(8.5) = 5.427 \quad \Rightarrow$ global maximum

**(4)** $x_2 = 6$ is the global minimum and
$b = 8.5$ is the global maximum of $f$.



## Global Extrema in $(a,b)$*

Extrema of $f(x)$ in **open** interval $(a,b)$ (or $(-\infty,\infty)$).

**Procedure** for differentiable functions:

**(1)** Compute $f'(x)$.

**(2)** Find all stationary points $x_i$ (i.e., $f'(x_i) = 0$).

**(3)** Evaluate $f(x)$ for all *stationary* points $x_i$.

**(4)** Determine $\lim_{x\to a} f(x)$ and $\lim_{x\to b} f(x)$.

**(5)** Largest of these values is **global maximum**,
smallest of these values is **global minimum**.

**(6)** A global extremum exists **only if** the largest (smallest) value
is obtained in a *stationary point*!



## Global Extrema in $(a,b)$*

Compute all *global* extrema of

$$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto e^{-x^2}$$

**(1)** $f'(x) = -2x\,e^{-x^2}$.

**(2)** $f'(x) = -2x\,e^{-x^2} = 0$ has unique root $x_1 = 0$.

**(3)**
$f(0) = 1 \quad \Rightarrow$ global maximum
$\lim_{x\to-\infty} f(x) = 0 \quad \Rightarrow$ no global minimum
$\lim_{x\to\infty} f(x) = 0$

**(4)** The function has a global maximum in $x_1 = 0$,
but no global minimum.



## Existence and Uniqueness

- A function need not have maxima or minima:

  $$f\colon (0,1) \to \mathbb{R},\ x \mapsto x$$

  (Points 0 and 1 are not in domain $(0,1)$.)

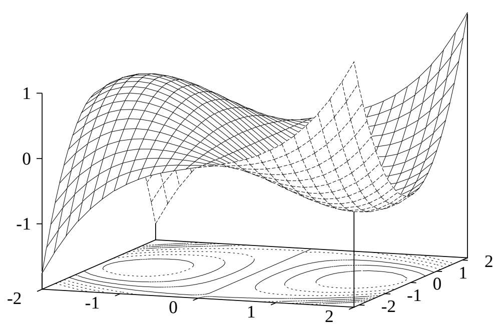
- (Global) maxima need not be unique:

  $$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto x^4 - 2x^2$$

  has two global minima at $-1$ and $1$.



## Example – Local Extrema




$$f(x,y) = \frac{1}{6}x^3 - x + \frac{1}{4}xy^2$$



## Local Extremum

A point $x_0$ is a **local maximum** (or *local minimum*) of $f$, if
- $x_0$ is a **critical point** of $f$,
- $f$ is **locally concave** (and *locally convex*, resp.) around $x_0$.

*Sufficient condition* for two times differentiable functions:

Let $x_0$ be a critical point of $f$. Then
- $f''(x_0)$ *negative definite* $\Rightarrow$ $x_0$ is local maximum
- $f''(x_0)$ *positive definite* $\Rightarrow$ $x_0$ is local minimum

It is sufficient to evaluate $f''(x)$ at the critical point $x_0$.
(In opposition to the condition for global extrema.)

## Necessary and Sufficient

We again want to explain two important concepts using the example of local minima.

Condition "$f'(x_0) = 0$" is **necessary** for a local minimum:

Every local minimum must have this properties.
However, not every point with such a property is a local minimum (e.g. $x_0 = 0$ in $f(x) = x^3$).
Stationary points are *candidates* for local extrema.

Condition "$f'(x_0) = 0$ *and* $f''(x_0)$ *is positive definite*" is **sufficient** for a local minimum.

If it is satisfied, then $x_0$ is a local minimum.
However, there are local minima where this condition does not hold (e.g. $x_0 = 0$ in $f(x) = x^4$).
If it is *not* satisfied, we cannot draw *any conclusion*.



## Procedure – Univariate Functions*

*Sufficient condition*
for local extrema of a differentiable function in *one* variable:

1. Compute $f'(x)$ and $f''(x)$.
2. Find all roots $x_i$ of $f'(x_i) = 0$ (critical points).
3. If $f''(x_i) < 0 \;\Rightarrow\;$ $x_i$ is a *local maximum*.
   If $f''(x_i) > 0 \;\Rightarrow\;$ $x_i$ is a *local minimum*.
   If $f''(x_i) = 0 \;\Rightarrow\;$ *no conclusion possible*!

If $f''(x_i) = 0$ we need more sophisticated methods!
(E.g., terms of higher order of the Taylor series expansion around $x_i$.)

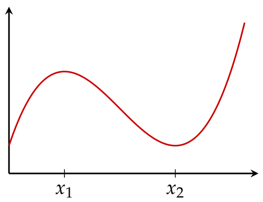


## Example – Local Extrema*

Find all local extrema of

$$f(x) = \frac{1}{12}x^3 - x^2 + 3x + 1$$

1. $f'(x) = \frac{1}{4}x^2 - 2x + 3$,
   $f''(x) = \frac{1}{2}x - 2$.
2. $\frac{1}{4}x^2 - 2x + 3 = 0$
   has roots
   $x_1 = 2$ and $x_2 = 6$.




3. $f''(2) = -1 \;\Rightarrow\;$ $x_1$ is a local maximum.
   $f''(6) = 1 \;\Rightarrow\;$ $x_2$ is a local minimum.



## Example – Critical Points

Compute all critical points of

$$f(x,y) = \frac{1}{6}x^3 - x + \frac{1}{4}xy^2$$

Partial derivatives:

$$\begin{array}{lll} (I) & f_x = \frac{1}{2}x^2 - 1 + \frac{1}{4}y^2 & = 0 \\ (II) & f_y = \frac{1}{2}xy & = 0 \end{array}$$

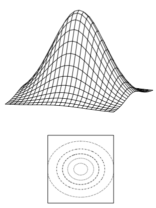
$$\begin{array}{llrcr} (II) & \Rightarrow & x = 0 & \text{or} & y = 0 \\ (I) & \Rightarrow & -1 + \frac{1}{4}y^2 = 0 & \Big| & \frac{1}{2}x^2 - 1 = 0 \\ & & y = \pm 2 & \Big| & x = \pm\sqrt{2} \end{array}$$

Critical points:

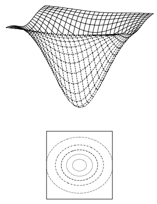
$$\mathbf{x}_1 = (0, 2) \qquad \mathbf{x}_3 = (\sqrt{2}, 0)$$
$$\mathbf{x}_2 = (0, -2) \qquad \mathbf{x}_4 = (-\sqrt{2}, 0)$$



## Critical Point – Local Extrema



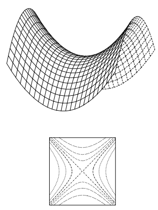
local maximum



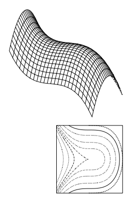
local minimum



## Critical Point – Saddle Point



saddle point



example for higher order



## Procedure – Local Extrema

1. Compute gradient $\nabla f(x)$ and Hessian matrix $\mathbf{H}_f$.
2. Find all $\mathbf{x}_i$ with $\nabla f(\mathbf{x}_i) = 0$ (critical points).
3. Compute leading principle minors $H_k$ for all *critical points* $\mathbf{x}_i$:
   **(a)** All leading principle minors $H_k > 0$
   $\Rightarrow$ $\mathbf{x}_0$ is a **locale minimum** of $f$.
   **(b)** For all leading principle minors, $(-1)^k H_k > 0$
   [ i.e., $H_1, H_3, \ldots < 0$ and $H_2, H_4, \ldots > 0$ ]
   $\Rightarrow$ $\mathbf{x}_0$ is a **locale maximum** of $f$.
   **(c)** $\det(\mathbf{H}_f(\mathbf{x}_i)) \neq 0$ but neither (a) nor (b) is satisfied
   $\Rightarrow$ $\mathbf{x}_0$ is a **saddle point** of $f$.
   **(d)** Otherwise *no conclusion* can be drawn,
   i.e., $\mathbf{x}_i$ may or may not be an extremum or saddle point.



## Procedure – Bivariate Function

1. Compute gradient $\nabla f(x)$ and Hessian matrix $\mathbf{H}_f$.
2. Find all $\mathbf{x}_i$ with $\nabla f(\mathbf{x}_i) = 0$ (critical points).
3. Compute leading principle minors $H_k$ for all *critical points* $\mathbf{x}_i$:
   **(a)** $H_2 > 0$ and $H_1 > 0$
   $\Rightarrow$ $\mathbf{x}_0$ is a **locale minimum** of $f$.
   **(b)** $H_2 > 0$ and $H_1 < 0$
   $\Rightarrow$ $\mathbf{x}_0$ is a **locale maximum** of $f$.
   **(c)** $H_2 < 0$
   $\Rightarrow$ $\mathbf{x}_0$ is a **saddle point** of $f$.
   **(d)** $H_2 = \det(\mathbf{H}_f(\mathbf{x}_0)) = 0$
   $\Rightarrow$ *no conclusion* can be drawn,
   i.e., $\mathbf{x}_i$ may or may not be an extremum or saddle point.

## Example – Bivariate Function

Compute all local extrema of of

$$f(x,y) = \frac{1}{6}x^3 - x + \frac{1}{4}xy^2$$

1. $\nabla f = (\frac{1}{2}x^2 - 1 + \frac{1}{4}y^2, \frac{1}{2}xy)$

   $$\mathbf{H}_f(x,y) = \begin{pmatrix} x & \frac{1}{2}y \\ \frac{1}{2}y & \frac{1}{2}x \end{pmatrix}$$

2. Critical points:

   $$\mathbf{x}_1 = (0,2),\ \mathbf{x}_2 = (0,-2),\ \mathbf{x}_3 = (\sqrt{2},0),\ \mathbf{x}_4 = (-\sqrt{2},0)$$



## Example – Bivariate Function / Cont.

3. Leading principle minors:

   $$\mathbf{H}_f(\mathbf{x}_1) = \mathbf{H}_f(0,2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

   $H_2 = -1 < 0 \quad \Rightarrow \quad \mathbf{x}_1$ is a *saddle point*

   $$\mathbf{H}_f(\mathbf{x}_2) = \mathbf{H}_f(0,-2) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

   $H_2 = -1 < 0 \quad \Rightarrow \quad \mathbf{x}_2$ is a *saddle point*



## Example – Bivariate Function / Cont.

3. Leading principle minors:

   $$\mathbf{H}_f(\mathbf{x}_3) = \mathbf{H}_f(\sqrt{2},0) = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

   $H_2 = 1 > 0$ and $H_1 = \sqrt{2} > 0$

   $\Rightarrow \quad \mathbf{x}_3$ is a *local minimum*

   $$\mathbf{H}_f(\mathbf{x}_4) = \mathbf{H}_f(-\sqrt{2},0) = \begin{pmatrix} -\sqrt{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} \end{pmatrix}$$

   $H_2 = 1 > 0$ and $H_1 = -\sqrt{2} < 0$

   $\Rightarrow \quad \mathbf{x}_4$ is a *local maximum*



## Derivative of Optimal Value

Let $p, r > 0$ and $f\colon D = [0,\infty)^2 \to \mathbb{R},\ f(x,y) = 4\,x^{\frac{1}{4}}y^{\frac{1}{4}} - px - ry$

Hessian matrix: $\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} -\frac{3}{4}x^{-\frac{7}{4}}y^{\frac{1}{4}} & \frac{1}{4}x^{-\frac{3}{4}}y^{-\frac{3}{4}} \\ \frac{1}{4}x^{-\frac{3}{4}}y^{-\frac{3}{4}} & -\frac{3}{4}x^{\frac{1}{4}}y^{-\frac{7}{4}} \end{pmatrix}$

Leading principle minors:

$H_1 = -\frac{3}{4}x^{-\frac{7}{4}}y^{\frac{1}{4}} < 0$

$H_2 = \frac{1}{2}x^{-\frac{3}{2}}y^{-\frac{3}{2}} > 0$

$f$ is strictly concave in $D$.

Critical point: $\nabla f = (x^{-\frac{3}{4}}y^{\frac{1}{4}} - p, x^{\frac{1}{4}}y^{-\frac{3}{4}} - r) = 0$

$f_x = x^{-\frac{3}{4}}y^{\frac{1}{4}} - p = 0$

$f_y = x^{\frac{1}{4}}y^{-\frac{3}{4}} - r = 0$

$\Rightarrow \quad \mathbf{x}_0 = \left(\sqrt{\frac{1}{rp^3}}, \sqrt{\frac{1}{r^3p}}\right)$

$\mathbf{x}_0$ is the global maximum of $f$.

**Question:**
What is the derivative of optimal value $f^* = f(\mathbf{x}_0)$ w.r.t. $r$ or $p$?



## Envelope Theorem

We are given function

$\mathbf{x} \mapsto f(\mathbf{x}, \mathbf{r})$

$\mathbf{x} = (x_1, \dots, x_n)$ ... variable (endogeneous)

$\mathbf{r} = (r_1, \dots, r_k)$ ... parameter (exogeneous)

with extremum $\mathbf{x}^*$.

This extremum depends on parameter $\mathbf{r}$:

$\mathbf{x}^* = \mathbf{x}^*(\mathbf{r})$

and so does the optimal value $f^*$:

$f^*(\mathbf{r}) = f(\mathbf{x}^*(\mathbf{r}), \mathbf{r})$

We have:

$$\boxed{\frac{\partial f^*(\mathbf{r})}{\partial r_j} = \left.\frac{\partial f(\mathbf{x},\mathbf{r})}{\partial r_j}\right|_{\mathbf{x}=\mathbf{x}^*(\mathbf{r})}}$$



## Envelope Theorem / Proof Idea

$$\frac{\partial f^*(\mathbf{r})}{\partial r_j} = \left.\frac{\partial f(\mathbf{x}^*(\mathbf{r}),\mathbf{r})}{\partial r_j}\right|_{\mathbf{x}=\mathbf{x}^*(\mathbf{r})} \qquad \text{[ chain rule ]}$$

$$= \sum_{i=1}^n \underbrace{f_{x_i}(\mathbf{x}^*(\mathbf{r}),\mathbf{r})}_{\substack{=0 \\ \text{as } \mathbf{x}^* \text{ is a critical point}}} \cdot \frac{\partial x_i^*(\mathbf{r})}{\partial r_j} + \left.\frac{\partial f(\mathbf{x},\mathbf{r})}{\partial r_j}\right|_{\mathbf{x}=\mathbf{x}^*(\mathbf{r})}$$

$$= \left.\frac{\partial f(\mathbf{x},\mathbf{r})}{\partial r_j}\right|_{\mathbf{x}=\mathbf{x}^*(\mathbf{r})}$$



## Example – Envelope Theorem

The (unique) maximum of

$$f\colon D = [0,\infty)^2 \to \mathbb{R},\quad f(x,y) = 4\,x^{\frac{1}{4}}y^{\frac{1}{4}} - px - ry$$

is $\mathbf{x}^*(p,r) = (x^*(p,r), y^*(p,r)) = \left(\sqrt{\frac{1}{rp^3}}, \sqrt{\frac{1}{r^3p}}\right)$.

**Question:**
What is the derivative of optimal value $f^* = f(\mathbf{x}_0)$ w.r.t. $r$ or $p$?

$$\frac{\partial f^*(p,r)}{\partial p} = \left.\frac{\partial f(\mathbf{x};p,r)}{\partial p}\right|_{\mathbf{x}=\mathbf{x}^*(p,r)} = -x\Big|_{\mathbf{x}=\mathbf{x}^*(p,r)} = -\sqrt{\frac{1}{rp^3}}$$

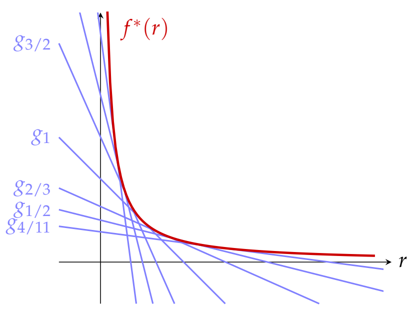
$$\frac{\partial f^*(p,r)}{\partial r} = \left.\frac{\partial f(\mathbf{x};p,r)}{\partial r}\right|_{\mathbf{x}=\mathbf{x}^*(p,r)} = -y\Big|_{\mathbf{x}=\mathbf{x}^*(p,r)} = -\sqrt{\frac{1}{r^3p}}$$



## A Geometric Interpretation

Let $f(x,r) = \sqrt{x} - rx$. We want $f^*(r) = \max_x f(x,r)$.

Graphs of $g_x(r) = f(x,r)$ for various values of $x$.



## Summary

- global extremum
- local extremum
- minimum, maximum and saddle point
- critical point
- hessian matrix and principle minors
- envelope theorem



Chapter 15

# Lagrange Function



## Constraint Optimization

Find the extrema of function

$$f(x, y)$$

subject to

$$g(x, y) = c$$

Example:
Find the extrema of function

$$f(x, y) = x^2 + 2\,y^2$$

subject to

$$g(x, y) = x + y = 3$$



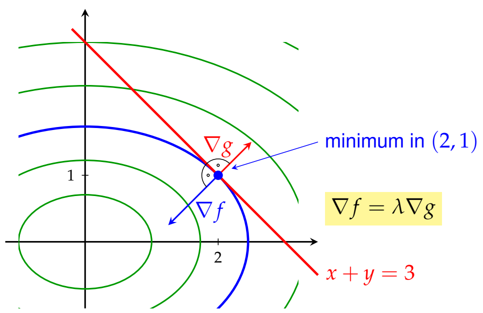
## Graphical Solution

For the case of two variables we can find a solution graphically.

1. Draw the constraint $g(x, y) = c$ in the $xy$-plain.
   (The *feasible region* is a curve in the plane)
2. Draw *appropriate* contour lines of objective function $f(x, y)$.
3. Investigate which contour lines of the objective function intersect with the feasible region.
   Estimate the (approximate) location of the extrema.



## Example – Graphical Solution




Extrema of $f(x, y) = x^2 + 2\,y^2$ subject to $g(x, y) = x + y = 3$



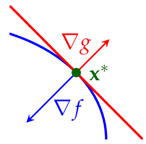
## Lagrange Approach

Let $\mathbf{x}^*$ be an extremum of $f(x, y)$ subject to $g(x, y) = c$.
Then $\nabla f(\mathbf{x}^*)$ and $\nabla g(\mathbf{x}^*)$ are proportional, i.e.,

$$\nabla f(\mathbf{x}^*) = \lambda\, \nabla g(\mathbf{x}^*)$$

where $\lambda$ is some proportionality factor.

$$\begin{aligned} f_x(\mathbf{x}^*) &= \lambda\, g_x(\mathbf{x}^*) \\ f_y(\mathbf{x}^*) &= \lambda\, g_y(\mathbf{x}^*) \\ g(\mathbf{x}^*) &= c \end{aligned}$$




Transformation yields

$$\begin{aligned} f_x(\mathbf{x}^*) - \lambda\, g_x(\mathbf{x}^*) &= 0 \\ f_y(\mathbf{x}^*) - \lambda\, g_y(\mathbf{x}^*) &= 0 \\ c - g(\mathbf{x}^*) &= 0 \end{aligned}$$

The l.h.s. is the gradient of $\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda\,(c - g(x, y))$ .



## Lagrange Function

We create a new function from $f$, $g$ and an auxiliary variable $\lambda$, called **Lagrange function**:

$$\boxed{\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda\,(c - g(x, y))}$$

Auxiliary variable $\lambda$ is called **Lagrange multiplier**.

Local extrema of $f$ subject to $g(x, y) = c$ are critical points of Lagrange function $\mathcal{L}$:

$$\begin{aligned} \mathcal{L}_x &= f_x - \lambda\, g_x &&= 0 \\ \mathcal{L}_y &= f_y - \lambda\, g_y &&= 0 \\ \mathcal{L}_\lambda &= c - g(x, y) &&= 0 \end{aligned}$$



## Example – Lagrange Function

Compute the local extrema of

$$f(x, y) = x^2 + 2y^2 \quad \text{subject to} \quad g(x, y) = x + y = 3$$

Lagrange function:

$$\mathcal{L}(x, y, \lambda) = (x^2 + 2y^2) + \lambda(3 - (x + y))$$

Critical points:

$$\begin{aligned} \mathcal{L}_x &= 2x - \lambda && = 0 \\ \mathcal{L}_y &= 4y - \lambda && = 0 \\ \mathcal{L}_\lambda &= 3 - x - y && = 0 \end{aligned}$$

$\Rightarrow$ unique critical point: $(\mathbf{x}_0; \lambda_0) = (2, 1; 4)$

## Bordered Hessian Matrix

Matrix

$$\boxed{\bar{\mathbf{H}}(\mathbf{x};\lambda) = \begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{pmatrix}}$$

is called the **bordered Hessian Matrix**.

*Sufficient condition for local extremum:*

Let $(\mathbf{x}_0;\lambda_0)$ be a critical point of $\mathcal{L}$.

- $|\bar{\mathbf{H}}(\mathbf{x}_0;\lambda_0)| > 0 \quad\Rightarrow\quad$ $\mathbf{x}_0$ is a *local maximum*
- $|\bar{\mathbf{H}}(\mathbf{x}_0;\lambda_0)| < 0 \quad\Rightarrow\quad$ $\mathbf{x}_0$ is a *local minimum*
- $|\bar{\mathbf{H}}(\mathbf{x}_0;\lambda_0)| = 0 \quad\Rightarrow\quad$ no conclusion possible



## Example – Bordered Hessian Matrix

Compute the local extrema of

$$f(x,y) = x^2 + 2y^2 \qquad \text{subject to} \qquad g(x,y) = x + y = 3$$

Lagrange function: $\mathcal{L}(x,y,\lambda) = (x^2+2y^2) + \lambda(3-x-y)$

Critical point: $(\mathbf{x}_0;\lambda_0) = (2,1;4)$

Determinant of the bordered Hessian:

$$|\bar{\mathbf{H}}(\mathbf{x}_0;\lambda_0)| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 4 \end{vmatrix} = -6 < 0$$

$\Rightarrow$ $\mathbf{x}_0 = (2,1)$ is a local minimum.



## Many Variables and Constraints

Objective function

$$f(x_1,\dots,x_n)$$

and constraints

$$\begin{aligned} g_1(x_1,\dots,x_n) &= c_1 \\ &\vdots \qquad\qquad (k<n) \\ g_k(x_1,\dots,x_n) &= c_k \end{aligned}$$

**Optimization problem:** $\min / \max \; f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) = \mathbf{c}$.

**Lagrange Function**:

$$\boxed{\mathcal{L}(\mathbf{x};\boldsymbol{\lambda}) = f(\mathbf{x}) + \boldsymbol{\lambda}^\mathsf{T}(\mathbf{c} - \mathbf{g}(\mathbf{x}))}$$



## Recipe – Critical Points

1. Create Lagrange Function $\mathcal{L}$:

$$\boxed{\begin{aligned} &\mathcal{L}(x_1,\dots,x_n;\lambda_1,\dots,\lambda_k) \\ &\quad = f(x_1,\dots,x_n) + \sum_{i=1}^k \lambda_i\,(c_i - g_i(x_1,\dots,x_n)) \end{aligned}}$$

2. Compute all first partial derivatives of $\mathcal{L}$.
3. We get a system of $n+k$ equations in $n+k$ unknowns. Find all solutions.
4. The first $n$ components $(x_1,\dots,x_n)$ are the elements of the critical points.



## Example – Critical Points

Compute all critical points of

$$f(x_1,x_2,x_3) = (x_1-1)^2 + (x_2-2)^2 + 2\,x_3^2$$

subject to

$$x_1 + 2\,x_2 = 2 \quad\text{and}\quad x_2 - x_3 = 3$$

Lagrange function:

$$\begin{aligned} \mathcal{L}(x_1,x_2,x_3;\lambda_1,\lambda_2) = {}& ((x_1-1)^2 + (x_2-2)^2 + 2\,x_3^2) \\ & + \lambda_1\,(2 - x_1 - 2\,x_2) + \lambda_2\,(3 - x_2 + x_3) \end{aligned}$$



## Example – Critical Points

Partial derivatives (gradient):

$$\begin{aligned} \mathcal{L}_{x_1} &= 2\,(x_1-1) - \lambda_1 && = 0 \\ \mathcal{L}_{x_2} &= 2\,(x_2-2) - 2\,\lambda_1 - \lambda_2 && = 0 \\ \mathcal{L}_{x_3} &= 4\,x_3 + \lambda_2 && = 0 \\ \mathcal{L}_{\lambda_1} &= 2 - x_1 - 2\,x_2 && = 0 \\ \mathcal{L}_{\lambda_2} &= 3 - x_2 + x_3 && = 0 \end{aligned}$$

We get the critical points of $\mathcal{L}$ by solving this system of equations.

$$x_1 = -\tfrac{6}{7},\ x_2 = \tfrac{10}{7},\ x_3 = -\tfrac{11}{7};\quad \lambda_1 = -\tfrac{26}{7},\ \lambda_2 = \tfrac{44}{7}\,.$$

The unique critical point of $f$ subject to these constraints is

$$\mathbf{x}_0 = (-\tfrac{6}{7}, \tfrac{10}{7}, -\tfrac{11}{7})\,.$$



## Bordered Hessian Matrix

$$\boxed{\bar{\mathbf{H}}(\mathbf{x};\boldsymbol{\lambda}) = \begin{pmatrix} 0 & \dots & 0 & \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial g_k}{\partial x_1} & \dots & \frac{\partial g_k}{\partial x_n} \\ \frac{\partial g_1}{\partial x_1} & \dots & \frac{\partial g_k}{\partial x_1} & \mathcal{L}_{x_1x_1} & \dots & \mathcal{L}_{x_1x_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \dots & \frac{\partial g_k}{\partial x_n} & \mathcal{L}_{x_nx_1} & \dots & \mathcal{L}_{x_nx_n} \end{pmatrix}}$$

For $r = k+1,\dots,n$
let $B_r(\mathbf{x};\boldsymbol{\lambda})$ denote the $(k+r)$-th leading principle minor of $\bar{\mathbf{H}}(\mathbf{x};\boldsymbol{\lambda})$.



## Sufficient Condition for Local Extrema

Assume that $(\mathbf{x}_0;\boldsymbol{\lambda}_0)$ is a critical point of $\mathcal{L}$. Then

- $(-1)^k\,B_r(\mathbf{x}_0;\boldsymbol{\lambda}_0) > 0$ for all $r = k+1,\dots,n$
  $\Rightarrow$ $\mathbf{x}_0$ is a *local minimum*
- $(-1)^r\,B_r(\mathbf{x}_0;\boldsymbol{\lambda}_0) > 0$ for all $r = k+1,\dots,n$
  $\Rightarrow$ $\mathbf{x}_0$ is a *local maximum*

($n$ is the number of variables $x_i$ and $k$ is the number of constraints.)

## Example – Sufficient Condition for Local Extrema

Compute all extrema of $f(x_1, x_2, x_3) = (x_1 - 1)^2 + (x_2 - 2)^2 + 2\,x_3^2$
subject to constraints $x_1 + 2\,x_2 = 2$ and $x_2 - x_3 = 3$

Lagrange Function:

$$\mathcal{L}(x_1, x_2, x_3; \lambda_1, \lambda_2) = ((x_1 - 1)^2 + (x_2 - 2)^2 + 2\,x_3^2) + \lambda_1\,(2 - x_1 - 2\,x_2) + \lambda_2\,(3 - x_2 + x_3)$$

Critical point of $\mathcal{L}$:

$$x_1 = -\tfrac{6}{7},\ x_2 = \tfrac{10}{7},\ x_3 = -\tfrac{11}{7};\ \lambda_1 = -\tfrac{26}{7},\ \lambda_2 = \tfrac{44}{7}\ .$$



## Example – Sufficient Condition for Local Extrema

Bordered Hessian matrix:

$$\bar{\mathbf{H}}(\mathbf{x};\boldsymbol{\lambda}) = \begin{pmatrix} 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 2 & 0 & 0 \\ 2 & 1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 4 \end{pmatrix}$$

3 variables, 2 constraints: $n = 3,\ k = 2 \quad \Rightarrow \quad r = 3$

$B_3 = |\bar{\mathbf{H}}(\mathbf{x};\boldsymbol{\lambda})| = 14$

$(-1)^k B_r = (-1)^2 B_3 = 14 \quad > 0$ condition satisfied
$(-1)^r B_r = (-1)^3 B_3 = -14 \quad < 0$ not satisfied

Critical point $\mathbf{x}_0 = (-\frac{6}{7}, \frac{10}{7}, -\frac{11}{7})$ is a *local minimum*.



## Sufficient Condition for Global Extrema

Let $(\mathbf{x}^*, \boldsymbol{\lambda}^*)$ be a critical point of the Lagrange function $\mathcal{L}$ of optimization problem

$$\min / \max \ f(\mathbf{x}) \ \text{ subject to } \ \mathbf{g}(\mathbf{x}) = \mathbf{c}$$

If $\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}^*)$ is *concave* (convex) in $\mathbf{x}$, then $\mathbf{x}^*$ is a **global maximum** (global minimum) of $f(\mathbf{x})$ subject to $\mathbf{g}(\mathbf{x}) = \mathbf{c}$.



## Example – Sufficient Condition for Global Extrema

$(x^*, y^*; \lambda^*) = (2, 1; 4)$ is a critical point of the Lagrange function $\mathcal{L}$ of optimization problem

$$\min / \max \ f(x, y) = x^2 + 2y^2 \ \text{ subject to } \ g(x, y) = x + y = 3$$

Lagrange function:

$$\mathcal{L}(x, y, \lambda^*) = (x^2 + 2y^2) + 4 \cdot (3 - (x + y))$$

Hessian matrix:

$$\mathbf{H}_{\mathcal{L}}(x, y) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \qquad \begin{aligned} H_1 &= 2 \quad > 0 \\ H_2 &= 8 \quad > 0 \end{aligned}$$

$\mathcal{L}$ is convex in $(x, y)$.

Thus $(x^*, y^*) = (2, 1)$ is a global minimum.



## Example – Sufficient Condition for Global Extrema

$$(\mathbf{x}^*; \boldsymbol{\lambda}^*) = (-\tfrac{6}{7}, \tfrac{10}{7}, -\tfrac{11}{7}; -\tfrac{26}{7}, \tfrac{44}{7})$$

is a critical point of the Lagrange function of optimization problem

$$\begin{aligned} \min / \max \ & f(x_1, x_2, x_3) = (x_1 - 1)^2 + (x_2 - 2)^2 + 2\,x_3^2 \\ \text{subject to} \ & g_1(x_1, x_2, x_3) = x_1 + 2\,x_2 = 2 \\ & g_2(x_1, x_2, x_3) = x_2 - x_3 = 3 \end{aligned}$$

Lagrange function:

$$\mathcal{L}(\mathbf{x}; \boldsymbol{\lambda}^*) = ((x_1 - 1)^2 + (x_2 - 2)^2 + 2\,x_3^2) - \tfrac{26}{7}\,(2 - x_1 - 2\,x_2) + \tfrac{44}{7}\,(3 - x_2 + x_3)$$



## Example – Sufficient Condition for Global Extrema

Hessian matrix:

$$\mathbf{H}_{\mathcal{L}}(x_1, x_2, x_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \begin{aligned} H_1 &= 2 \quad > 0 \\ H_2 &= 4 \quad > 0 \\ H_3 &= 16 \quad > 0 \end{aligned}$$

$\mathcal{L}$ is convex in $\mathbf{x}$.

$\mathbf{x}^* = (-\frac{6}{7}, \frac{10}{7}, -\frac{11}{7})$ is a global minimum.



## Interpretation of Lagrange Multiplier

Extremum $\mathbf{x}^*$ of optimization problem

$$\min / \max \ f(\mathbf{x}) \ \text{ subject to } \ \mathbf{g}(\mathbf{x}) = \mathbf{c}$$

depends on $\mathbf{c}$, $\mathbf{x}^* = \mathbf{x}^*(\mathbf{c})$, and so does the extremal value

$$f^*(\mathbf{c}) = f(\mathbf{x}^*(\mathbf{c}))$$

How does $f^*(\mathbf{c})$ change with varying $\mathbf{c}$?

$$\boxed{\frac{\partial f^*}{\partial c_j}(\mathbf{c}) = \lambda_j^*(\mathbf{c})}$$

That is, Lagrange multiplier $\lambda_j$ is the derivative of the extremal value w.r.t. exogeneous variable $c_j$ in constraint $g_j(\mathbf{x}) = c_j$.



## Proof Idea

Lagrange function $\mathcal{L}$ and objective function $f$ coincide in extemum $\mathbf{x}^*$.

$$\begin{aligned} \frac{\partial f^*(\mathbf{c})}{\partial c_j} &= \frac{\partial \mathcal{L}(\mathbf{x}^*(\mathbf{c}), \boldsymbol{\lambda}(\mathbf{c}))}{\partial c_j} \qquad \text{[ chain rule ]} \\ &= \sum_{i=1}^{n} \underbrace{\mathcal{L}_{x_i}(\mathbf{x}^*(\mathbf{c}), \boldsymbol{\lambda}(\mathbf{c}))}_{\substack{=0 \\ \text{as } \mathbf{x}^* \text{ is a critical point}}} \cdot \frac{\partial x_i^*(\mathbf{c})}{\partial c_j} + \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{c})}{\partial c_j} \right|_{(\mathbf{x}^*(\mathbf{c}), \boldsymbol{\lambda}^*(\mathbf{c}))} \\ &= \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{c})}{\partial c_j} \right|_{(\mathbf{x}^*(\mathbf{c}), \boldsymbol{\lambda}^*(\mathbf{c}))} \\ &= \left. \frac{\partial}{\partial c_j} \left( f(\mathbf{x}) + \sum_{i=1}^{k} \lambda_i (c_i - g_i(\mathbf{x})) \right) \right|_{(\mathbf{x}^*(\mathbf{c}), \boldsymbol{\lambda}^*(\mathbf{c}))} \\ &= \lambda_j^*(\mathbf{c}) \end{aligned}$$

## Example – Lagrange Multiplier

$(x^*, y^*) = (2, 1)$ is a minimum of optimization problem

$$\begin{array}{ll} \min / \max & f(x,y) = x^2 + 2y^2 \\ \text{subject to} & g(x,y) = x + y = c = 3 \end{array}$$

with $\lambda^* = 4$.

How does the minimal value $f^*(c)$ change with varying $c$?

$$\frac{df^*}{dc} = \lambda^* = 4$$



## Envelope Theorem

What is the derivative of the extremal value $f^*$ of optimization problem

$$\min / \max \quad f(\mathbf{x}, \mathbf{p}) \quad \text{subject to} \quad \mathbf{g}(\mathbf{x}, \mathbf{p}) = \mathbf{c}$$

w.r.t. parameters (exogeneous variables) $\mathbf{p}$?

$$\boxed{\frac{\partial f^*(\mathbf{p})}{\partial p_j} = \left. \frac{\partial \mathcal{L}(\mathbf{x}, \mathbf{p})}{\partial p_j} \right|_{(\mathbf{x}^*(\mathbf{p}), \boldsymbol{\lambda}^*(\mathbf{p}))}}$$



## Example – Roy's Identity

Maximize utility function

$$\max\ U(\mathbf{x}) \quad \text{subject to} \quad \mathbf{p}^\mathsf{T} \cdot \mathbf{x} = w$$

The maximal utility depends on prices $\mathbf{p}$ and income $w$ ab:

$$U^* = U^*(\mathbf{p}, w) \qquad \text{[ indirect utility function ]}$$

Lagrange function $\quad \mathcal{L}(\mathbf{x}, \lambda) = U(\mathbf{x}) + \lambda\,(w - \mathbf{p}^\mathsf{T} \cdot \mathbf{x})$

$$\frac{\partial U^*}{\partial p_j} = \frac{\partial \mathcal{L}}{\partial p_j} = -\lambda^* x_j^* \qquad \text{and} \qquad \frac{\partial U^*}{\partial w} = \frac{\partial \mathcal{L}}{\partial w} = \lambda^*$$

and thus

$$x_j^* = -\frac{\partial U^* / \partial p_j}{\partial U^* / \partial w} \qquad \text{[ Marshallian demand function ]}$$



## Example – Shephard's Lemma

Minimize expenses

$$\min\ \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \quad \text{subject to} \quad U(\mathbf{x}) = \bar{u}$$

The *expenditure function* (minimal expenses) depend on prices $\mathbf{p}$ and level $\bar{u}$ of utility: $\quad e = e(\mathbf{p}, \bar{u})$

Lagrange function $\quad \mathcal{L}(\mathbf{x}, \lambda) = \mathbf{p}^\mathsf{T} \cdot \mathbf{x} + \lambda\,(\bar{u} - U(\mathbf{x}))$

$$\frac{\partial e}{\partial p_j} = \frac{\partial \mathcal{L}}{\partial p_j} = x_j^* \qquad \text{[ Hicksian demand function ]}$$



## Summary

- constraint optimization
- graphical solution
- Lagrange function and Lagrange multiplier
- extremum and critical point
- bordered Hessian matrix
- global extremum
- interpretation of Lagrange multiplier
- envelope theorem



Chapter 16

# Kuhn Tucker Conditions



## Constraint Optimization

Find the maximum of function

$$f(x,y)$$

subject to

$$g(x,y) \leq c, \qquad x, y \geq 0$$

Example:
Find the maxima of

$$f(x,y) = -(x-5)^2 - (y-5)^2$$

subject to

$$x^2 + y \leq 9, \qquad x, y \geq 0$$

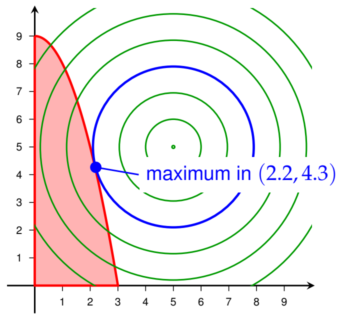


## Graphical Solution

For the case of two variables we can find a solution graphically.

1. Draw the constraint $g(x,y) \leq c$ in the $xy$-plain (*feasible region*).
2. Draw *appropriate* contour lines of objective function $f(x,y)$.
3. Investigate which contour lines of the objective function intersect with the feasible region.
   Estimate the (approximate) location of the maxima.

## Example – Graphical Solution

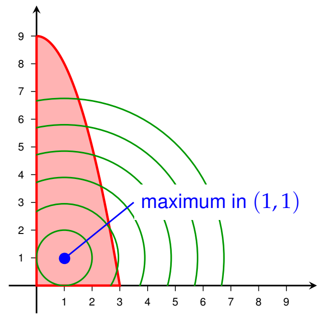



Maximum of $f(x,y) = -(x-5)^2 - (y-5)^2$

subject to $g(x,y) = x^2 + y \le 9, \quad x, y \ge 0.$



## Example – Graphical Solution




Maximum of $f(x,y) = -(x-1)^2 - (y-1)^2$

subject to $g(x,y) = x^2 + y \le 9, \quad x, y \ge 0.$



## Constraint Optimization

Compute the maximum of function

$$f(x_1, \dots, x_n)$$

subject to

$$g_1(x_1, \dots, x_n) \le c_1$$
$$\vdots$$
$$g_k(x_1, \dots, x_n) \le c_k$$
$$x_1, \dots, x_n \ge 0 \qquad \textbf{(non-negativity constraint)}$$

**Optimization problem:**

$$\max \; f(\mathbf{x}) \quad \text{subject to} \quad \mathbf{g}(\mathbf{x}) \le \mathbf{c} \quad \text{and} \quad \mathbf{x} \ge 0.$$



## Non-Negativity Constraint

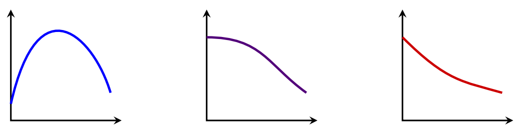
Univariate function $f$ with non-negativity constraint.

We find for the maximum $x^*$ of $f$:

- $x^*$ is an *interior* point of the feasible region: $x^* > 0$ and $f'(x^*) = 0$; or
- $x^*$ is a boundary point of the feasible region: $x^* = 0$ and $f'(x^*) \le 0$.

Summary:

$$f'(x^*) \le 0, \qquad x^* \ge 0 \qquad \text{and} \qquad x^* f'(x^*) = 0$$





## Non-Negativity Constraint

For the case of a multivariate function $f(\mathbf{x})$ with non-negativity constraints $x_j \ge 0$, we obtain such a condition for each of the variables:

$$f_{x_j}(\mathbf{x}^*) \le 0, \qquad x_j^* \ge 0 \qquad \text{and} \qquad x_j^* f_{x_j}(\mathbf{x}^*) = 0$$



## Slack Variables

Maximize

$$f(x_1, \dots, x_n)$$

subject to

$$g_1(x_1, \dots, x_n) + \mathbf{s_1} = c_1$$
$$\vdots$$
$$g_k(x_1, \dots, x_n) + \mathbf{s_k} = c_k$$
$$x_1, \dots, x_n \ge 0$$
$$\mathbf{s_1}, \dots, \mathbf{s_k} \ge 0 \qquad \text{(new non-negativity constraint)}$$

**Lagrange function**:

$$\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = f(x_1, \dots, x_n) + \sum_{i=1}^{k} \lambda_i (c_i - g_i(x_1, \dots, x_n) - \mathbf{s_i})$$



## Slack Variables

$$\tilde{\mathcal{L}}(\mathbf{x}, \mathbf{s}, \boldsymbol{\lambda}) = f(x_1, \dots, x_n) + \sum_{i=1}^{k} \lambda_i (c_i - g_i(x_1, \dots, x_n) - s_i)$$

Apply non-negativity conditions:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial x_j} \le 0, \quad x_j \ge 0 \quad \text{and} \quad x_j \frac{\partial \tilde{\mathcal{L}}}{\partial x_j} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial s_i} \le 0, \quad s_i \ge 0 \quad \text{and} \quad s_i \frac{\partial \tilde{\mathcal{L}}}{\partial s_i} = 0$$

$$\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_i} = 0 \qquad \text{(no non-negativity constraint)}$$



## Elimination of Slack Variables

Because of $\frac{\partial \tilde{\mathcal{L}}}{\partial s_i} = -\lambda_i$ the second line is equivalent to

$$\lambda_i \ge 0, \qquad s_i \ge 0 \qquad \text{and} \qquad \lambda_i s_i = 0$$

Equations $\frac{\partial \tilde{\mathcal{L}}}{\partial \lambda_i} = c_i - g_i(\mathbf{x}) - s_i = 0$ imply $s_i = c_i - g_i(\mathbf{x})$

and consequently the second line is equivalent to

$$\lambda_i \ge 0, \quad c_i - g_i(\mathbf{x}) \ge 0 \quad \text{and} \quad \lambda_i (c_i - g_i(\mathbf{x})) = 0\,.$$

Therefore there is no need of slack variables any more.

## Elimination of Slack Variables

So we replace $\tilde{\mathcal{L}}$ by Lagrange function

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(x_1, \dots, x_n) + \sum_{i=1}^{k} \lambda_i (c_i - g_i(x_1, \dots, x_n))$$

Observe that

$$\frac{\partial \mathcal{L}}{\partial x_j} = \frac{\partial \tilde{\mathcal{L}}}{\partial x_j} \quad \text{and} \quad \frac{\partial \mathcal{L}}{\partial \lambda_i} = c_i - g_i(\mathbf{x})$$

So the second line of the condition for a maximum now reads

$$\lambda_i \geq 0, \quad \frac{\partial \mathcal{L}}{\partial \lambda_i} \geq 0 \quad \text{and} \quad \lambda_i \frac{\partial \mathcal{L}}{\partial \lambda_i} = 0$$



## Kuhn-Tucker Conditions

$$\boxed{\mathcal{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(x_1, \dots, x_n) + \sum_{i=1}^{k} \lambda_i (c_i - g_i(x_1, \dots, x_n))}$$

The **Kuhn-Tucker conditions** for a (global) maximum are:

$$\boxed{\begin{array}{l} \dfrac{\partial \mathcal{L}}{\partial x_j} \leq 0, \quad x_j \geq 0 \quad \text{and} \quad x_j \dfrac{\partial \mathcal{L}}{\partial x_j} = 0 \\[2ex] \dfrac{\partial \mathcal{L}}{\partial \lambda_i} \geq 0, \quad \lambda_i \geq 0 \quad \text{and} \quad \lambda_i \dfrac{\partial \mathcal{L}}{\partial \lambda_i} = 0 \end{array}}$$

Notice that these Kuhn-Tucker conditions are not sufficient.
(Analogous to critical points.)



## Example – Kuhn-Tucker Conditions

Find the maximum of

$$f(x, y) = -(x-5)^2 - (y-5)^2$$

subject to

$$x^2 + y \leq 9, \qquad x, y \geq 0$$



## Example – Kuhn-Tucker Conditions

Lagrange function:

$$\mathcal{L}(x, y; \lambda) = -(x-5)^2 - (y-5)^2 + \lambda(9 - x^2 - y)$$

Kuhn-Tucker Conditions:

$$\begin{array}{rlcll}
(A) & \mathcal{L}_x & = & -2(x-5) - 2\lambda x & \leq 0 \\
(B) & \mathcal{L}_y & = & -2(y-5) - \lambda & \leq 0 \\
(C) & \mathcal{L}_\lambda & = & 9 - x^2 - y & \geq 0 \\[2ex]
(N) & & & x,\ y,\ \lambda & \geq 0 \\[2ex]
(I) & x\,\mathcal{L}_x & = & -x(2(x-5) + 2\lambda x) & = 0 \\
(II) & y\,\mathcal{L}_y & = & -y(2(y-5) + \lambda) & = 0 \\
(III) & \lambda\,\mathcal{L}_\lambda & = & \lambda(9 - x^2 - y) & = 0
\end{array}$$



## Example – Kuhn-Tucker Conditions

Express equations $(I)$–$(III)$ as

$$\begin{array}{lll}
(I) & x = 0 & \text{or} \quad 2(x-5) + 2\lambda x = 0 \\
(II) & y = 0 & \text{or} \quad 2(y-5) + \lambda = 0 \\
(III) & \lambda = 0 & \text{or} \quad 9 - x^2 - y = 0
\end{array}$$

We have to compute all 8 combinations and check whether the resulting solutions satisfy inequalities $(A)$, $(B)$, $(C)$, and $(N)$.

- If $\lambda = 0$ $(III, \text{left})$, then by $(I)$ and $(II)$ there exist four solutions for $(x, y; \lambda)$:

  $(0, 0; 0)$, $(0, 5; 0)$, $(5, 0; 0)$, and $(5, 5; 0)$.

  However, none of these points satisfies all inequalities $(A)$, $(B)$, $(C)$.

  Hence $\lambda \neq 0$.



## Example – Kuhn-Tucker Conditions

If $\lambda \neq 0$, then $(III, \text{right})$ implies $\quad y = 9 - x^2$.

- If $\lambda \neq 0$ and $x = 0$, then $y = 9$ and because of $(II, \text{right})$, $\lambda = -8$. A contradiction to $(N)$.
- If $\lambda \neq 0$ and $y = 0$, then $x = 3$ and because of $(I, \text{right})$, $\lambda = \frac{2}{3}$. A contradiction to $(B)$.
- Consequently all three variables must be non-zero.
  Thus $y = 9 - x^2$ and $\lambda = -2(y-5) = -2(4 - x^2)$.
  Substituted in $(I)$ yields $2(x-5) - 4(4-x^2)x = 0$ and

  $$x = \tfrac{\sqrt{11}+1}{2} \approx 2.158 \qquad y = \tfrac{12-\sqrt{11}}{2} \approx 4.342$$
  $$\lambda = \sqrt{11} - 2 \approx 1.317$$

The Kuhn-Tucker conditions are thus satisfied only in point

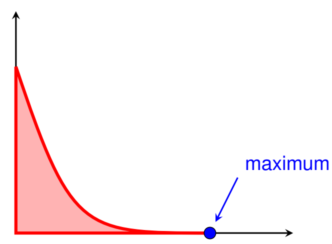
$$(x, y; \lambda) = \left(\tfrac{\sqrt{11}+1}{2}, \tfrac{12-\sqrt{11}}{2}; \sqrt{11} - 2\right) .$$



## Kuhn-Tucker Conditions

Unfortunately the Kuhn-Tucker conditions are not necessary!

That is, there exist optimization problems where the maximum does *not* satisfy the Kuhn-Tucker conditions.






## Kuhn-Tucker Theorem

We need a tool to determine whether a point is a (global) maximum.

The **Kuhn-Tucker theorem** provides a *sufficient* condition:

**(1)** Objective function $f(\mathbf{x})$ is differentiable and **concave**.

**(2)** All functions $g_i(\mathbf{x})$ from the constraints are differentiable and **convex**.

**(3)** Point $\mathbf{x}^*$ satisfy the Kuhn-Tucker conditions.

Then $\mathbf{x}^*$ is a *global maximum* of $f$ subject to constraints $g_i \leq c_i$.

The maximum is unique, if function $f$ is *strictly concave*.

## Example – Kuhn-Tucker Theorem

Find the maximum of

$$f(x,y) = -(x-5)^2 - (y-5)^2$$

subject to

$$x^2 + y \le 9, \qquad x, y \ge 0$$

The respective Hessian matrices of $f(x,y)$ and $g(x,y) = x^2 + y$ are

$$\mathbf{H}_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_g = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$



## Example – Kuhn-Tucker Theorem

$$\mathbf{H}_f = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{H}_g = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$$

**(1)** $f$ is strictly concave.

**(2)** $g$ is convex.

**(3)** Point $(x, y; \lambda) = \left(\frac{\sqrt{11}+1}{2}, \frac{12-\sqrt{11}}{2}; \sqrt{11}-2\right)$ satisfy the Kuhn-Tucker conditions.

Thus by the Kuhn-Tucker theorem, $\mathbf{x}^* = (\frac{\sqrt{11}+1}{2}, \frac{12-\sqrt{11}}{2})$ is the maximum we sought for.



## Summary

- constraint optimization
- graphical solution
- Lagrange function
- Kuhn-Tucker conditions
- Kuhn-Tucker theorem



Chapter 17

# Differential Equation



## A Simple Growth Model (Domar)

In Domar's growth model we have the following assumptions:

**(1)** An increase of the rate of investments $I(t)$ increases income $Y(t)$:

$$\frac{dY}{dt} = \frac{1}{s} \cdot \frac{dI}{dt} \qquad (s = \text{constant})$$

**(2)** Ratio of capital stock $K(t)$ and production capacity $\kappa(t)$ is constant:

$$\frac{\kappa(t)}{K(t)} = \varrho \qquad (= \text{constant})$$

**(E)** In equilibrium we have:

$$Y = \kappa$$

**Problem:** Which flow of investment causes our model to remain in equilibrium for all times $t \ge 0$?



## A simple Growth Model (Domar)

We search for a function $I(t)$ which satisfies model assumptions and equilibrium condition for all times $t \ge 0$.

$Y(t) = \kappa(t)$ for all $t$ implies $Y'(t) = \kappa'(t)$.

We thus find

$$\frac{1}{s} \cdot \frac{dI}{dt} \overset{(1)}{=} \frac{dY}{dt} \overset{(E)}{=} \frac{d\kappa}{dt} \overset{(2)}{=} \varrho \frac{dK}{dt} = \varrho\, I(t)$$

or in short

$$\frac{1}{s} \cdot \frac{dI}{dt} = \varrho\, I(t)$$

This equation contains a **function** and its **derivative**.
It must hold *for all* $t \ge 0$.
The unknown in this equation is a **function**.



## Differential Equation of First Order

An **ordinary differential equation (ODE) of first order** is an equation where the unknown is a univariate function and which contains the first (but not any higher) derivative of that function.

$$\boxed{y' = F(t, y)}$$

Examples:

$$y' = a\,y$$

$$y' + a\,y = b$$

$$y' + a\,y = b\,y^2$$

are ODEs of first order which describe exponential, exponentially bounded, and logistic growth, resp.



## Remarks

- When time $t$ is the independent variable of a function $y(t)$, then often Newton's notation is used for its derivatives:

$$\dot{y}(t) = \frac{dy}{dt} \quad \text{and} \quad \ddot{y}(t) = \frac{d^2y}{dt^2}$$

- The independent variable is often not given explicitly:

$$y' = a\,y \quad \text{is short for} \quad y'(t) = a\,y(t).$$

## Solution of Domar's Model

Transformation of the differential equation yields

$$\frac{1}{I(t)} I'(t) = \varrho\, s$$

This equation must hold for all $t$:

$$\ln(I) = \int \frac{1}{I} dI = \int \frac{1}{I(t)} I'(t)\, dt = \int \varrho\, s\, dt = \varrho\, s\, t + c$$

Substitution: $\quad I = I(t) \quad \Rightarrow \quad dI = I'(t)\, dt$

Thus we get

$$I(t) = e^{\varrho s t} \cdot e^{c} = C\, e^{\varrho s t} \qquad (C > 0)$$



## General Solution

All solutions of ODE $\quad I' = \varrho s I \quad$ can be written as

$$I(t) = C\, e^{\varrho s t} \qquad (C > 0)$$

This representation is called the **general solution** of the ODE.

We obtain *infinitely many* solutions!

We can easily verify the correctness of these solutions:

$$\frac{dI}{dt} = \varrho s \cdot C\, e^{\varrho s t} = \varrho s \cdot I(t)$$



## Initial Value Problem

In our model investment rate $I(t)$ is known at time $t = 0$ (i.e., "now").
So we have *two* equations:

$$\begin{cases} I'(t) = \varrho s \cdot I \\ I(0) = I_0 \end{cases}$$

We have to find some function $I(t)$ which satisfies both the ODE and the *initial value*.

We have to solve the so called **initial value problem**.

We obtain the so called **particular solution** of the initial value problem by substituting the initial values into the general solution of the ODE.



## Solution of Domar's Model

We obtain the particular solution of initial value problem

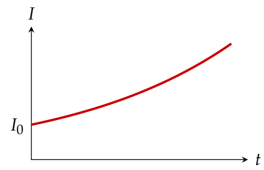
$$\begin{cases} I'(t) = \varrho s \cdot I \\ I(0) = I_0 \end{cases}$$

by substituting into the general solution:

$$I_0 = I(0) = C\, e^{\varrho s 0} = C$$

and thus

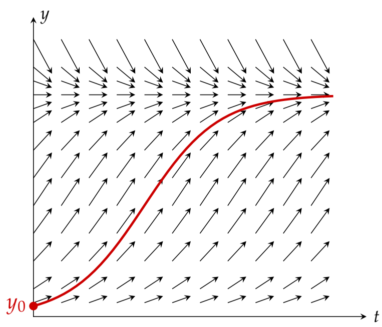
$$I(t) = I_0\, e^{\varrho s t}$$






## Graphical Interpretation

Equation $y' = F(t, y)$ assigns the slope of a tangent to each point $(t, y)$. We get a so called **vector field**.






## Separation of Variables

Differential equations of the form

$$\boxed{y' = f(t) \cdot g(y)}$$

can be solved by means of **separation of variables**:

$$\frac{dy}{dt} = f(t) \cdot g(y) \quad \Longleftrightarrow \quad \frac{1}{g(y)} dy = f(t)\, dt$$

Integration of either side yields:

$$\int \frac{1}{g(y)} dy = \int f(t)\, dt + c$$

We thus obtain the solution of the ODE as *implicit* function.

We have solved the ODE of Domar's model by separation of variables.



## Example – Separation of Variables

Find the solutions of ODE

$$y' + t\, y^2 = 0$$

Separation of variables:

$$\frac{dy}{dt} = -t\, y^2 \quad \Rightarrow \quad -\frac{dy}{y^2} = t\, dt$$

Integration yields

$$-\int \frac{dy}{y^2} = \int t\, dt + c \quad \Rightarrow \quad \frac{1}{y} = \frac{1}{2} t^2 + c$$

and thus we obtain the general solution as

$$y(t) = \frac{2}{t^2 + 2c}$$



## Example – Initial Value Problem

Compute the solution of the initial value problem

$$y' + t\, y^2 = 0, \quad y(0) = 1$$

Particular solution by substitution:

$$1 = y(0) = \frac{2}{0^2 + 2c} \quad \Rightarrow \quad c = 1$$

and thus

$$y(t) = \frac{2}{t^2 + 2}$$

## Linear ODE of First Order

A **linear differential equation of first order** is of form

$$\boxed{y'(t) + a(t)\, y(t) = s(t)}$$

It is called

- **homogeneous** ODE, if $s = 0$, and
- **inhomogeneous** ODE, if $s \neq 0$.

Homogeneous linear ODE of first order can be solved by separation of variables.



## Example – Homogeneous Linear ODE

Find the general solution of the homogeneous linear ODE

$$y' + 3\,t^2 y = 0$$

Separation of variables:

$$\frac{dy}{dt} = -3\,t^2\,y \quad\Rightarrow\quad \frac{1}{y}dy = -3t^2\,dt \quad\Rightarrow\quad \ln y = -t^3 + c$$

General solution thus is

$$y(t) = C\,e^{-t^3}$$



## Inhomogeneous Linear ODE of First Order

The general solution of inhomogeneous linear ODE

$$y'(t) + a\,y(t) = s$$

can be written as

$$\boxed{y(t) = y_h(t) + y_p(t)}$$

where

- $y_h(t)$ is the general solution of the corresponding homogeneous equation $y'(t) + a\,y(t) = 0$, and
- $y_p(t)$ is some particular solution of the inhomogeneous equation.

If coefficients $a$ and $b$ are *constants* we set $y_p(t) = \text{const}$.
Then $y_p' = 0$ and $y_p(t) = \frac{s}{a}$.



## Inhomogeneous Linear ODE of First Order

For the case where all coefficients $a$ and $b$ are *constants* and *non-zero* the general solution of

$$y'(t) + a\,y(t) = s$$

is given as

$$\boxed{y(t) = C\,e^{-at} + \frac{s}{a}}$$

Observe that $C\,e^{-at}$ is just the solution of the corresponding homogeneous ODE $y'(t) + a\,y(t) = 0$.



## Inhomogeneous Linear ODE of First Order

For the initial value problem

$$y'(t) + a\,y(t) = s, \quad y(0) = y_0$$

we obtain the particular solution

$$\boxed{y(t) = (y_0 - \bar{y})\;e^{-at} + \bar{y} \qquad \text{with } \bar{y} = \frac{s}{a}}$$

We find this solution by substituting the initial value into the particular solution.



## Example – Inhomogeneous Linear ODE

Find the solution of the initial value problem

$$y' - 3y = 6, \quad y(0) = 1$$

We find

$$\bar{y} = \frac{s}{a} = \frac{6}{-3} = -2$$

$$y(t) = (y_0 - \bar{y})\,e^{-at} + \bar{y} = (1 - (-2))\,e^{3t} - 2 = 3e^{3t} - 2$$

The particular solution thus is

$$y(t) = 3e^{3t} - 2$$



## Model – Dynamic of Market Price

Assume that demand and supply functions are linear:

$$\begin{aligned} q_d(t) &= \alpha - \beta\,p(t) && (\alpha, \beta > 0) \\ q_s(t) &= -\gamma + \delta\,p(t) && (\gamma, \delta > 0) \end{aligned}$$

The rate of price change is directly proportional to the difference $(q_d - q_s)$:

$$\frac{dp}{dt} = j\,(q_d(t) - q_s(t)) \qquad (j > 0)$$

How does price $p(t)$ evolve in time?

$$\begin{aligned} \frac{dp}{dt} &= j\,(q_d - q_s) = j\,(\alpha - \beta p - (-\gamma + \delta p)) \\ &= j\,(\alpha + \gamma) - j\,(\beta + \delta)p \end{aligned}$$

i.e., we obtain the inhomogeneous linear ODE of first order

$$p'(t) + j\,(\beta + \delta)\,p(t) = j\,(\alpha + \gamma)$$



## Model – Dynamic of Market Price

The solution of initial value problem

$$p'(t) + j\,(\beta + \delta)\,p(t) = j\,(\alpha + \gamma), \qquad p(0) = p_0$$

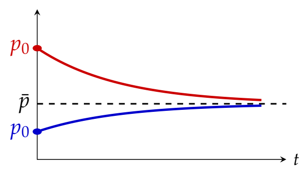
is

$$p(t) = (p_0 - \bar{p})\,e^{-j(\beta+\delta)t} + \bar{p}$$

with

$$\bar{p} = \frac{s}{a} = \frac{j(\alpha + \gamma)}{j(\beta + \delta)} = \frac{\alpha + \gamma}{\beta + \delta}$$

Observe that $\bar{p}$ is just the price in market equilibrium.



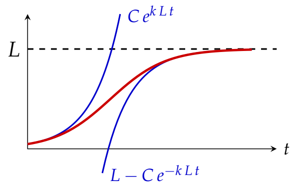
## Logistic Differential Equation

A **logistic differential equation** is of form

$$\boxed{y'(t) - k\,y(t)\,(L - y(t)) = 0}$$

where $k, L > 0$ and $0 \le y(t) \le L$.

- $y \approx 0$: $\quad y'(t) - k\,L\,y(t) \approx 0 \quad \Rightarrow \quad y(t) \approx C\,e^{k\,L\,t}$
- $y \approx L$: $\quad y'(t) + k\,L\,y(t) \approx k\,L^2 \quad \Rightarrow \quad y(t) \approx L - C\,e^{-k\,L\,t}$



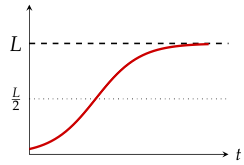



## Logistic Differential Equation

We can find general solution by separation of variables:

$$\boxed{y(t) = \frac{L}{1 + C\,e^{-L\,k\,t}}}$$

All solutions have an inflection point in $y = \frac{L}{2}$.






## Example – Logistic Differential Equation

A flu epidemic happens in a city with 8100 inhabitants. When the epidemic has been detected 100 persons have been infected. Twenty days later 1000 persons have been infected. It is expected that all inhabitants eventually will be infected.
Give a model for the number of infected persons.

We use a logistic ODE with $L = 8100$.
Let $q(t)$ denote the number of infected persons,
where $q(0) = 100$ and $q(20) = 1000$.

The general solution of this ODE is

$$q(t) = \frac{8100}{1 + C\,e^{-8100kt}}$$

We have to estimate parameters $k$ and $C$.



## Example – Logistic Differential Equation

$$\begin{aligned} q(0) &= 100 && \Rightarrow \quad \frac{8100}{1+C} = 100 && \Rightarrow \quad C = 80 \\ q(20) &= 1000 && \Rightarrow \quad \frac{8100}{1 + 80\,e^{-8100 \cdot 20\,k}} = 1000 && \Rightarrow \quad k = 0.00001495 \end{aligned}$$

The number of infected persons can be described by means of function

$$q(t) = \frac{8100}{1 + 80\,e^{-0.121\,t}}\,.$$



## Differential Equation of Second Order

An **ordinary differential equation (ODE) of second order** is an equation where the unknown is a univariate function and which contains the second (but not any higher) derivative of that function.

$$\boxed{y'' = F(t, y, y')}$$

We restrict our interest to **linear differential equations of second order with constant coefficients**:

$$\boxed{y''(t) + a_1\,y'(t) + a_2\,y(t) = s}$$



## Homogeneous Linear ODE of Second Order

We obtain the general solution of the homogeneous linear ODE

$$y''(t) + a_1\,y'(t) + a_2\,y(t) = 0$$

by means of the ansatz

$$y(t) = C\,e^{\lambda t}$$

where $\lambda$ satisfies the **characteristic equation**

$$\boxed{\lambda^2 + a_1\lambda + a_2 = 0}$$

This condition immediately follows from

$$\begin{aligned} y''(t) + a_1\,y'(t) + a_2\,y(t) &= \lambda^2\,C\,e^{\lambda t} + a_1\,\lambda\,C\,e^{\lambda t} + a_2\,C\,e^{\lambda t} \\ &= C\,e^{\lambda t}(\lambda^2 + a_1\,\lambda + a_2) = 0 \end{aligned}$$



## Characteristic Equation

The characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

has solutions

$$\lambda_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}$$

We have three cases:

1. $\frac{a_1^2}{4} - a_2 > 0$: two distinct real solutions
2. $\frac{a_1^2}{4} - a_2 = 0$: exactly one real solution
3. $\frac{a_1^2}{4} - a_2 < 0$: two complex (non-real) solutions



## Case: $\frac{a_1^2}{4} - a_2 > 0$

The general solution of the homogeneous ODE is given by

$$\boxed{y(t) = C_1\,e^{\lambda_1 t} + C_2\,e^{\lambda_2 t}\,, \qquad \text{with} \quad \lambda_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}}$$

where $C_1$ and $C_2$ are arbitrary real numbers.

## Example: $\frac{a_1^2}{4} - a_2 > 0$

Compute the general solution of ODE

$$y'' - y' - 2y = 0\,.$$

Characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

has distinct real solutions

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 2\,.$$

Thus the general solution of the homogeneous ODE is given by

$$y(t) = C_1 e^{-t} + C_2 e^{2t}\,.$$



## Case: $\frac{a_1^2}{4} - a_2 = 0$

The general solution of the homogeneous ODE is given by

$$\boxed{y(t) = (C_1 + C_2\,t)\,e^{\lambda t}\,, \qquad \text{with} \quad \lambda = -\frac{a_1}{2}}$$

We can verify the validity of solution $t\,e^{\lambda t}$ by a simple (but tedious) straight-forward computation.



## Example: $\frac{a_1^2}{4} - a_2 = 0$

Compute the general solution of ODE

$$y'' + 4y' + 4y = 0\,.$$

Characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

has the unique solution

$$\lambda = -2\,.$$

The general solution of the homogeneous ODE is thus given by

$$y(t) = (C_1 + C_2 t)\,e^{-2t}\,.$$



## Case: $\frac{a_1^2}{4} - a_2 < 0$

In this case root $\sqrt{\frac{a_1^2}{4} - a_2}$ is a non-real (imaginary) number.

From the rules for complex numbers one can derive purely real solutions:

$$\boxed{\begin{gathered} y(t) = e^{at}\left[C_1\,\cos(bt) + C_2\,\sin(bt)\right] \\ \text{with} \quad a = -\tfrac{a_1}{2} \quad \text{and} \quad b = \sqrt{\left|\tfrac{a_1^2}{4} - a_2\right|} \end{gathered}}$$

Notice that $a$ is the real part of the solution of the characteristic equation and $b$ the imaginary part.
Computations with complex numbers however are beyond the scope of this course.



## Example: $\frac{a_1^2}{4} - a_2 < 0$

Compute the general solution of ODE

$$y'' + y' + y = 0\,.$$

Characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

does not have real solutions as $\frac{a_1^2}{4} - a_2 = \frac{1}{4} - 1 = -\frac{3}{4} < 0$.

$$a = -\tfrac{a_1}{2} = -\tfrac{1}{2} \quad \text{and} \quad b = \sqrt{\left|\tfrac{a_1^2}{4} - a_2\right|} = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2}$$

The general solution of the homogeneous ODE is thus given by

$$y(t) = e^{-\frac{1}{2}t}\left[C_1 \cos\left(\tfrac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]\,.$$



## Inhomogeneous Linear ODE of Second Order

We obtain the general solution of the inhomogeneous ODE

$$y''(t) + a_1\,y'(t) + a_2\,y(t) = s$$

by mean so (provide that $a_2 \neq 0$)

$$\boxed{y(t) = y_h(t) + \frac{s}{a_2}}$$

where $y_h(t)$ is the general solution of the corresponding homogeneous ODE

$$y_h''(t) + a_1\,y_h'(t) + a_2\,y_h(t) = 0\,.$$



## Example – Inhomogeneous Linear ODE of Second Order

Compute the general solution of ODE

$$y''(t) + y'(t) - 2y(t) = -10$$

Characteristic equation of the homogeneous ODE

$$\lambda^2 + \lambda - 2 = 0$$

has real solutions

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -2\,.$$

The general solution of the inhomogeneous ODE is thus given by

$$y(t) = C_1\,e^{\lambda_1 t} + C_2\,e^{\lambda_2 t} + \frac{s}{a_2} = C_1\,e^{t} + C_2\,e^{-2t} + \frac{-10}{-2}\,.$$



## Initial Value Problem

All general solutions of linear ODEs of second order contain two independent integration constants $C_1$ and $C_2$.

Consequently we need two initial values for the particular solution of the **initial value problem**

$$\boxed{\begin{cases} y''(t) + a_1\,y'(t) + a_2\,y(t) = s \\ y(t_0) = y_0 \\ y'(t_0) = y_0' \end{cases}}$$

## Example – Initial Value Problem

Find the particular solution of initial value problem

$$y''(t) + y'(t) - 2y(t) = -10, \qquad y(0) = 12, \quad y'(0) = -2\,.$$

Its general solution is given by

$$y(t) = C_1\,e^t + C_2\,e^{-2t} + 5$$
$$y'(t) = C_1\,e^t - 2C_2\,e^{-2t}$$

Substitution of the initial values yields equations

$$12 = y(0) = C_1 + C_2$$
$$-2 = y'(0) = C_1 - 2C_2$$

with solutions $C_1 = 4$ and $C_2 = 3$.
Thus the particular solution of the initial value problem is given by

$$y(t) = 4e^t + 3e^{-2t} + 5\,.$$



## Fixed Point of an ODE

The inhomogeneous linear ODE

$$y''(t) + a_1\,y'(t) + a_2\,y(t) = s$$

has the special constant solution

$$y(t) = \bar{y} = \frac{s}{a_2} \qquad \text{(= constant)}$$

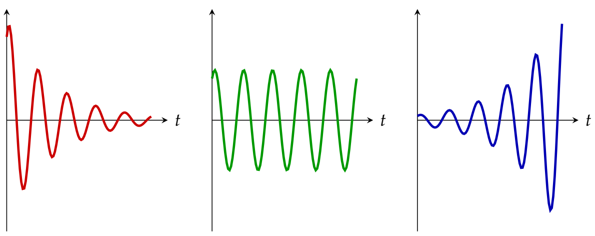
Point $\bar{y}$ is called **fixed point**, **stationary point**, or **equilibrium point** of the ODE.



## Stable and Unstable Fixed Points

The value of $a$ determines the qualitative behavior of solution curve

$$y(t) = e^{at}\left[C_1\,\cos(bt) + C_2\,\sin(bt)\right] + \bar{y}\,.$$




$a < 0$ $\qquad$ $a = 0$ $\qquad$ $a > 0$

stable fixed point $\qquad$ unstable fixed point

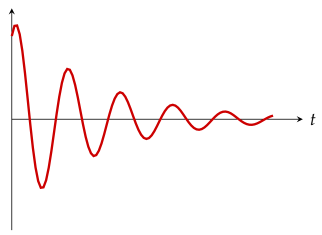


## Asymptotically Stable Fixed Point

If $a < 0$, then every solution

$$y(t) = e^{at}\left[C_1\,\cos(bt) + C_2\,\sin(bt)\right] + \bar{y}$$

*converges* to $\bar{y}$. The fixed point $\bar{y}$ is then **asymptotically stable**.



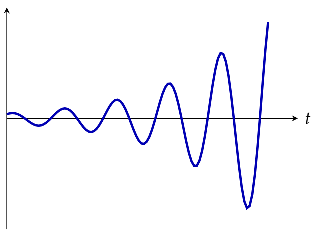



## Unstable Fixed Point

If $a > 0$, then every solution

$$y(t) = e^{at}\left[C_1\,\cos(bt) + C_2\,\sin(bt)\right] + \bar{y}$$

with initial value $y(0) = y_0 \neq \bar{y}$ diverges.
Such a fixed point $\bar{y}$ is called **unstable**.






## Example – Asymptotically Stable Fixed Point

The general solution of

$$y'' + y' + y = 2$$

is given

$$y(t) = 2 + e^{-\frac{1}{2}t}\left[C_1\cos\left(\tfrac{\sqrt{3}}{2}t\right) + C_2\sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]$$

Fixed point $\bar{y} = 2$ is asymptotically stable as $a = -\frac{1}{2} < 0$.



## Summary

- differential equation of first order
- ODE
- vector field
- separation of variables
- homogeneous and inhomogeneous linear ODE of first order
- logistic ODE
- homogeneous and inhomogeneous linear ODE of second order
- stable and unstable equilibrium points



Chapter 18

# Difference Equation

## First Difference

Suppose a state variable $y$ can only be estimated at **discrete** time points $t_1, t_2, t_3, \dots$. In particular we assume that $t_i \in \mathbb{N}$. Thus we can describe the behavior of such a variable by means of a map

$$\mathbb{N} \to \mathbb{R},\ t \mapsto y(t)$$

i.e., a *sequence*. We write $y_t$ instead of $y(t)$.

For the marginal changes of $y$ we have to replace the differential quotient $\frac{dy}{dt}$ by the **difference quotient** $\frac{\Delta y}{\Delta t}$.

So if $\Delta t = 1$ this reduces to the **first difference**

$$\boxed{\Delta y_t = y_{t+1} - y_t}$$



## Rules for Differences

For differences similar rules can be applied as for derivatives:

- $\Delta(c\, y_t) = c\, \Delta y_t$
- $\Delta(y_t + z_t) = \Delta y_t + \Delta z_t$ *Summation rule*
- $\Delta(y_t \cdot z_t) = y_{t+1}\, \Delta z_t + z_t\, \Delta y_t$ *Product rule*
- $\Delta\left(\frac{y_t}{z_t}\right) = \frac{z_t\, \Delta y_t - y_t\, \Delta z_t}{z_t\, z_{t+1}}$ *Quotient rule*



## Differences of Higher Order

The $k$-th derivative $\frac{d^k y}{dt^k}$ has to be replaced by the **difference of order $k$**:

$$\boxed{\Delta^k y_t = \Delta(\Delta^{k-1} y_t) = \Delta^{k-1} y_{t+1} - \Delta^{k-1} y_t}$$

For example the **second difference** is then

$$\begin{aligned} \Delta^2 y_t &= \Delta(\Delta y_t) = \Delta y_{t+1} - \Delta y_t \\ &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\ &= y_{t+2} - 2\, y_{t+1} + y_t \end{aligned}$$



## Difference Equation

A **difference equation** is an equation that contains the differences of a sequence. It is of order $n$ if it contains a difference of order $n$ (but not higher).

| | |
|---|---|
| $\Delta y_t = 3$ | difference equation of first order |
| $\Delta y_t = \frac{1}{2} y_t$ | difference equation of first order |
| $\Delta^2 y_t + 2\, \Delta y_t = -3$ | difference equation of second order |

If in addition an initial value $y_0$ is given
we have a so called **initial value problem**.



## Equivalent Representation

Difference equations can equivalently written without $\Delta$-notation.

$$\Delta y_t = 3 \Leftrightarrow y_{t+1} - y_t = 3 \quad \Leftrightarrow \quad y_{t+1} = y_t + 3$$

$$\Delta y_t = \tfrac{1}{2} y_t \Leftrightarrow y_{t+1} - y_t = \tfrac{1}{2} y_t \quad \Leftrightarrow \quad y_{t+1} = \tfrac{3}{2} y_t$$

$$\begin{aligned} &\Delta^2 y_t + 2\, \Delta y_t = -3 \Leftrightarrow \\ &\quad \Leftrightarrow (y_{t+2} - 2\, y_{t+1} + y_t) + 2\,(y_{t+1} - y_t) = -3 \\ &\quad \Leftrightarrow \quad y_{t+2} = y_t - 3 \end{aligned}$$

These can be seen as *recursion formulæ* for sequences.

**Problem:**
Find a sequence $y_t$ which satisfies the given recursion formula *for all* $t \in \mathbb{N}$.



## Initial Value Problem and Iterations

Difference equations of first order can be solved by iteratively computing the elements of the sequence if the initial value $y_0$ is given.

Compute the solution of $y_{t+1} = y_t + 3$ with initial value $y_0$.

$$\begin{aligned} &y_1 = y_0 + 3 \\ &y_2 = y_1 + 3 = (y_0 + 3) + 3 = y_0 + 2 \cdot 3 \\ &y_3 = y_2 + 3 = (y_0 + 2 \cdot 3) + 3 = y_0 + 3 \cdot 3 \\ &\dots \\ &y_t = y_0 + 3\, t \end{aligned}$$

For initial value $y_0 = 5$ we obtain $y_t = 5 + 3\, t$.



## Example – Iterations

Compute the solution of $y_{t+1} = \frac{3}{2} y_t$ with initial value $y_0$.

$$\begin{aligned} &y_1 = \tfrac{3}{2} y_0 \\ &y_2 = \tfrac{3}{2} y_1 = \tfrac{3}{2}\left(\tfrac{3}{2} y_0\right) = \left(\tfrac{3}{2}\right)^2 y_0 \\ &y_3 = \tfrac{3}{2} y_2 = \tfrac{3}{2}\left(\tfrac{3}{2}^2 y_0\right) = \left(\tfrac{3}{2}\right)^3 y_0 \\ &\dots \\ &y_t = \left(\tfrac{3}{2}\right)^t y_0 \end{aligned}$$

For initial value $y_0 = 5$ we obtain $y_t = 5 \cdot \left(\frac{3}{2}\right)^t$.



## Homogeneous Linear Difference Equation of First Order

A **homogeneous linear difference equation of first order** is of form

$$\boxed{y_{t+1} + a\, y_t = 0}$$

Ansatz for general solution:

$$y_t = C\, \beta^t, \quad C\, \beta \neq 0, \quad \text{for some fixed } C \in \mathbb{R}.$$

It has to satisfy the difference equation for all $t$:

$$y_{t+1} + a\, y_t = C\, \beta^{t+1} + a\, C\, \beta^t = 0.$$

Division by $C\, \beta^t$ yields $\beta + a = 0$ and thus $\beta = -a$ and

$$\boxed{y_t = C\,(-a)^t}$$

## Example – Homogeneous Equation

Homogeneous linear difference equation

$$y_{t+1} - \frac{3}{2} y_t = 0$$

has general solution

$$y_t = C \left(\frac{3}{2}\right)^t .$$



## Properties of Solutions

The behavior of solution

$$y_t = C\,\beta^t = C\,(-a)^t$$

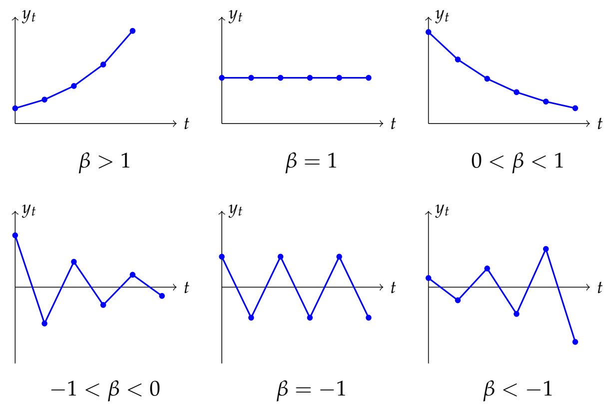
obviously depends on parameter $\beta = -a$ which can be summarized as following:

| | | |
|---|---|---|
| oscillating | $\Leftrightarrow$ | $\beta < 0$ |
| convergent | $\Leftrightarrow$ | $\lvert\beta\rvert < 1$ |

We want to note that $\beta$ is the root of the *characteristic equation* $\beta + a = 0$.



## Properties of Solutions






## Inhomogeneous Linear Difference Equation

The general solution of **inhomogeneous linear difference equation**

$$\boxed{y_{t+1} + a\,y_t = s}$$

can be written as

$$\boxed{y_t = y_{h,t} + y_{p,t}}$$

where

- $p_{h,t}$ is the general solution of the corresponding homogeneous equation $y_{t+1} + a\,y_t = 0$, and
- $y_{h,t}$ is some particular solution of the inhomogeneous equation.

How can we find $y_{p,t}$?



## Inhomogeneous Linear Difference Equation

As parameters $a$ and $s$ are constant we may set $y_{h,t} = c =$ const.

Then

$$y_{p,t+1} + a\,y_{p,t} = c + a\,c = s$$

which implies

$$y_{p,t} = c = \frac{s}{1+a} \quad \text{if } a \neq -1.$$

If $a = -1$ we set $y_{p,t} = c\,t$. Then

$$c\,(t+1) + (-1)\,c\,t = s$$

which implies $c = s$ and

$$y_{p,t} = s\,t\,.$$



## Inhomogeneous Linear Difference Equation

An **inhomogeneous linear difference equation of first order** with *constant coefficients* is of form

$$\boxed{y_{t+1} + a\,y_t = s}$$

The general solution is given by

$$\boxed{y_t = \begin{cases} C\,(-a)^t + \dfrac{s}{1+a} & \text{if } a \neq -1, \\ C + s\,t & \text{if } a = -1. \end{cases}}$$

Observe that $C\,(-a)^t$ is just the solution of the corresponding homogeneous difference equation $y_{t+1} + a\,y_t = 0$.



## Asymptotically Stable

Observe that $y_{p,t} = \bar{y} = \frac{s}{1+a}$ is a *fixed point* (or **equilibrium point**) of the inhomogeneous equation $y_{t+1} + a\,y_t = s$.
Obviously solution

$$y_t = C\,(-a)^t + \bar{y} \qquad (C \neq 0)$$

converges to $\bar{y}$ if and only if $\lvert a\rvert < 1$.

In this case $\bar{y}$ is (*locally*) **asymptotically stable**.

Otherwise if $\lvert a\rvert > 1$, $y_t$ diverges and $\bar{y}$ is called **unstable**.



## Example – Inhomogeneous Equation

The inhomogeneous linear difference equation

$$y_{t+1} - 2\,y_t = 2$$

has general solution

$$y_t = C\,2^t - 2\,.$$

We get the particular solution of the initial value problem with $y_0 = 1$ by

$$1 = y_0 = C\,2^0 - 2\,.$$

Thus $C = 3$ and consequently

$$y_t = 3 \cdot 2^t - 2\,.$$

## Example – Inhomogeneous Equation

The inhomogeneous linear difference equation

$$y_{t+1} - y_t = 3$$

has general solution

$$y_t = C + 3\,t.$$

We get the particular solution of the initial value problem with $y_0 = 4$ by

$$4 = y_0 = C + 3 \cdot 0\,.$$

Thus $C = 4$ and consequently

$$y_t = 4 + 3\,t\,.$$



## Model – Dynamic of Market Price

Assume that demand and supply functions are linear:

$$\begin{aligned} q_{d,t} &= \alpha - \beta\, p_t && (\alpha, \beta > 0) \\ q_{s,t} &= -\gamma + \delta\, p_t && (\gamma, \delta > 0) \end{aligned}$$

and the change of price is directly proportional to the difference $(q_d - q_s)$:

$$p_{t+1} - p_t = j\,(q_{d,t} - q_{s,t}) \qquad (j > 0)$$

How does price $p_t$ evolve in time?

$$\begin{aligned} p_{t+1} - p_t = j\,(q_{d,t} - q_{s,t}) &= j\,(\alpha - \beta p_t - (-\gamma + \delta p_t)) \\ &= j\,(\alpha + \gamma) - j\,(\beta + \delta) p_t \end{aligned}$$

i.e., we obtain the inhomogeneous linear difference equation

$$p_{t+1} + (j(\beta + \gamma) - 1)\, p_t = j(\alpha + \gamma)$$



## Model – Dynamic of Market Price

The general solution

$$p_{t+1} + (j(\beta + \gamma) - 1)\, p_t = j(\alpha + \gamma)$$

is then

$$p_t = C\,(1 - j(\beta + \delta))^t + \bar{p}$$

where $\bar{p} = \frac{\alpha + \gamma}{\beta + \delta}$ is the price in market equilibrium.

For initial value $p_0$ we finally obtain the particular solution

$$p_t = (p_0 - \bar{p})(1 - j(\beta + \delta))^t + \bar{p}$$

The difference equation has fixed point $\bar{p}$.
It is asymptotically stable if and only if $j(\beta + \delta) < 2$.



## Model – Dynamic of Market Price

Consider the following market model:

$$\begin{aligned} q_{d,t} &= q_{s,t} \\ q_{d,t} &= \alpha - \beta p_t && (\alpha, \beta > 0) \\ q_{s,t} &= -\gamma + \delta p_{t-1} && (\gamma, \delta > 0) \end{aligned}$$

Observe that we have market equilibrium in each period.
The supply depends on the price of the preceding period.
Substituting of the second and third equation onto the first yields the inhomogeneous linear difference equation

$$\beta\, p_t + \delta\, p_{t-1} = \alpha + \gamma \quad \Leftrightarrow \quad p_{t+1} + \frac{\delta}{\beta}\, p_t = \frac{\alpha + \gamma}{\beta}$$



## Model – Dynamic of Market Price

Inhomogeneous linear first order difference equation

$$p_{t+1} + \frac{\delta}{\beta}\, p_t = \frac{\alpha + \gamma}{\beta}$$

with initial value $p_0$ has solution

$$p_t = (p_0 - \bar{p}) \left(-\frac{\delta}{\beta}\right)^t + \bar{p} \qquad \text{where } \bar{p} = \frac{\alpha + \gamma}{\beta + \delta}.$$

As all constants are positive, root $-\frac{\delta}{\beta} < 0$ and thus all solutions of such a market model oscillate.

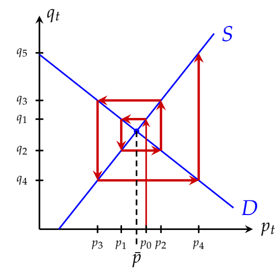
The solution converges to the $\bar{p}$ if $\left|\frac{\delta}{\beta}\right| < 1$.



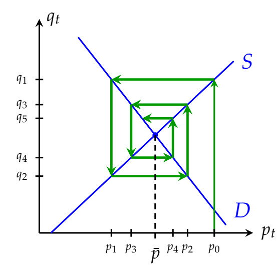
## Cobweb Model

We also can analyze this model *graphically*.
Demand and supply are functions of price $p$:

$$D(p) = \alpha - \beta p, \quad \text{and} \quad S(p) = -\gamma + \delta p$$




$\delta > \beta$




$\delta < \beta$



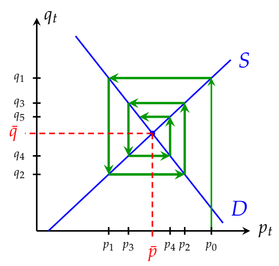
## Cobweb Model

↑ We start in period $0$ with price $p_0$ and get supply $q_1 = S(p_0)$ in period $1$.

← Market equilibrium implies new price $p_1$ given implicitly by $D(p_1) = q_1$.

↓ In period 2 price $p_1$ yields supply $q_2 = S(p_1)$.

→ Market equilibrium implies new price $p_2$ given implicitly by $D(p_2) = q_2$.




Iterating this procedure spins a **cobweb** around *equilibrium point* $(\bar{p}, \bar{q})$ with $\bar{q} = S(\bar{p}) = D(\bar{p})$.



## Cobweb Model – Nonlinear Functions

Cobweb models also work when functions $D(p)$ and $S(p)$ are nonlinear.

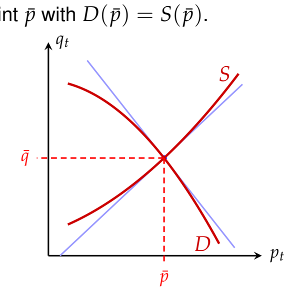
Then there may not exist a solution in closed form.
However, we still have an equilibrium point $\bar{p}$ with $D(\bar{p}) = S(\bar{p})$.

Linearized versions of $D$ and $S$:

$$\begin{aligned} \widehat{D}(p) &= D(\bar{p}) + D'(\bar{p})(p - \bar{p}) \\ \widehat{S}(p) &= S(\bar{p}) + S'(\bar{p})(p - \bar{p}) \end{aligned}$$

Equilibrium point $\bar{p}$ is *locally asymptotically stable* if

- $D'(\bar{p}) < 0 < S'(\bar{p})$, and
- $|S'(\bar{p})| < |D'(\bar{p})|$.



## Linear Difference Equation of Second Order

A **difference equation** is an equation that contains the differences of second order of a sequence.

We restrict our interest to **linear difference equations of second order with constant coefficients**:

$$\boxed{y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s}$$



## Homogeneous Linear Difference Equation

We obtain the general solution of the homogeneous linear ODE

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

by means of the ansatz

$$y_t = C\,\beta^t, \qquad C\,\beta \neq 0$$

which has to satisfies the difference equation :

$$C\,\beta^{t+2} + a_1\, C\,\beta^{t+1} + a_2\, C\,\beta^t = 0\,.$$

Hence $\beta$ has to satisfy the **characteristic equation**

$$\boxed{\beta^2 + a_1\,\beta + a_2 = 0}$$



## Characteristic Equation

The characteristic equation

$$\beta^2 + a_1\,\beta + a_2 = 0$$

has solutions

$$\beta_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}$$

We have three cases:

1. $\frac{a_1^2}{4} - a_2 > 0$: two distinct real solutions
2. $\frac{a_1^2}{4} - a_2 = 0$: exactly one real solution
3. $\frac{a_1^2}{4} - a_2 < 0$: two complex (non-real) solutions



## Case: $\frac{a_1^2}{4} - a_2 > 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

is given by

$$\boxed{y(t) = C_1\,\beta_1^t + C_2\,\beta_2^t\,, \qquad \text{with} \quad \beta_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}}$$

where $C_1$ and $C_2$ are arbitrary real numbers.



## Example: $\frac{a_1^2}{4} - a_2 > 0$

Compute the general solution of difference equation

$$y_{t+2} - 3\,y_{t+1} + 2\,y_t = 0\,.$$

Characteristic equation

$$\beta^2 - 3\,\beta + 2 = 0$$

has distinct real solutions

$$\beta_1 = 1 \quad \text{and} \quad \beta_2 = 2\,.$$

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1\,1^t + C_2\,2^t = C_1 + C_2\,2^t\,.$$



## Case: $\frac{a_1^2}{4} - a_2 = 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

is given by

$$\boxed{y_t = C_1\,\beta^t + C_2\,t\,\beta^t\,, \qquad \text{with} \quad \beta = -\frac{a_1}{2}}$$

We can verify the validity of solution $t\,\beta^t$ by a simple (but tedious) straight-forward computation.



## Example: $\frac{a_1^2}{4} - a_2 = 0$

Compute the general solution of difference equation

$$y_{t+2} - 4\,y_{t+1} + 4\,y_t = 0\,.$$

Characteristic equation

$$\beta^2 - 4\,\beta + 4 = 0$$

has the unique solution

$$\beta = 2\,.$$

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1\,2^t + C_2\,t\,2^t\,.$$



## Case: $\frac{a_1^2}{4} - a_2 < 0$

In this case root $\sqrt{\frac{a_1^2}{4} - a_2}$ is a non-real (imaginary) number:

$$\beta_{1,2} = a \pm b\,i$$

where

- $a = -\frac{a_1}{2}$ is called the **real part**, and
- $b = \sqrt{\left|a_2 - \frac{a_1^2}{4}\right|}$ the **imaginary part** of root $\beta$.

Alternatively $\beta$ can be represent by so called polar coordinates

$$\beta_{1,2} = r(\cos\theta \pm i\,\sin\theta)$$

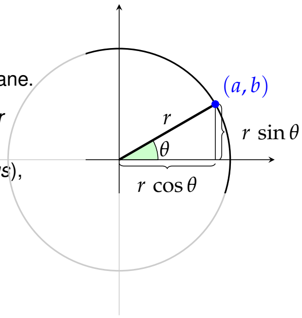
where

- $r = |\beta| = \sqrt{a^2 + b^2} = \sqrt{\frac{a_1^2}{4} + a_2 - \frac{a_1^2}{4}} = \sqrt{a_2}$ is called the **modulus** (or *absolute value*) of $\beta$, and
- $\theta = \arg(\beta)$ the *argument* of $\beta$.

## Modulus and Argument

A complex number $z = a + b\,i$ can be interpreted as point $(a, b)$ in the (real) plane.

This point can also can be given by *polar coordinates* with radius $r = |z|$ (*absolute value* or *modulus*), and angle $\theta$ (called the *argument* of $z$).




We then have

$$r = |z| = \sqrt{a^2 + b^2}$$

and

$$\tan\theta = \frac{b}{a} \qquad \text{because } \cos\theta = \tfrac{a}{r} \text{ and } \sin\theta = \tfrac{b}{r}.$$



## Case: $\frac{a_1^2}{4} - a_2 < 0$

From the rules for complex numbers one can derive purely real solutions of the homogeneous difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

given by

$$\boxed{y_t = r^t \left[C_1 \cos(\theta t) + C_2 \sin(\theta t)\right] \quad \text{with} \quad r = |\beta| = \sqrt{a_2} \quad \text{and} \quad \theta = \arg(\beta)}$$

Argument $\arg(\beta)$ is given by

$$\cos\theta = \tfrac{a}{r} = -\tfrac{a_1}{2\sqrt{a_2}}$$

$$\sin\theta = \tfrac{b}{r} = \sqrt{1 - \tfrac{a_1^2}{4a_2}}$$



## Example: $\frac{a_1^2}{4} - a_2 < 0$

Compute the general solution of difference equation

$$y_{t+2} + 2\, y_{t+1} + 4\, y_t = 0\,.$$

Characteristic equation

$$\beta^2 + 2\,\beta + 4 = 0$$

has the complex solutions

$$\beta_{1,2} = -1 \pm \sqrt{3}\, i$$

i.e., $a = -1$ and $b = \sqrt{3}$.



## Example: $\frac{a_1^2}{4} - a_2 < 0$

Complex root $\beta = a + b\,i$ with $a = -1$ and $b = \sqrt{3}$ has polar coordinates:

- $r = \sqrt{1^2 + 3} = \sqrt{4} = 2$, and
- $\theta = \frac{2\pi}{3}$, as $\sin\theta = \frac{a}{r} = -\frac{1}{2}$ and $\cos\theta = \frac{b}{r} = \frac{\sqrt{3}}{2}$.

Thus the general solution of the homogeneous equation is given by

$$y_t = 2^t \left[C_1 \cos\left(\frac{2\pi}{3} t\right) + C_2 \sin\left(\frac{2\pi}{3} t\right)\right]\,.$$

Argument $\theta$ can be computed by means of the *arcus tangens* function $\arctan(b/a)$.
A more convenient way is to use function `atan2` which is available in programs like **R**.



## Inhomogeneous Linear Difference Equation

The general solution of **inhomogeneous linear difference equation**

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s$$

can be written as

$$\boxed{y_t = y_{h,t} + y_{p,t}}$$

where

- $y_{h,t}$ is the general solution of the corresponding homogeneous equation $y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s$, and
- $y_{h,t}$ is some particular solution of the inhomogeneous equation.

How can we find $y_{p,t}$?



## Inhomogeneous Linear Difference Equation

By assumption all coefficients $a_1$, $a_2$, and $s$. So we may assume that $y_{p,t} = c = \text{const}$:

$$c + a_1\, c + a_2\, c = s$$

which implies

$$\boxed{y_{p,t} = c = \frac{s}{1 + a_1 + a_2} \qquad \text{if } a_1 + a_2 \neq -1.}$$

If $a_1 + a_2 \neq -1$ we may use $y_{p,t} = ct$ and get

$$\boxed{y_{p,t} = \frac{s}{a_1 + 2}\, t \qquad \text{if } a_1 + a_2 = -1 \text{ and } a_1 \neq -2.}$$



## Example – Inhomogeneous Equation

Compute the general solution of difference equation

$$y_{t+2} + 2\, y_{t+1} + 4\, y_t = 14\,.$$

General solution of homogeneous equation $y_{t+2} + 2\, y_{t+1} + 4\, y_t = 0$:

$$y_{h,t} = 2^t \left[C_1 \cos\left(\frac{2\pi}{3} t\right) + C_2 \sin\left(\frac{2\pi}{3} t\right)\right]\,.$$

As $a_1 + a_2 = 2 + 4 \neq -1$ we use $y_{p,t} = \frac{14}{1+2+4} = 2$ and obtain the general solution of the inhomogeneous equation as

$$y_t = y_{h,t} + y_{p,t} = 2^t \left[C_1 \cos\left(\frac{2\pi}{3} t\right) + C_2 \sin\left(\frac{2\pi}{3} t\right)\right] + 2\,.$$



## Example – Inhomogeneous Equation

Compute the general solution of difference equation

$$y_{t+2} - 3\, y_{t+1} + 2\, y_t = 2\,.$$

General solution of homogeneous equation $y_{t+2} - 3\, y_{t+1} + 2\, y_t = 0$:

$$y_{h,t} = C_1 + C_2 2^t\,.$$

As $a_1 + a_2 = -3 + 2 = -1$ and $a_1 \neq -2$ we use $y_{p,t} = \frac{2}{-3+2} t = -2t$ and obtain the general solution of the inhomogeneous equation as

$$y_t = y_{h,t} + y_{p,t} = C_1 + C_2 2^t - 2t\,.$$

## Fixed Point of a Difference Equation

The inhomogeneous linear difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s$$

has the special constant solution (for $a_1 + a_2 \neq -1$)

$$y_{p,t} = \bar{y} = \frac{s}{1 + a_1 + a_2} \quad \text{(= constant)}$$

Point $\bar{y}$ is called **fixed point**, or **equilibrium point** of the difference equation.



## Stable and Unstable Fixed Points

When we review general solutions of linear difference equations (with constant coefficients) we observe that these solutions converge to a fixed point $\bar{y}$ for all choices of constants $C$ if the absolute values of the roots $\beta$ of the characteristic equation are less than one:

$$\boxed{y_t \to \bar{y} \text{ for } t \to \infty \quad \text{if} \quad |\beta| < 1.}$$

In this case $\bar{y}$ is called an **asymptotically stable** fixed point.



## Summary

- differences of sequences
- difference equation
- homogeneous and inhomogeneous linear difference equation of first order with constant coefficients
- cobweb model
- homogeneous and inhomogeneous linear difference equation of second order with constant coefficients
- stable and unstable fixed points



Chapter 19

# Control Theory



## Economic Growth

**Problem:** Maximize consumption in period $[0, T]$:

$$\max_{0 \le s(t) \le 1} \int_0^T (1 - s(t))\, f(k(t))\, dt$$

$f(k)$ ... production function
$k(t)$ ... capital stock at time $t$
$s(t)$ ... rate of investment at time $t$ , $s \in [0, 1]$

We can control $s(t)$ at each time freely.
$s$ is called **control function.**

$k(t)$ follows the differential equation

$$k'(t) = s(t)\, f(k(t)), \quad k(0) = k_0, \quad k(T) \ge k_T \,.$$



## Oil Extraction

$y(t)$ ... amount of oil in reservoir at time $t$
$u(t)$ ... rate of extraction at time $t$: $y'(t) = -u(t)$
$p(t)$ ... market price of oil at time $t$
$C(t, y, u)$ ... extraction costs per unit of time
$r$ ... (constant) discount rate

**Problem I:** Maximize revenue in fixed time horizon $[0, T]$:

$$\max_{u(t) \ge 0} \int_0^T \big[p(t)u(t) - C(t, y(t), u(t))\big] e^{-rt}\, dt$$

We can control $u(t)$ freely at each time where $u(t) \ge 0$.

$y(t)$ follows the differential equation:

$$y'(t) = -u(t), \quad y(0) = K, \quad y(T) \ge 0 \,.$$



## Oil Extraction

**Problem I:**
Find an *extraction process* $u(t)$ for a fixed time period $[0, T]$ that optimizes the profit.

**Problem II:**
Find an *extraction process* $u(t)$ *and time horizon* $T$ that optimizes the profit.



## The Standard Problem ($T$ Fixed)

1. Maximize for **objective function** $f$

   $$\max_u \int_0^T f(t, y, u)\, dt, \qquad u \in \mathcal{U} \subseteq \mathbb{R}\,.$$

   $u$ is the **control function**, $\mathcal{U}$ is the **control region**.
2. *Controlled differential equation* (initial value problem)

   $$y' = g(t, y, u), \qquad y(0) = y_0\,.$$
3. Terminal value
   **(a)** $y(T) = y_1$
   **(b)** $y(T) \ge y_1$ [or: $y(T) \le y_1$]
   **(c)** $y(T)$ free

$(y, u)$ is called a **feasible pair** if (2) and (3) are satisfied.

## Hamiltonian

Analogous to the Lagrange function we define function

$$\mathcal{H}(t, y, u, \lambda) = \lambda_0 f(t, y, u) + \lambda(t) g(t, y, u)$$

which is called the **Hamiltonian** of the standard problem.

Function $\lambda(t)$ is called the **adjoint function**.

Scalar $\lambda_0 \in \{0, 1\}$ can be assumed to be $1$.
(However, there exist rare exceptions where $\lambda_0 = 0$.)

In the following we always assume that $\lambda_0 = 1$. Then

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda(t) g(t, y, u)$$



## Maximum Principle

Let $(y^*, u^*)$ be an *optimal pair* of the standard problem.
Then there exists a continuous function $\lambda(t)$ such that for all $t \in [0, T]$:

**(i)** $u^*$ maximizes $\mathcal{H}$ w.r.t. $u$, i.e.,

$$\mathcal{H}(t, y^*, u^*, \lambda) \geq \mathcal{H}(t, y^*, u, \lambda) \quad \text{for all } u \in \mathcal{U}$$

**(ii)** $\lambda$ satisfies the differential equation

$$\lambda' = -\frac{\partial}{\partial y} \mathcal{H}(t, y^*, u^*, \lambda)$$

**(iii) Transversality condition**
(a) $y(T) = y_1$: $\lambda(T)$ free
(b) $y(T) \geq y_1$: $\lambda(T) \geq 0$ [with $\lambda(T) = 0$ if $y^*(T) > y_1$]
(c) $y(T)$ free: $\lambda(T) = 0$



## A Necessary Condition

The maximum principle gives a *necessary* condition for an **optimal pair** of the standard problem, i.e., a feasible pair which solves the optimization problem.

That is, for every optimal pair we can find such a function $\lambda(t)$.

On the other hand if we can find such a function for some feasible pair $(y_0, u_0)$ then $(y_0, u_0)$ need not be optimal.

However, it is a *candidate* for an optimal pair.

(Comparable to the role of stationary points in static constraint optimization problems.)



## A Sufficient Condition

Let $(y^*, u^*)$ be a feasible pair of the standard problem and $\lambda(t)$ some function that satisfies the maximum principle.

If $\mathcal{U}$ is convex and $\mathcal{H}(t, y, u, \lambda)$ is concave in $(y, u)$ for all $t \in [0, T]$, then $(y^*, u^*)$ is an optimal pair.



## Recipe

1. For every triple $(t, y, \lambda)$ find a (global) maximum $\hat{u}(t, y, \lambda)$ of $\mathcal{H}(t, y, u, \lambda)$ w.r.t. $u$.
2. Solve system of differential equations

$$\begin{aligned} y' &= g(t, y, \hat{u}(t, y, \lambda), \lambda) \\ \lambda' &= -\mathcal{H}_y(t, y, \hat{u}(t, y, \lambda), \lambda) \end{aligned}$$

3. Find particular solutions $y^*(t)$ and $\lambda^*(t)$ which satisfy initial condition $y(0) = y_0$ and the transversality condition, resp.
4. We get candidates for an optimal pair by $y^*(t)$ and $u^*(t) = \hat{u}(t, y^*, \lambda^*)$.
5. If $\mathcal{U}$ is convex and $\mathcal{H}(t, y, u, \lambda^*)$ is concave in $(y, u)$, then $(y^*, u^*)$ is an optimal pair.



## Example 1

Find optimal control $u^*$ for

$$\max \int_0^2 y(t)\, dt, \quad u \in [0, 1]$$

$$y' = y + u, \quad y(0) = 0, \quad y(2) \text{ free}$$

**Heuristically:**
Objective function $y$ and thus $u$ should be as large as possible.
Therefore we expect that $u^*(t) = 1$ for all $t$.

Hamiltonian:

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda g(t, y, u) = y + \lambda(y + u)$$



## Example 1

$$\mathcal{H}(t, y, u, \lambda) = y + \lambda(y + u)$$

Maximum $\hat{u}$ of $\mathcal{H}$ w.r.t. $u$:

$$\hat{u} = \begin{cases} 1, & \text{if } \lambda \geq 0, \\ 0, & \text{if } \lambda < 0. \end{cases}$$

Solution of the (inhomogeneous linear) ODE

$$\lambda' = -\mathcal{H}_y = -(1 + \lambda), \quad \lambda(2) = 0$$
$$\Rightarrow \quad \lambda^*(t) = e^{2-t} - 1\,.$$

As $\lambda^*(t) = e^{2-t} - 1 \geq 0$ for all $t \in [0, 2]$ we have $\hat{u}(t) = 1$.



## Example 1

Solution of the (inhomogeneous linear) ODE

$$y' = y + \hat{u} = y + 1, \quad y(0) = 0$$
$$\Rightarrow \quad y^*(t) = e^t - 1\,.$$

We thus obtain

$$u^*(t) = \hat{u}(t) = 1\,.$$

Hamiltonian $\mathcal{H}(t, y, u, \lambda) = y + \lambda(y + u)$ is linear and thus concave in $(y, u)$.

$u^*(t) = 1$ is the optimal control we sought for.

## Example 2

Find the optimal control $u^*$ for

$$\min \int_0^T \left[y^2(t) + cu^2(t)\right]\, dt, \quad u \in \mathbb{R}, \quad c > 0$$
$$y' = u, \quad y(0) = y_0, \quad y(T) \text{ free}$$

We have to solve the maximization problem

$$\max \int_0^T -\left[y^2(t) + cu^2(t)\right]\, dt$$

Hamiltonian:

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda g(t, y, u) = -y^2 - cu^2 + \lambda u$$



## Example 2

Maximum $\hat{u}$ of $\mathcal{H}$ w.r.t. $u$:

$$0 = \mathcal{H}_u = -2c\hat{u} + \lambda \qquad \Rightarrow \qquad \hat{u} = \tfrac{\lambda}{2c}$$

Solution of the (system of) differential equations

$$y' = \hat{u} = \frac{\lambda}{2c}$$
$$\lambda' = -\mathcal{H}_y = 2y$$

By differentiating the second ODE we get

$$\lambda'' = 2y' = \frac{\lambda}{c} \qquad \Rightarrow \qquad \lambda'' - \frac{1}{c}\lambda = 0$$

Solution of the (homogeneous linear) ODE of second order

$$\lambda^*(t) = C_1 e^{rt} + C_2 e^{-rt}, \qquad \text{with } r = \tfrac{1}{\sqrt{c}}$$

($\pm\frac{1}{\sqrt{c}}$ are the two roots of the characteristic polynomial.)



## Example 2

Initial condition $y(0) = y_0$ and transversality condition, resp., yield

$$\lambda^{*\prime}(0) = 2y(0) = 2y_0$$
$$\lambda^*(T) = 0$$

and thus

$$r(C_1 - C_2) = 2y_0$$
$$C_1 e^{rT} + C_2 e^{-rT} = 0$$

with solutions

$$C_1 = \tfrac{2y_0 e^{-rT}}{r(e^{rT}+e^{-rT})}, \qquad C_2 = -\tfrac{2y_0 e^{rT}}{r(e^{rT}+e^{-rT})}\ .$$



## Example 2

Consequently we obtain

$$\lambda^*(t) = \tfrac{2y_0}{r(e^{rT}+e^{-rT})}\left(e^{-r(T-t)} - e^{r(T-t)}\right)$$
$$y^*(t) = \tfrac{1}{2}\lambda^*(t) = y_0 \tfrac{e^{-r(T-t)} - e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$$
$$u^*(t) = \hat{u}(t, y^*, \lambda^*) = \tfrac{1}{2c}\lambda^*(t) = \tfrac{y_0}{c}\,\tfrac{e^{-r(T-t)} - e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$$

It is easy to verify that Hamiltonian $\mathcal{H}(t, y, u, \lambda) = -y^2 - cu^2 + \lambda u$ is concave in $y$ and $u$.

$u^*(t) = \frac{y_0}{c}\,\frac{e^{-r(T-t)} - e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$ is the optimal control.



## Standard Problem ($T$ Variable)

If time horizon $[0, T]$ is not fixed in advanced we have to find an optimal time period $[0, T^*]$ in addition to the optimal control $u^*$.

For this purpose we have to add the following condition to the maximum principle (in addition to (i)–(iii)).

**(iv)** $\mathcal{H}(T^*, y^*(T^*), u^*(T^*), \lambda(T^*)) = 0$

The recipe for solving the optimization problem remains essentially the same.



## Summary

- standard problem
- Hamiltonian function
- maximum principle
- a sufficient condition