Chapter 8

Limits

Sequences*

A **sequence** is an enumerated collection of objects in which repetitions are allowed. These objects are called **members** or **terms** of the sequence.

In this chapter we are interested in *sequences of numbers*.

Formally a sequence is a special case of a *map*:

$$\boxed{a\colon \mathbb{N} \to \mathbb{R},\ n \mapsto a_n}$$

Sequences are denoted by $\big(a_n\big)_{n=1}^{\infty}$ or just $\big(a_n\big)$ for short.

An alternative notation used in literature is $\big\langle a_n\big\rangle_{n=1}^{\infty}$.

Sequences*

Sequences can be defined

- by **enumerating** of its terms,
- by a **formula**, or
- by **recursion**.
  Each term is determined by its predecessor(s).

Enumeration: $(a_n) = (1, 3, 5, 7, 9, \ldots)$

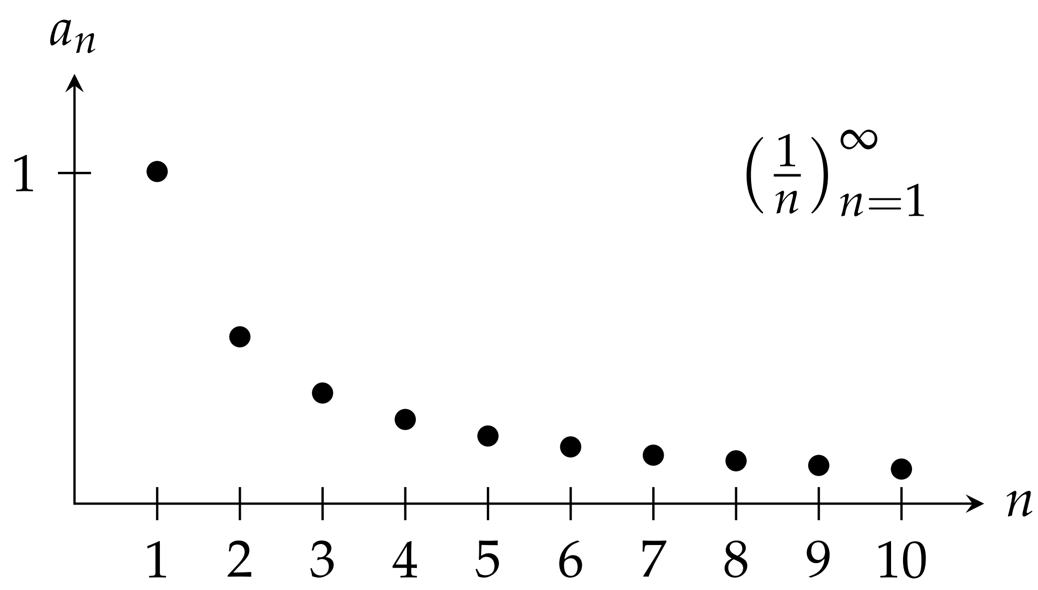
Formula: $(a_n) = (2\,n - 1)$

Recursion: $a_1 = 1,\;\; a_{n+1} = a_n + 2$

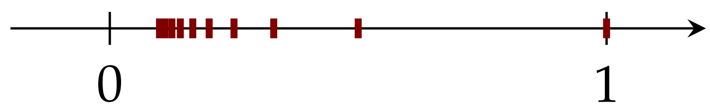
Graphical Representation*

A sequence (a_n) can by represented

(1) by drawing tuples (n, a_n) in the plane, or



(2) by drawing points on the number line.



Properties*

Properties of a sequence (a_n):

Property	Definition	
monotonically increasing	$a_{n+1} \geq a_n$	for all $n \in \mathbb{N}$
monotonically decreasing	$a_{n+1} \leq a_n$	
alternating	$a_{n+1} \cdot a_n < 0$	i.e. the sign changes
bounded	$\lvert a_n \rvert \leq M$	for some $M \in \mathbb{R}$

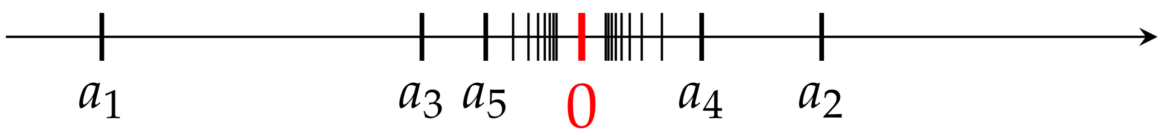
Sequence $\left(\frac{1}{n}\right)$ is

- monotonically decreasing; and
- bounded, as for all $n \in \mathbb{N}$, $|a_n| = |1/n| \leq M = 1$;
  (we could also choose $M = 1000$)
- but *not* alternating.

Limit of a Sequence*

Consider the following sequence of numbers

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \frac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \frac{1}{6}, \dots\right)$$



The terms of this sequence *tend* to 0 with increasing n.
We say that sequence (a_n) **converges** to 0.

We write

$$\boxed{(a_n) \to 0 \quad \text{or} \quad \lim_{n\to\infty} a_n = 0}$$

(read: “limit of a_n for n tends to ∞”)

Limit of a Sequence / Definition*

Definition:
A number $a \in \mathbb{R}$ is a **limit** of sequence (a_n), if *for every interval* $(a-\varepsilon, a+\varepsilon)$ there *exists an* N such that $a_n \in (a-\varepsilon, a+\varepsilon)$ for all $n \geq N$; i.e., all terms following a_N are contained in this interval.

Equivalent Definition: A sequence (a_n) converges to **limit** $a \in \mathbb{R}$ if *for every* $\varepsilon > 0$ there *exists an* N such that $|a_n - a| < \varepsilon$ for all $n \geq N$.

[Mathematicians like to use ε for a very small positive number.]

A sequence that has a *limit* is called **convergent**.
It **converges** to its limit.

It can be shown that a limit of a sequence is *unique*ly defined *(if it exists)*.

A sequence *without* a limit is called **divergent**.

Limit of a Sequence / Example*

Sequence

$$(a_n)_{n=1}^{\infty} = \left((-1)^n \tfrac{1}{n}\right)_{n=1}^{\infty} = \left(-1, \tfrac{1}{2}, -\tfrac{1}{3}, \tfrac{1}{4}, -\tfrac{1}{5}, \tfrac{1}{6}, \dots\right)$$

has limit $a = 0$.

For example, if we set $\varepsilon = 0.3$, then all terms following a_4 are contained in interval $(a - \varepsilon, a + \varepsilon)$.

If we set $\varepsilon = \frac{1}{1\,000\,000}$, then all terms starting with the 1 000 001-st term are contained in the interval.

Thus

$$\lim_{n \to \infty} \frac{(-1)^n}{n} = 0\ .$$

Limit of a Sequence / Example*

Sequence $(a_n)_{n=1}^{\infty} = \left(\frac{1}{2^n}\right)_{n=1}^{\infty} = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots\right)$ converges to 0:

$$\lim_{n\to\infty} a_n = 0$$

Sequence $(b_n)_{n=1}^{\infty} = \left(\frac{n-1}{n+1}\right)_{n=1}^{\infty} = \left(0, \frac{1}{3}, \frac{2}{4}, \frac{3}{5}, \frac{4}{6}, \frac{5}{7}, \ldots\right)$ is convergent:

$$\lim_{n\to\infty} b_n = 1$$

Sequence $(c_n)_{n=1}^{\infty} = \left((-1)^n\right)_{n=1}^{\infty} = \left(-1, 1, -1, 1, -1, 1, \ldots\right)$ is divergent.

Sequence $(d_n)_{n=1}^{\infty} = \left(2^n\right)_{n=1}^{\infty} = \left(2, 4, 8, 16, 32, \ldots\right)$ is divergent, but tends to ∞. By abuse of notation we write:

$$\lim_{n\to\infty} d_n = \infty$$

Limits of Important Sequences*

$$\lim_{n\to\infty} n^a = \begin{cases} 0 & \text{for } a < 0 \\ 1 & \text{for } a = 0 \\ \infty & \text{for } a > 0 \end{cases}$$

$$\lim_{n\to\infty} q^n = \begin{cases} 0 & \text{for } |q| < 1 \\ 1 & \text{for } q = 1 \\ \infty & \text{for } q > 1 \\ \nexists & \text{for } q \leq -1 \end{cases}$$

$$\lim_{n\to\infty} \frac{n^a}{q^n} = \begin{cases} 0 & \text{for } |q| > 1 \\ \infty & \text{for } 0 < q < 1 \\ \nexists & \text{for } -1 < q < 0 \end{cases} \qquad (|q| \notin \{0,1\})$$

Rules for Limits*

Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be convergent sequences with $\lim\limits_{n\to\infty} a_n = a$ and $\lim\limits_{n\to\infty} b_n = b$, resp., and let $(c_n)_{n=1}^{\infty}$ be a bounded sequence.
Then

(1)	$\lim\limits_{n\to\infty}(k \cdot a_n + d) = k \cdot a + d$	
(2)	$\lim\limits_{n\to\infty}(a_n + b_n) = a + b$	
(3)	$\lim\limits_{n\to\infty}(a_n \cdot b_n) = a \cdot b$	
(4)	$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \dfrac{a}{b}$	for $b \neq 0$
(5)	$\lim\limits_{n\to\infty}(a_n \cdot c_n) = 0$	provided $a = 0$
(6)	$\lim\limits_{n\to\infty} a_n^k = a^k$	

Example – Rules for Limits*

$$\lim_{n\to\infty}\left(2+\frac{3}{n^2}\right) = 2+3\underbrace{\lim_{n\to\infty} n^{-2}}_{=0} = 2+3\cdot 0 = 2$$

$$\lim_{n\to\infty}(2^{-n}\cdot n^{-1}) = \lim_{n\to\infty}\frac{n^{-1}}{2^n} = 0$$

$$\lim_{n\to\infty}\frac{1+\frac{1}{n}}{2-\frac{3}{n^2}} = \frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(2-\frac{3}{n^2}\right)} = \frac{1}{2}$$

$$\lim_{n\to\infty}\underbrace{\sin(n)}_{\text{bounded}}\cdot\underbrace{\frac{1}{n^2}}_{\to 0} = 0$$

Rules for Limits / Rational Terms*

Important!
When we apply these rules we have to take care that we never obtain expressions of the form $\frac{0}{0}$, $\frac{\infty}{\infty}$, or $0 \cdot \infty$.

These expressions are **not defined**!

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \frac{\lim_{n\to\infty} 3n^2+1}{\lim_{n\to\infty} n^2-1} = \frac{\infty}{\infty} \quad \text{(not defined)}$$

Trick: Reduce the fraction by the *largest power* in its **denominator**.

$$\lim_{n\to\infty} \frac{3n^2+1}{n^2-1} = \lim_{n\to\infty} \frac{\cancel{n^2}}{\cancel{n^2}} \cdot \frac{3+n^{-2}}{1-n^{-2}} = \frac{\lim_{n\to\infty} 3+n^{-2}}{\lim_{n\to\infty} 1-n^{-2}} = \frac{3}{1} = 3$$

Euler's Number*

$$\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e = 2.7182818284590\ldots}$$

This limit is very important in many applications including finance (continuous compounding).

$$\begin{aligned}
\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n &= \lim_{n\to\infty}\left(1+\frac{1}{n/x}\right)^n \\
&= \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} \qquad \left(m=\frac{n}{x}\right) \\
&= \left(\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right)^x = e^x
\end{aligned}$$

Cauchy Sequence

How can we determine that a sequence converges if we have no clue about the limit?

A sequence $(a_n)_{n=1}^{\infty}$ converges if and only if *for every* $\varepsilon > 0$ there *exists an* N such that $|a_n - a_m| < \varepsilon$ for all $n, m \geq N$.

A sequence with such a property is called a **Cauchy sequence**.

Series*

The sum of the first n terms of sequence $\big(a_i\big)_{i=1}^{\infty}$

$$s_n = \sum_{i=1}^{n} a_i$$

is called the n-th **partial sum** of the *sequence*.

The sequence $\big(s_n\big)_{n=1}^{\infty}$ of all partial sums is called the **series** of the sequence.

The series of sequence $\big(a_i\big) = \big(2\,i - 1\big)$ is

$$\big(s_n\big) = \left(\sum_{i=1}^{n}(2\,i-1)\right) = \big(1, 4, 9, 16, 25, \dots\big) = \big(n^2\big)\ .$$

Limit of a Series*

There are many cases where we have a summation over *infinitely* many terms, $\sum_{i=1}^{\infty} a_i$.

However, then the usual rules for addition (in particular associativity and commutativity) may not hold any more.
Applying them in such cases would result in contradicitions.

Thus an “infinite sum” is defined as the limit of a series:

$$\boxed{\sum_{i=1}^{\infty} a_i = \lim_{n\to\infty} \sum_{i=1}^{n} a_i}$$

For example, $$\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{\left(\frac{1}{2}\right)^n - 1}{\frac{1}{2} - 1} = 1.$$

Limit of a Function*

What happens with the value of a function f, if the argument x tends to some value x_0 (which need not belong to the domain of f)?

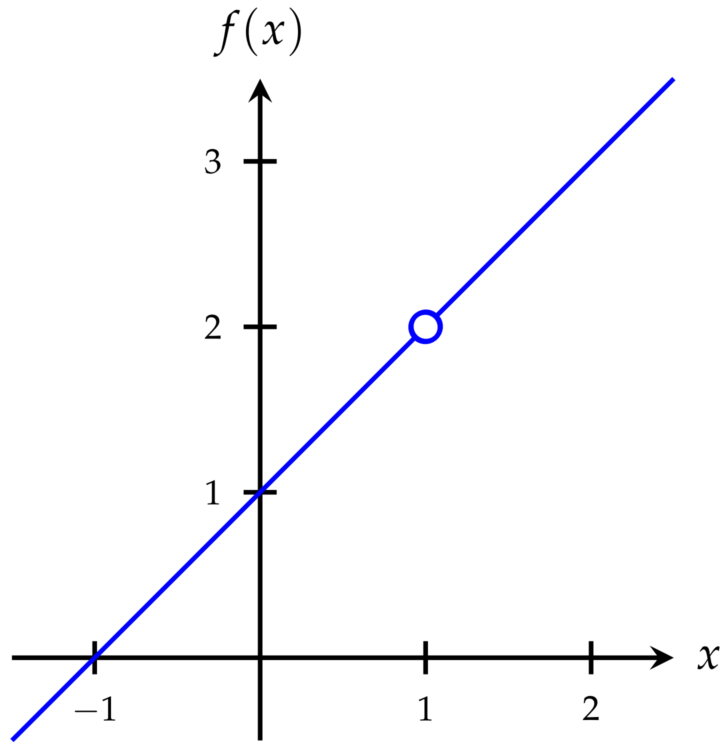
Function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

is not defined in $x = 1$.

By factorizing and reducing we get function

$$g(x) = x + 1 = \begin{cases} f(x), & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$$



Limit of a Function*

Suppose we approach argument $x_0 = 1$.

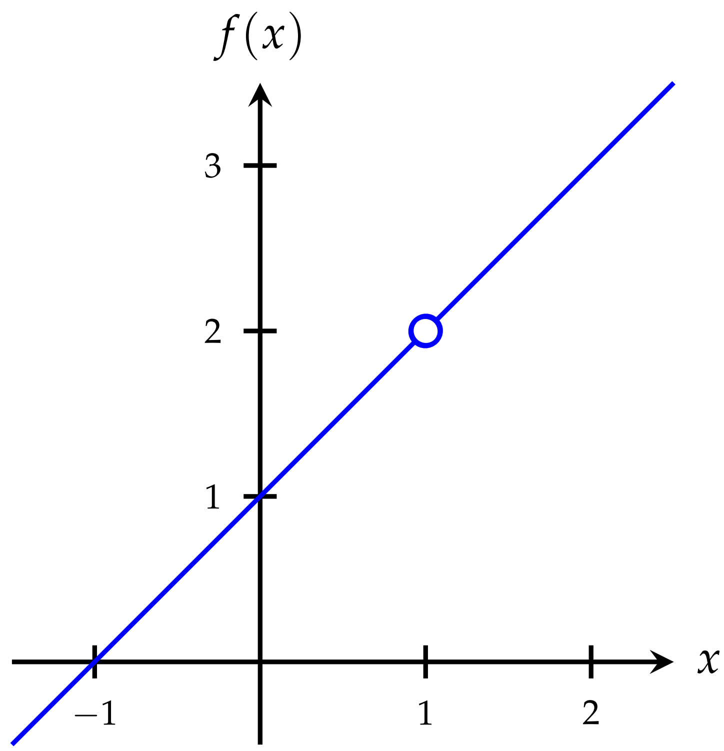
Then the value of function $f(x) = \dfrac{x^2 - 1}{x - 1}$ tends to 2.

We say:

$f(x)$ **converges** to 2 when x tends to 1

and write:

$$\boxed{\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2}$$



Limit of a Function*

Formal **definition**:
If sequence $\big(f(x_n)\big)_{n=1}^{\infty}$ of function values converges to number y_0 for every convergent sequence $(x_n)_{n=1}^{\infty} \to x_0$ of arguments, then y_0 is called the **limit** of f as x approaches x_0.

We write

$$\boxed{\lim_{x \to x_0} f(x) = y_0 \quad \text{or} \quad f(x) \to y_0 \ \text{ for } x \to x_0}$$

x_0 need not belong to the domain of f.
y_0 need not belong to the codomain of f.

Rules for Limits*

Rules for limits of functions are analogous to rules for sequences.

Let $\lim\limits_{x \to x_0} f(x) = a$ and $\lim\limits_{x \to x_0} g(x) = b$.

(1)	$\lim\limits_{x \to x_0} (c \cdot f(x) + d) = c \cdot a + d$	
(2)	$\lim\limits_{x \to x_0} (f(x) + g(x)) = a + b$	
(3)	$\lim\limits_{x \to x_0} (f(x) \cdot g(x)) = a \cdot b$	
(4)	$\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \dfrac{a}{b}$	for $b \neq 0$
(6)	$\lim\limits_{x \to x_0} (f(x))^k = a^k$	for $k \in \mathbb{N}$

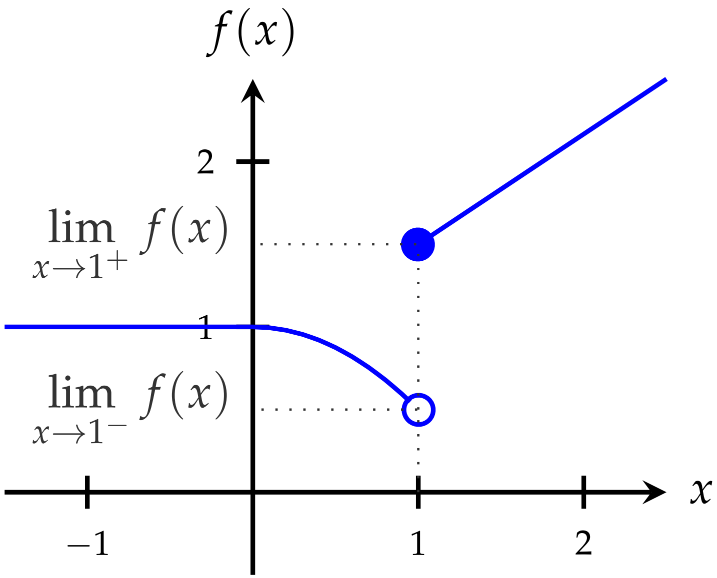
How to Find Limits?*

The following recipe is suitable for “simple” functions:

1. Draw the graph of the function.
2. Mark x_0 on the x-axis.
3. Follow the graph with your pencil until we reach x_0 starting from *right* of x_0.
4. The y-coordinate of your pencil in this point is then the so called **right-handed limit** of f as x approaches x_0 (from above):
   $\lim\limits_{x \to x_0^+} f(x)$. (Other notations: $\lim\limits_{x \downarrow x_0} f(x)$ or $\lim\limits_{x \searrow x_0} f(x)$)
5. Analogously we get the **left-handed limit** of f as x approaches x_0 (from below): $\lim\limits_{x \to x_0^-} f(x)$.
6. If both limits *coincide*, then the limit exists and we have

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0^-} f(x) = \lim_{x \to x_0^+} f(x)$$

Example – How to Find Limits?*

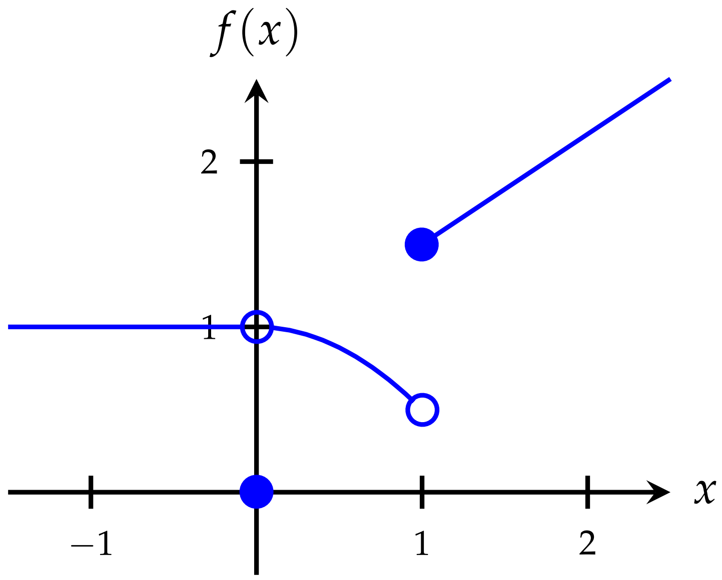


$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

$0.5 = \lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x) = 1.5$
i.e., the limit of f at $x_0 = 1$ does not exist.

The limits at other points, however, do exist,
e.g. $\lim\limits_{x \to 0} f(x) = 1$.

Example – How to Find Limits?*



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

The only difference is to above is the function value at $x_0 = 0$.
Nevertheless, the limit does exist:

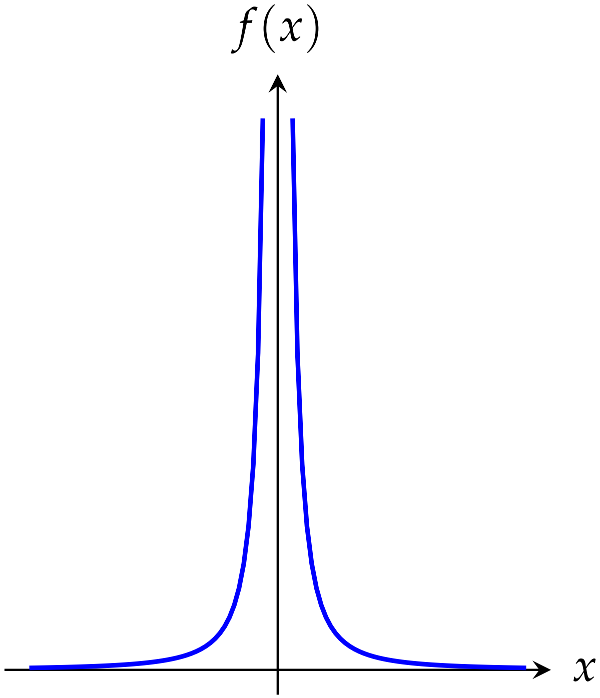
$$\lim_{x \to 0^-} f(x) = 1 = \lim_{x \to 0^+} f(x) \quad \Rightarrow \quad \lim_{x \to 0} f(x) = 1\,.$$

Unbounded Function*

It may happen that $f(x)$ tends to ∞ (or $-\infty$) if x tends to x_0.

We then write (by abuse of notation):

$$\boxed{\lim_{x \to x_0} f(x) = \infty}$$



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Limit as $x \to \infty$*

By abuse of language we can define the *limit* analogously for $x_0 = \infty$ and $x_0 = -\infty$, resp.

Limit

$$\lim_{x \to \infty} f(x)$$

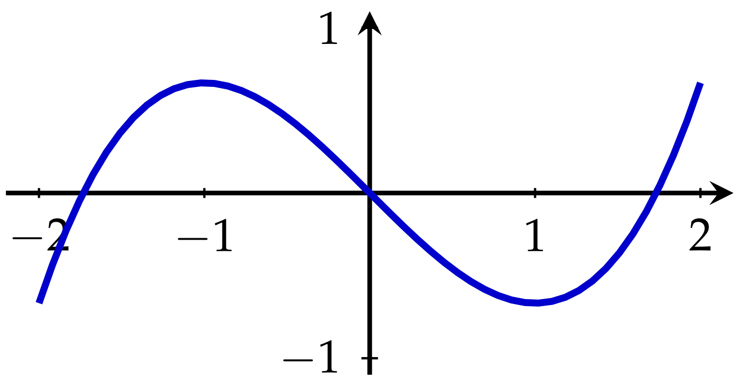
exists, if $f(x)$ converges whenever x tends to infinity.

$$\lim_{x \to \infty} \frac{1}{x^2} = 0 \qquad \text{and} \qquad \lim_{x \to -\infty} \frac{1}{x^2} = 0$$

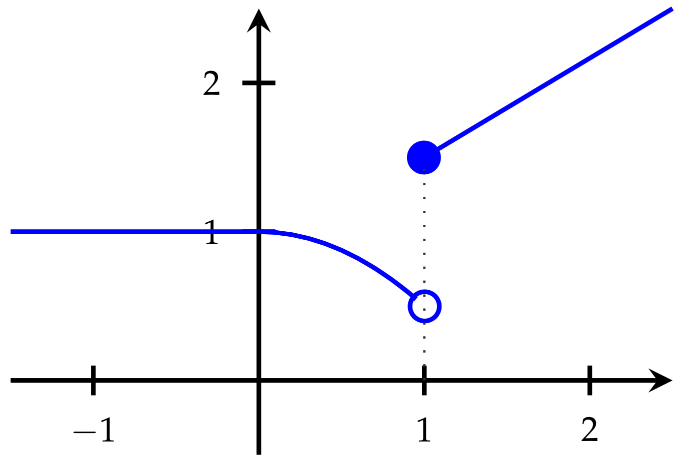
Continuous Functions*

We may observe that we can draw the graph of a function *without removing the pencil from paper*. We call such functions **continuous**.

For some other functions we *have to remove* the pencil. At such points the function has a **jump discontinuity**.



continuous



jump discontinuity at $x = 1$

Continuous Functions*

Formal **Definition:**

Function $f\colon D \to \mathbb{R}$ is called **continuous** *at* $x_0 \in D$, if

1. $\lim_{x \to x_0} f(x)$ exists, and

2. $\boxed{\lim_{x \to x_0} f(x) = f(x_0)}$.

The function is called **continuous** if it is continuous *at all* points of its domain.

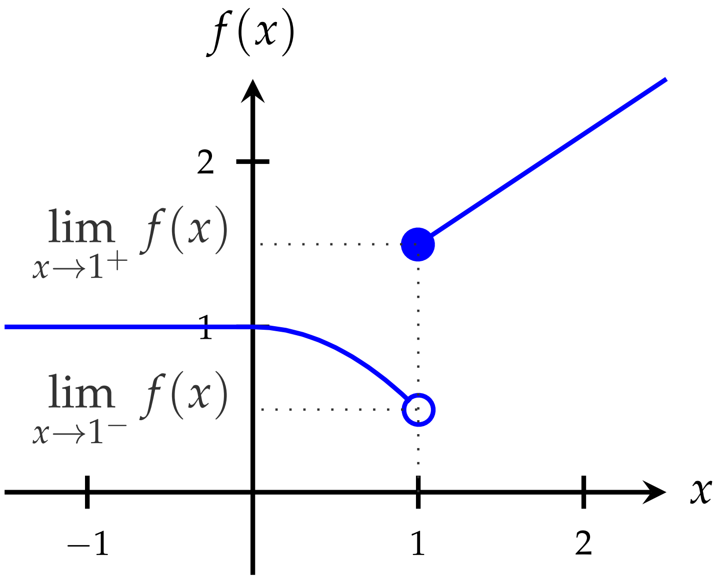
Note that continuity is a *local* property of a function.

Continuous Function and Limit*

Continuous functions have to important property that we can exchange function evaluation and the limiting process.

$$\boxed{\lim_{x \to x_0} f(x) = f(\lim_{x \to x_0} x)}$$

Discontinuous Function*



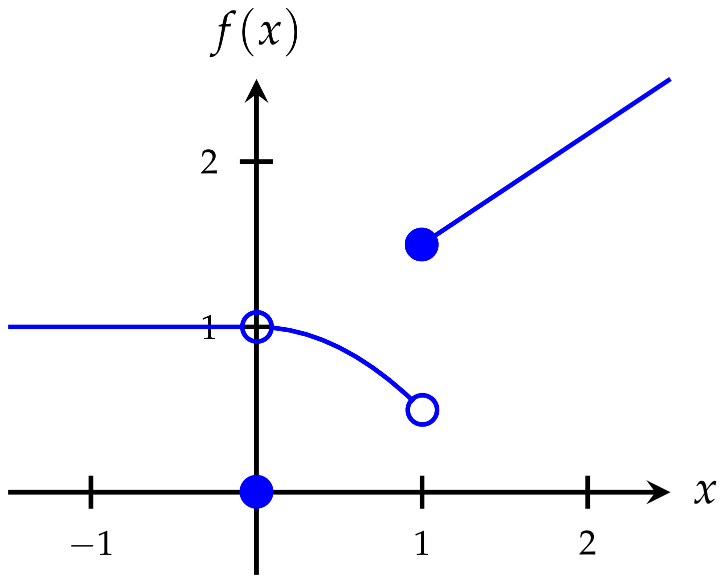
$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1 \\ \frac{x}{2} + 1, & \text{for } x \ge 1 \end{cases}$$

Not continuous in $x = 1$ as $\lim_{x\to 1} f(x)$ does not exist.

So f is not a continuous function.

However, it is still continuous in all $x \in \mathbb{R} \setminus \{1\}$.
For example at $x = 0$, $\lim_{x\to 0} f(x)$ does exist and $\lim_{x\to 0} f(x) = 1 = f(0)$.

Discontinuous Function*



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Not continuous in all $x = 0$, either.

$\lim_{x \to 0} f(x) = 1$ does exist but $\lim_{x \to 0} f(x) \neq f(0)$.

So f is not a continuous function.

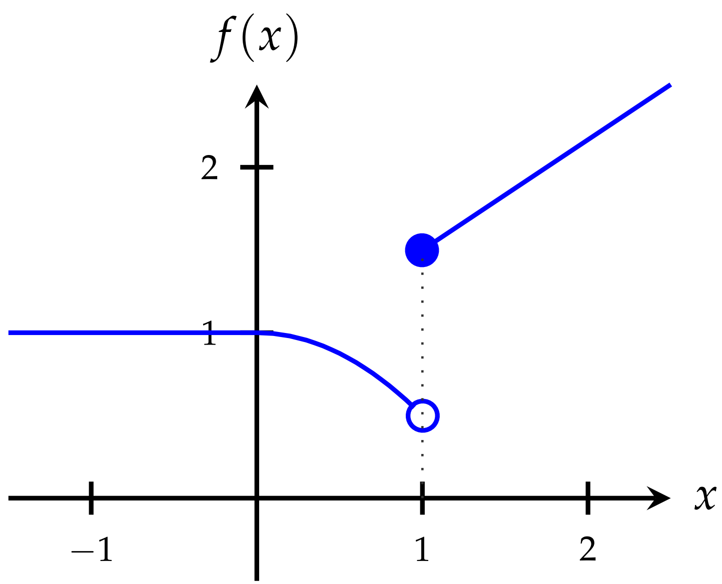
However, it is still continuous in all $x \in \mathbb{R} \setminus \{0, 1\}$.

Recipe for “Nice” Functions*

(1) Draw the graph of the given function.

(2) At all points of the *domain*, where we *have to remove* the pencil from paper the function is *not continuous*.

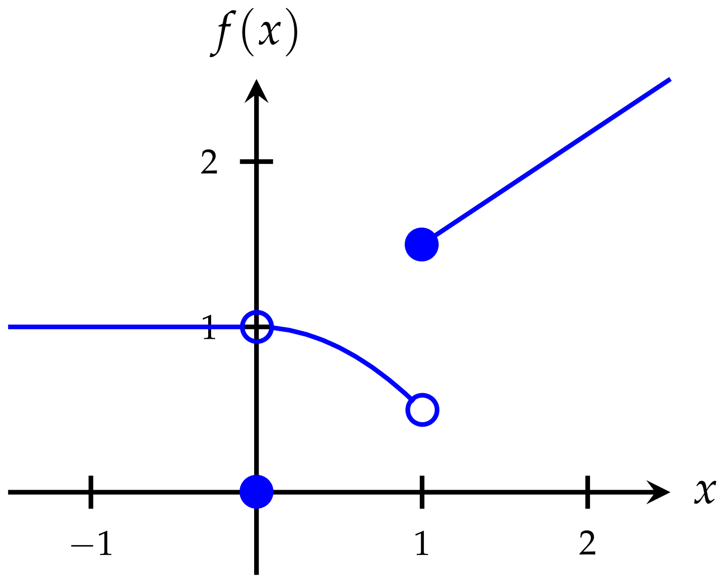
(3) At all other points of the domain (where we need not remove the pencil) the function is *continuous*.



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \leq x < 1, \\ \frac{x}{2} + 1, & \text{for } x \geq 1. \end{cases}$$

f is continuous
except at point $x = 1$.

Discontinuous Function*



$$f(x) = \begin{cases} 1, & \text{for } x < 0 \\ 0, & \text{for } x = 0 \\ 1 - \frac{x^2}{2}, & \text{for } 0 < x < 1 \\ \frac{x}{2} + 1, & \text{for } x \geq 1 \end{cases}$$

Function f is continuous *except* at points $x = 0$ and $x = 1$.

Summary

- sequence
- limit of a sequence
- limit of a function
- convergent and divergent
- Euler's number
- rules for limits
- continuous functions