Chapter 7

Real Functions

Real Function*

Real functions are maps where both *domain* and *codomain* are (unions of) intervals in $\mathbb{R}$.

Often only function terms are given but neither domain nor codomain. Then domain and codomain are implicitly given as following:

- *Domain* of the function is the largest *sensible* subset of the domain of the function terms (i.e., where the terms are defined).
- *Codomain* is the image (range) of the function

$$f(D) = \{y \mid y = f(x) \text{ for ein } x \in D_f\} .$$

Implicit Domain*

Production function $f(x) = \sqrt{x}$ is an abbreviation for

$$f\colon [0,\infty) \to [0,\infty),\ x \mapsto f(x) = \sqrt{x}$$

(There are no negative amounts of goods.
Moreover, $\sqrt{x}$ is not real for $x < 0$.)

Its derivative $f'(x) = \frac{1}{2\sqrt{x}}$ is an abbreviation for

$$f'\colon (0,\infty) \to (0,\infty),\ x \mapsto f'(x) = \frac{1}{2\sqrt{x}}$$

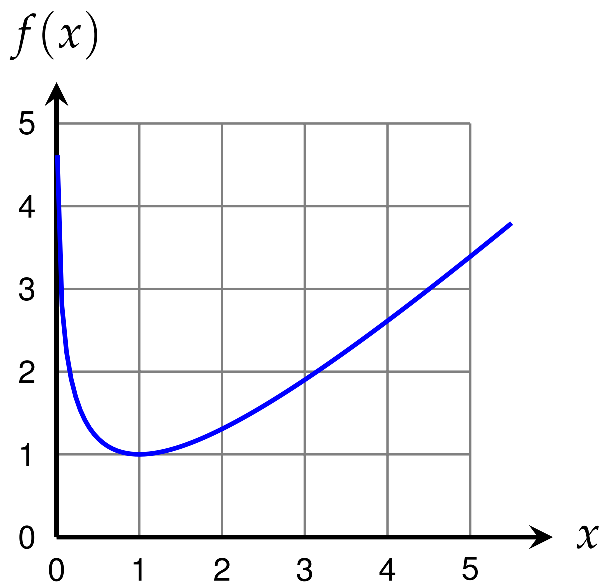
(Note the open interval $(0,\infty)$; $\frac{1}{2\sqrt{x}}$ is not defined for $x = 0$.)

Graph of a Function*

Each tuple $(x, f(x))$ corresponds to a point in the xy-plane.
The set of all these points forms a curve called the **graph** of function f.

$$\mathcal{G}_f = \{(x,y) \mid x \in D_f,\ y = f(x)\}$$

Graphs can be used to visualize functions.
They allow to detect many properties of the given function.



$$f(x) = x - \ln(x)$$

How to Draw a Graph*

1. Get an idea about the possible shape of the graph. One should be able to sketch graphs of elementary functions by heart.

2. Find an appropriate range for the x-axis.
   (It should show a characteristic detail of the graph.)

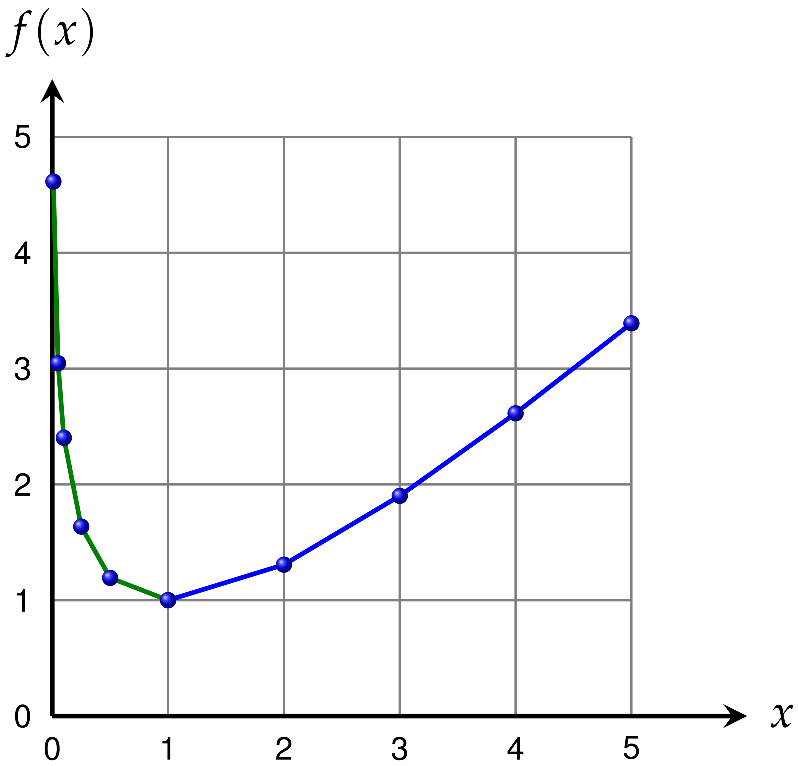
3. Create a table of function values and draw the corresponding points into the xy-plane.

   If known, use characteristic points like local extrema or inflection points.

4. Check if the curve can be constructed from the drawn points.
   If not add adapted points to your table of function values.

5. Fit the curve of the graph through given points in a proper way.

Example – How to Draw a Graph*



Graph of function

$$f(x) = x - \ln x$$

Table of values:

x	$f(x)$
0	ERROR
1	1
2	1.307
3	1.901
4	2.614
5	3.391
0.5	1.193
0.25	1.636
0.1	2.403
0.05	3.046

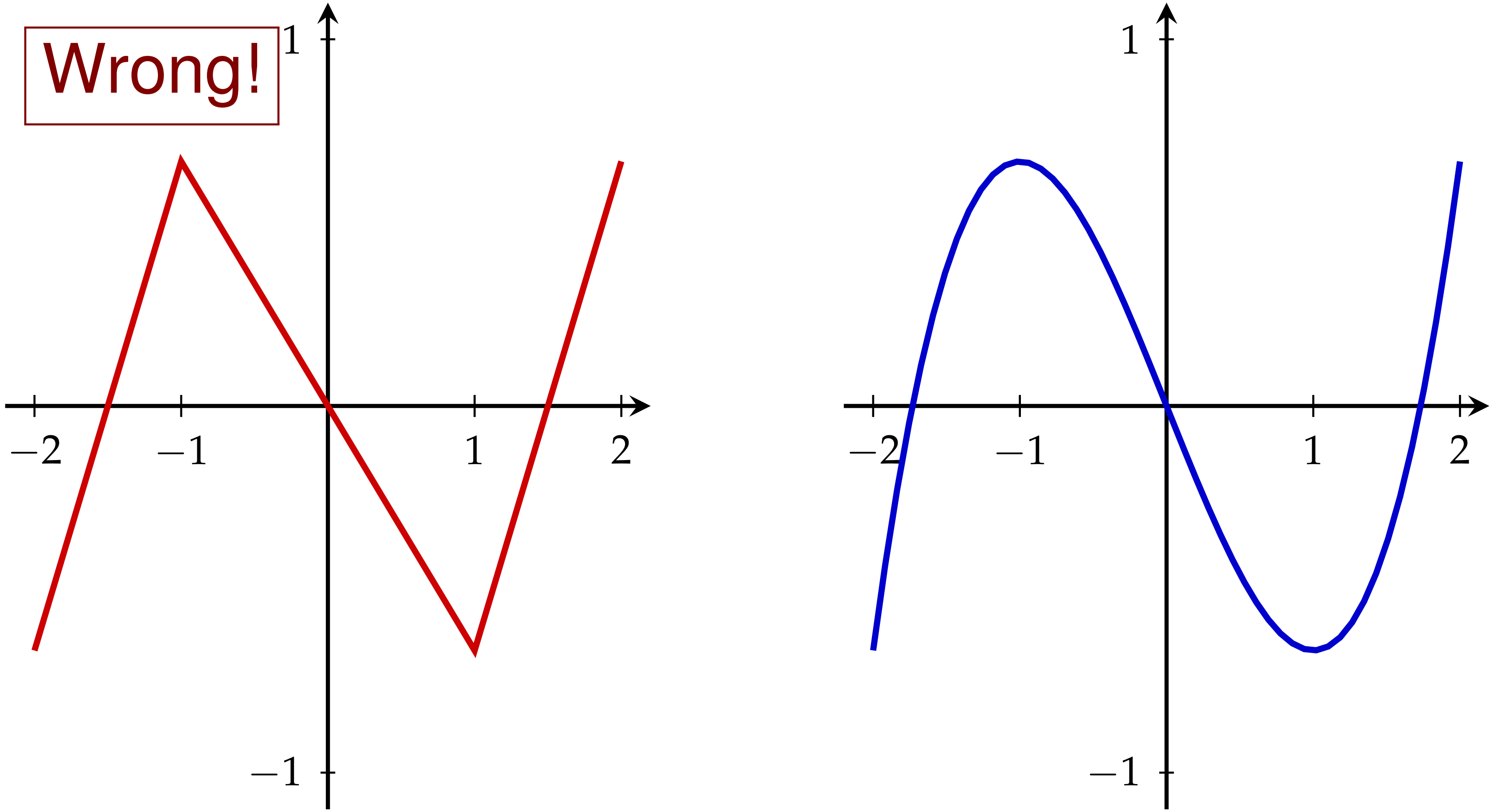
Sources of Errors

Most frequent errors when drawing function graphs:

- **Table of values is too small:**
  It is not possible to construct the curve from the computed function values.
- **Important points are ignored:**
  Ideally extrema and inflection points should be known and used.
- **Range for x and y-axes not suitable:**
  The graph is tiny or important details vanish in the “noise” of handwritten lines (or pixel size in case of a computer program).

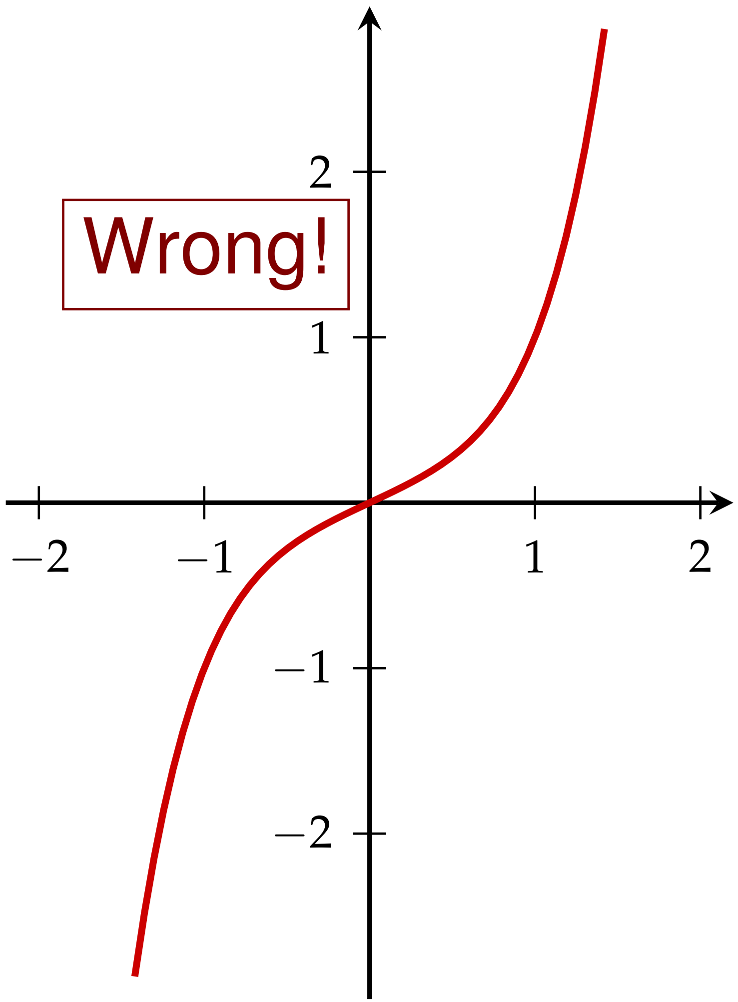
Sources of Errors*

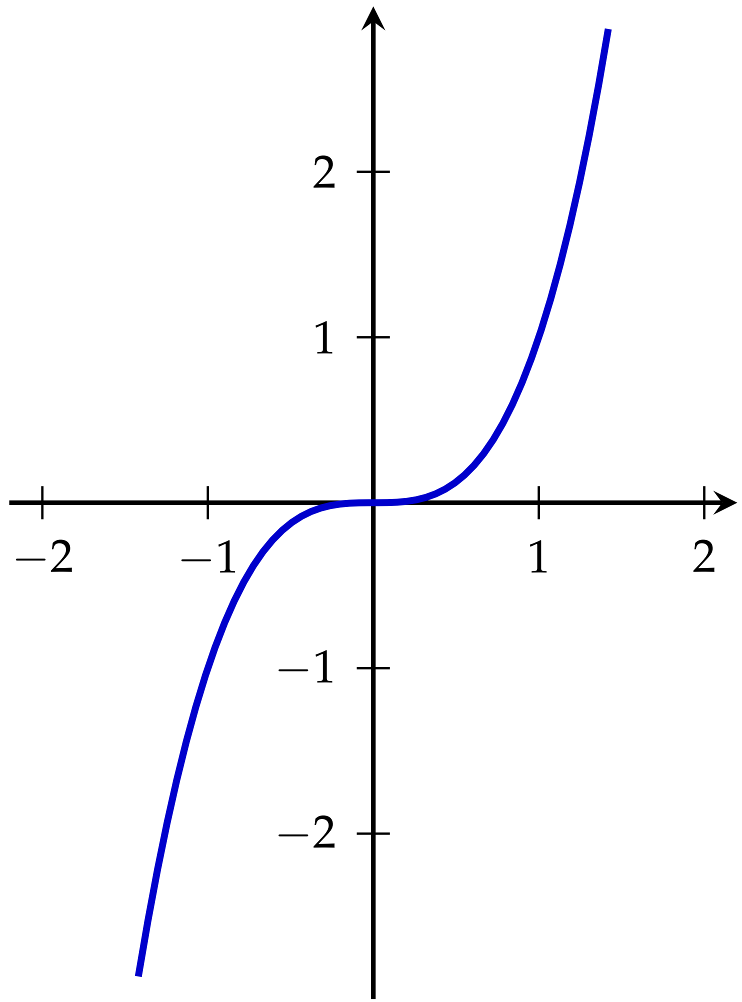
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:



Sources of Errors*

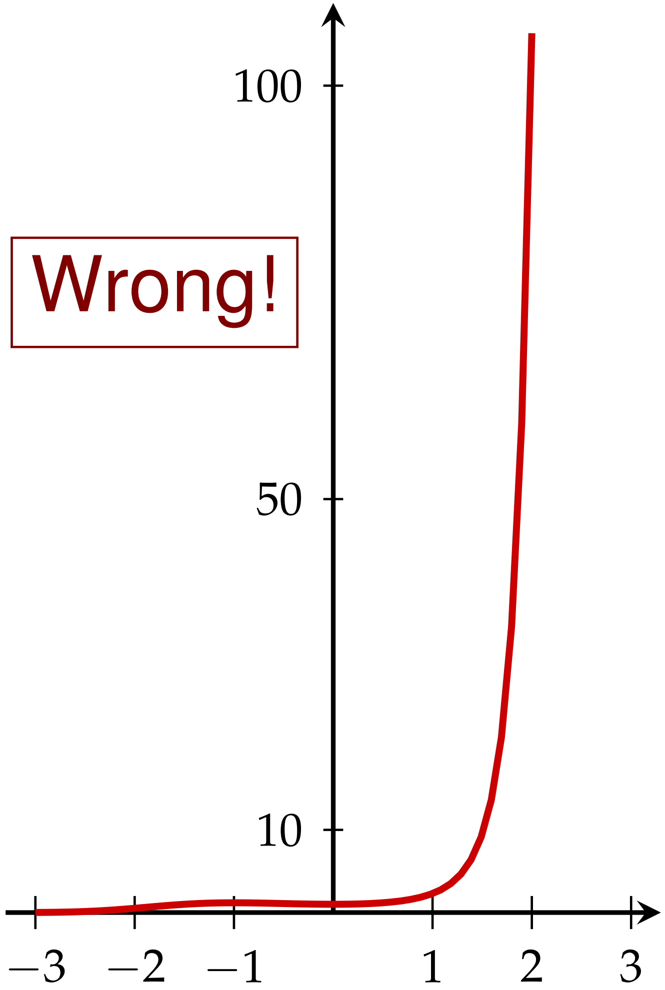
Graph of $f(x) = x^3$ has slope 0 in $x = 0$:

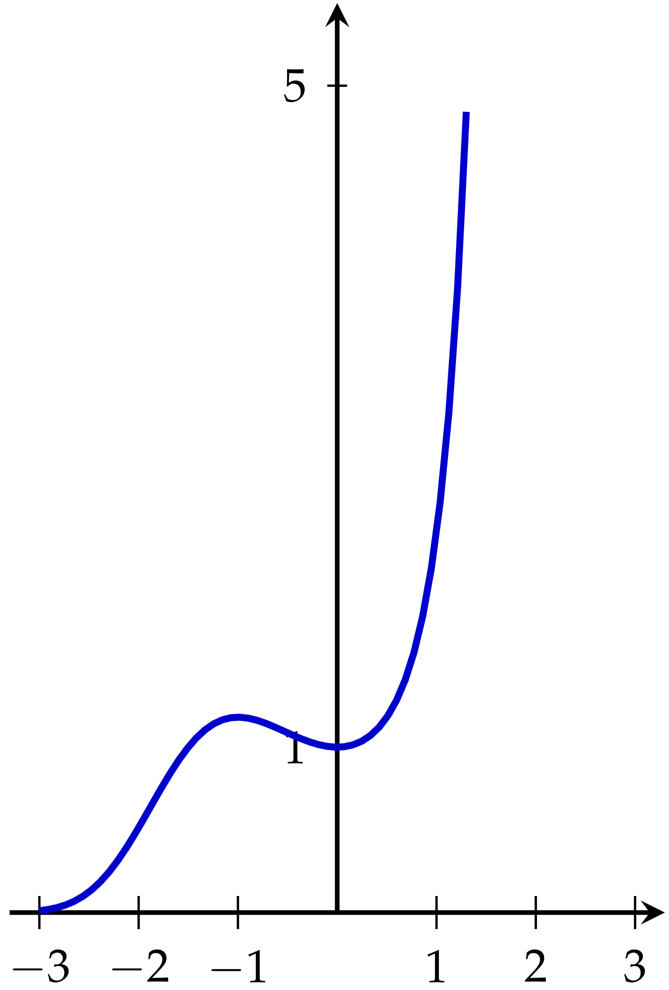




Sources of Errors*

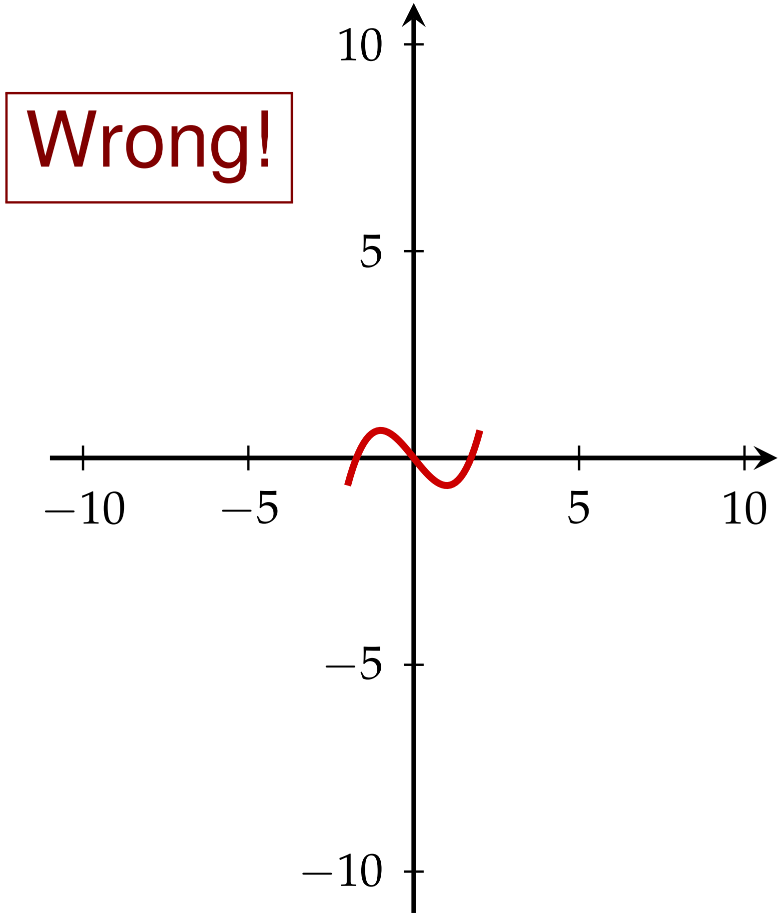
Function $f(x) = \exp(\frac{1}{3}x^3 + \frac{1}{2}x^2)$ has a local maximum in $x = -1$:

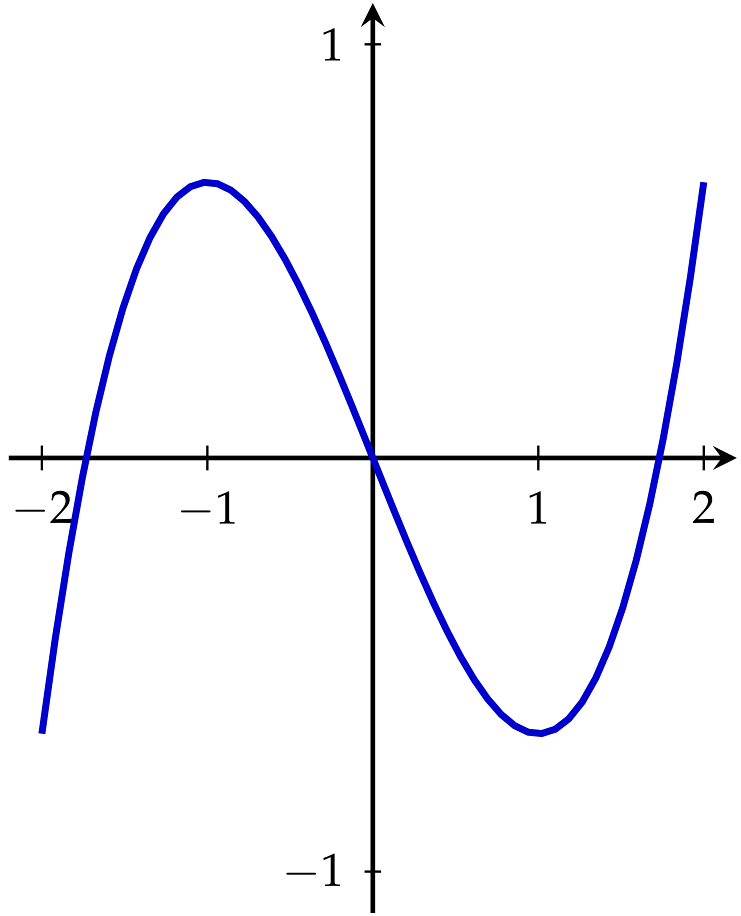




Sources of Errors*

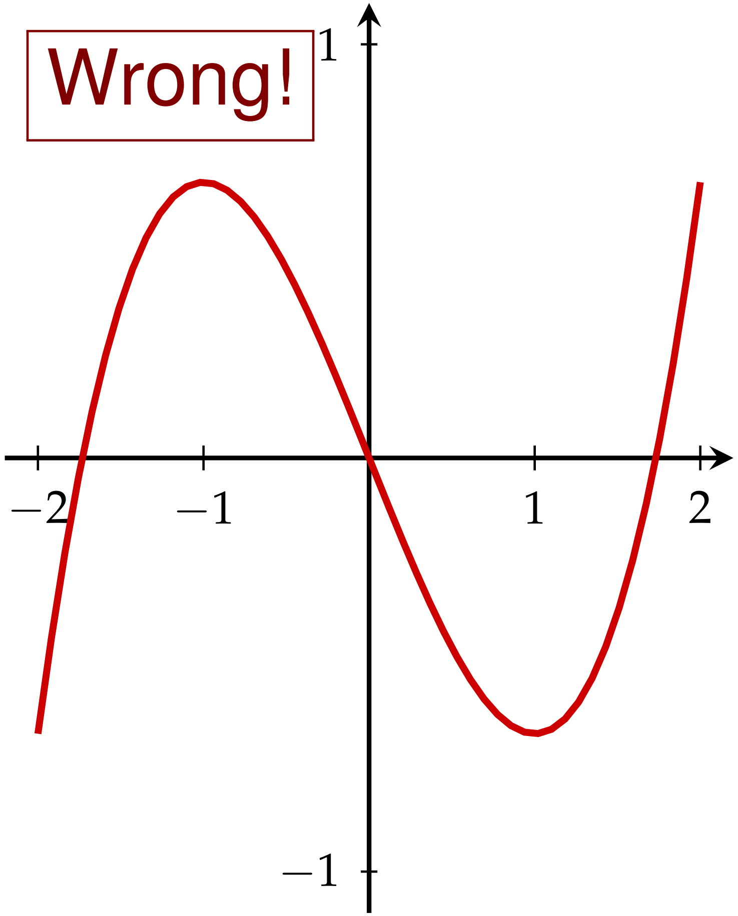
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[-2, 2]$:

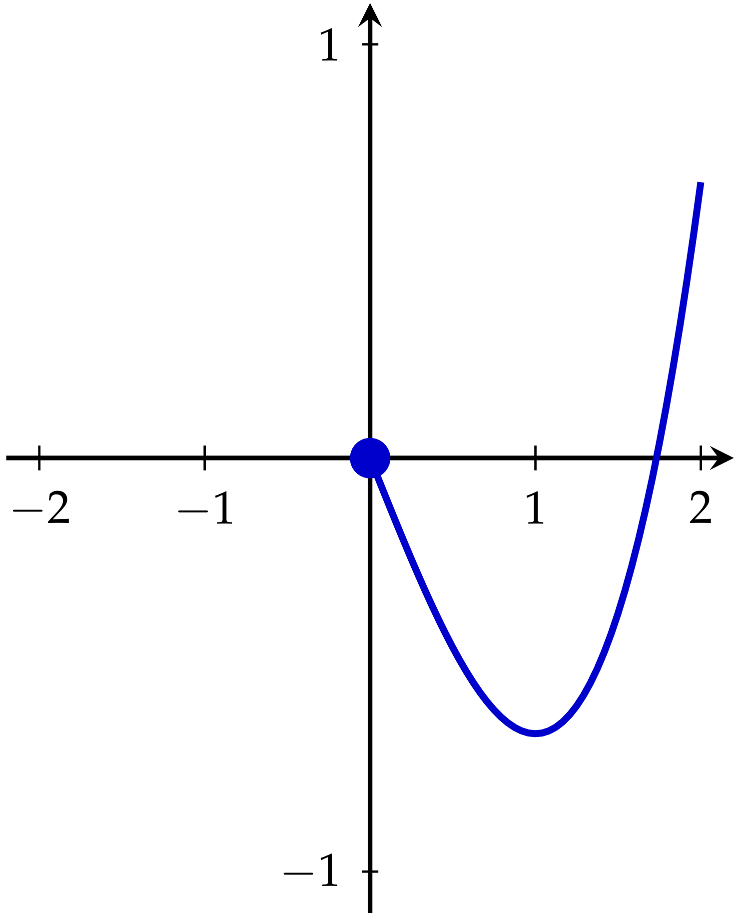




Sources of Errors*

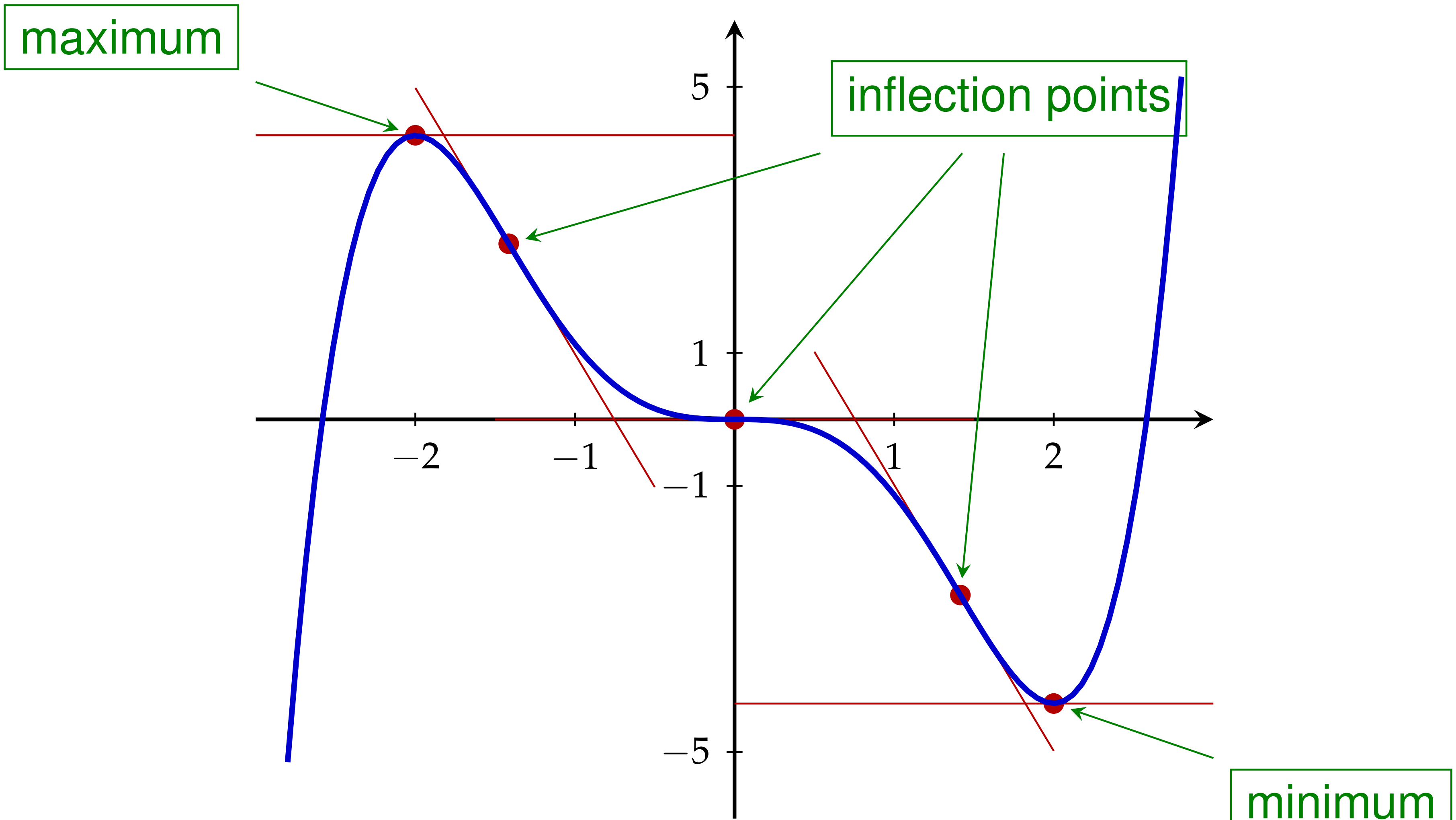
Graph of function $f(x) = \frac{1}{3}x^3 - x$ in interval $[0, 2]$: (not in $[-2, 2]$!)





Extrema and Inflection Points*

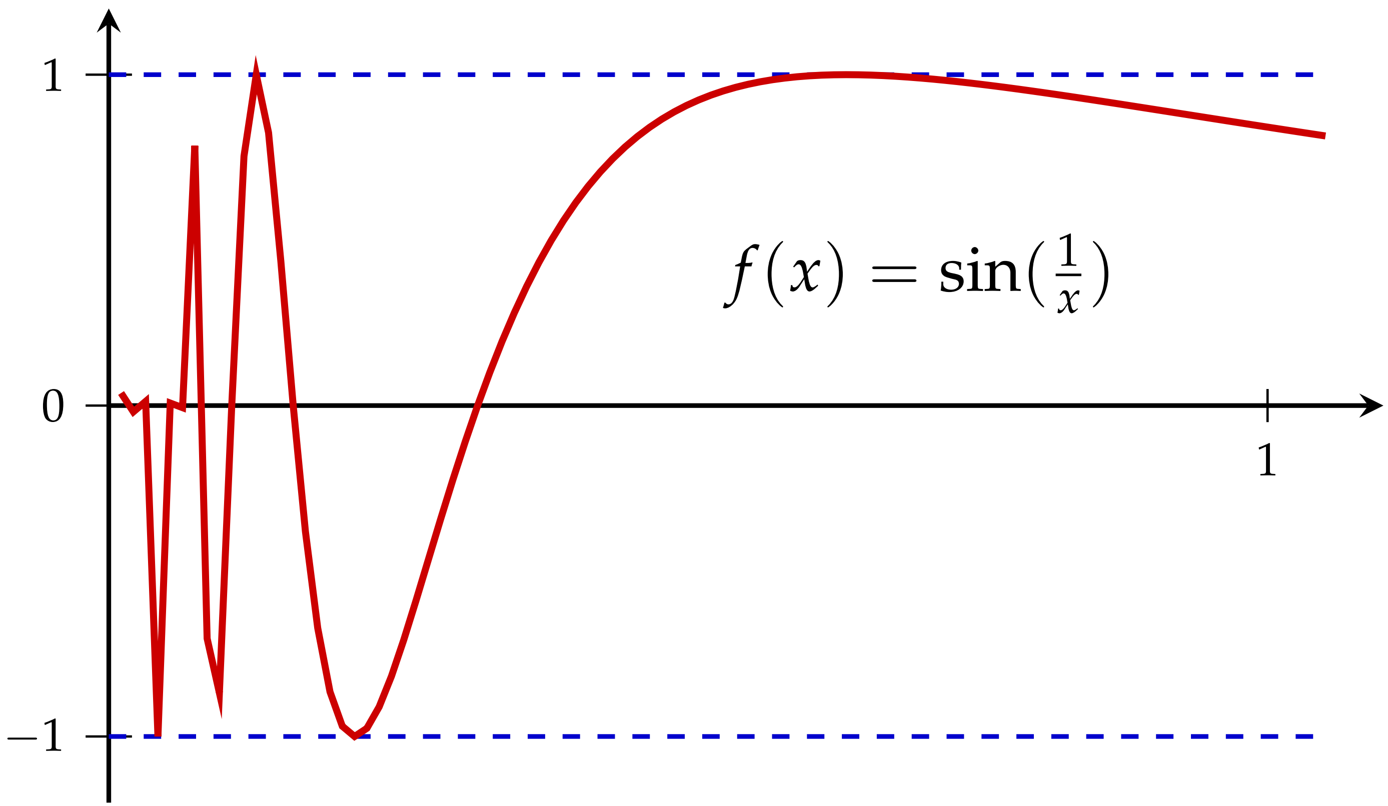
Graph of function $f(x) = \frac{1}{15}(3x^5 - 20x^3)$:



Sources of Errors

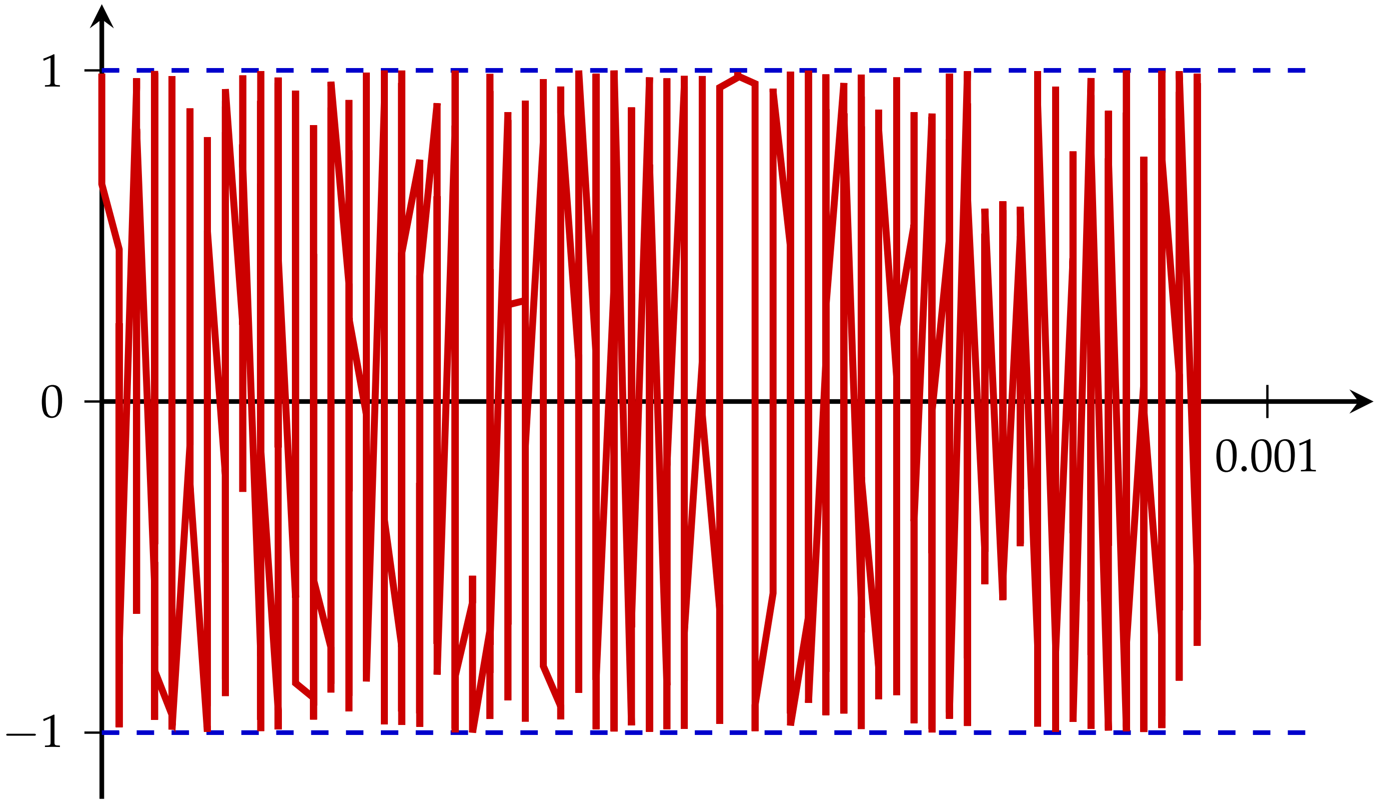
It is *important* that one already has an *idea of the shape* of the function graph **before** drawing the curve.

Even a graph drawn by means of a computer program can differ significantly from the true curve.



Sources of Errors*

$$f(x) = \sin(\tfrac{1}{x})$$



Sketch of a Function Graph*

Often a **sketch** of the graph is sufficient. Then the exact function values are not so important. Axes may not have scales.

However, it is important that the sketch clearly shows all characteristic details of the graph (like extrema or important function values).

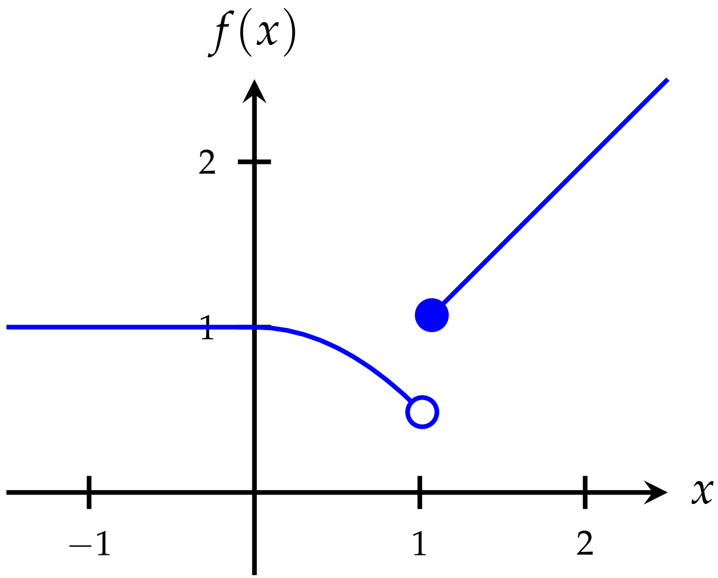
Sketches can also be drawn like a caricature:
They stress prominent parts and properties of the function.

Piece-wise Defined Functions*

The function term can be defined differently in subintervals of the domain.

At the boundary points of these subintervals we have to mark which points belong to the graph and which do not:

• (belongs) and ○ (does not belong).



$$f(x) = \begin{cases} 1, & \text{for } x < 0, \\ 1 - \frac{x^2}{2}, & \text{for } 0 \le x < 1, \\ x, & \text{for } x \ge 1. \end{cases}$$

Bijectivity*

Recall that each argument has exactly one image and that the number of preimages of an element in the codomain can vary.
Thus we can characterize maps by their possible number of preimages.

- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Also recall that a function has an *inverse* if and only if it is one-to-one and onto (i.e., *bijective*).

A Simple Horizontal Test*

How can we determine whether a real function is one-to-one or onto?
I.e., how many preimage may a $y \in W_f$ have?

(1) Draw the graph of the given function.

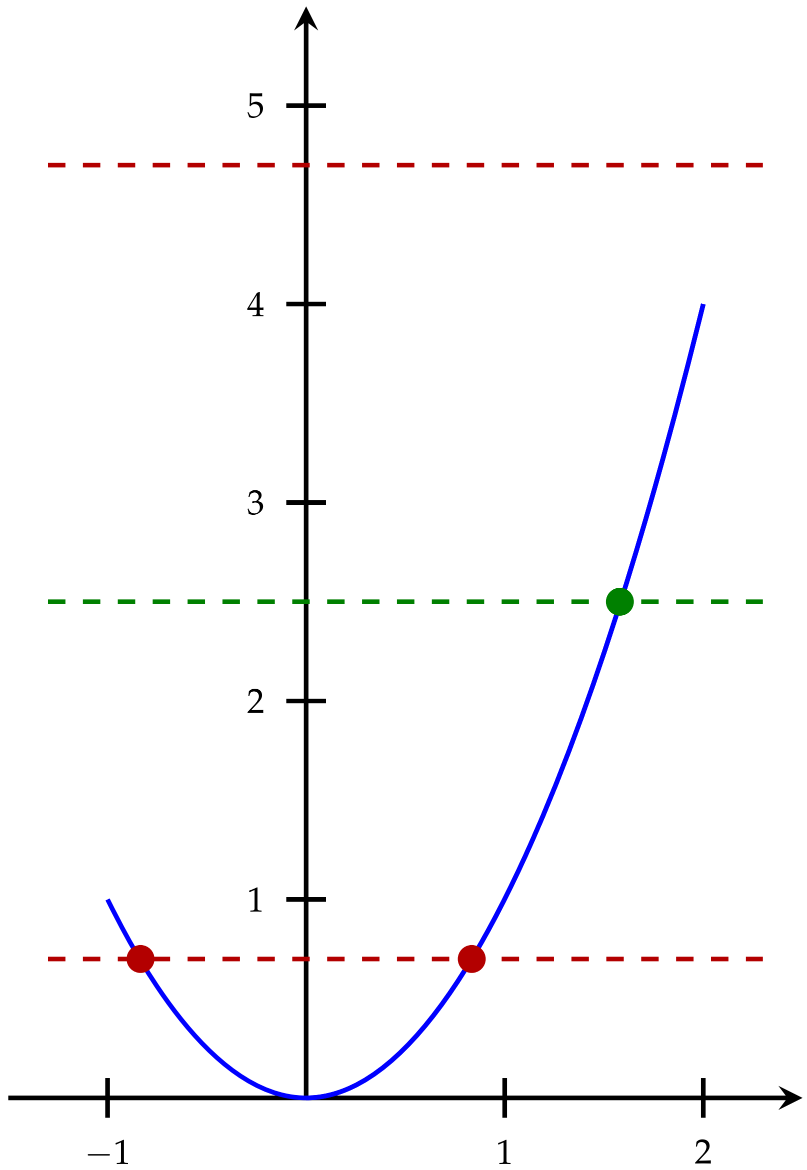
(2) Mark some $y \in W$ on the y-axis and draw a line parallel to the x-axis (*horizontal*) through this point.

(3) The number of intersection points of horizontal line and graph coincides with the number of preimages of y.

(4) Repeat Steps (2) and (3) for a *representative* set of y-values.

(5) Interpretation: If all horizontal lines intersect the graph in
- **(a)** *at most one* point, then f is **one-to-one**;
- **(b)** *at least one* point, then f is **onto**;
- **(c)** *exactly one* point, then f is **bijective**.

Example – Horizontal Test*



$f : [-1, 2] \to \mathbb{R}, x \mapsto x^2$

- is not one-to-one;
- is not onto.

$f : [0, 2] \to \mathbb{R}, x \mapsto x^2$

- is one-to-one;
- is not onto.

$f : [0, 2] \to [0, 4], x \mapsto x^2$

- is one-to-one and onto.

Beware! *Domain* and *codomain* are part of the function!

Function Composition*

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.

$$g \circ f\colon D_f \to W_g,\ x \mapsto (g \circ f)(x) = g(f(x))$$

is called **composite function.**
(read: "g composed with f", "g circle f", or "g after f")

Let $g\colon \mathbb{R} \to [0,\infty),\ x \mapsto g(x) = x^2,$
$f\colon \mathbb{R} \to \mathbb{R},\ x \mapsto f(x) = 3x - 2.$

Then $(g \circ f)\colon \mathbb{R} \to [0,\infty),$
$x \mapsto (g \circ f)(x) = g(f(x)) = g(3x-2) = (3x-2)^2$

and $(f \circ g)\colon \mathbb{R} \to \mathbb{R},$
$x \mapsto (f \circ g)(x) = f(g(x)) = f(x^2) = 3x^2 - 2$

Inverse Function*

If $f\colon D_f \to W_f$ is a **bijection**, then there exists a so called **inverse function**

$$f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)$$

with the property

$$f^{-1} \circ f = \mathrm{id} \quad \text{and} \quad f \circ f^{-1} = \mathrm{id}$$

We get the function term of the inverse by *interchanging* the roles of *argument* x and *image* y.

Example – Inverse Function*

We get the term for the inverse function by expressing x as function of y

We need the inverse function of

$$y = f(x) = 2x - 1$$

By rearranging we obtain

$$y = 2x - 1 \quad \Leftrightarrow \quad y + 1 = 2x \quad \Leftrightarrow \quad \frac{1}{2}(y+1) = x$$

Thus the term of the inverse function is $f^{-1}(y) = \frac{1}{2}(y+1)$.

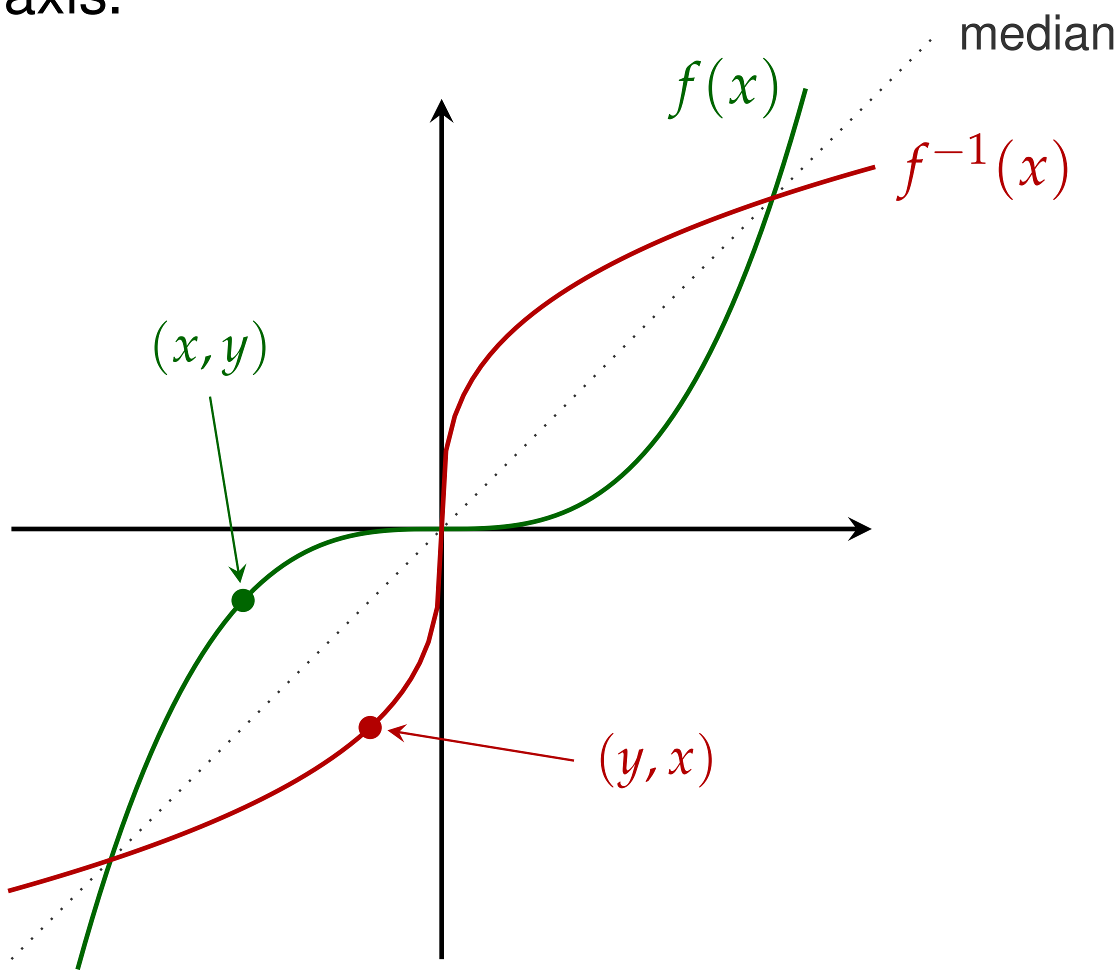
Arguments are usually denoted by x. So we write

$$f^{-1}(x) = \frac{1}{2}(x+1)\ .$$

The inverse function of $f(x) = x^3$ is $f^{-1}(x) = \sqrt[3]{x}$.

Geometric Interpretation*

Interchanging of x and y corresponds to reflection across the median between x and y-axis.



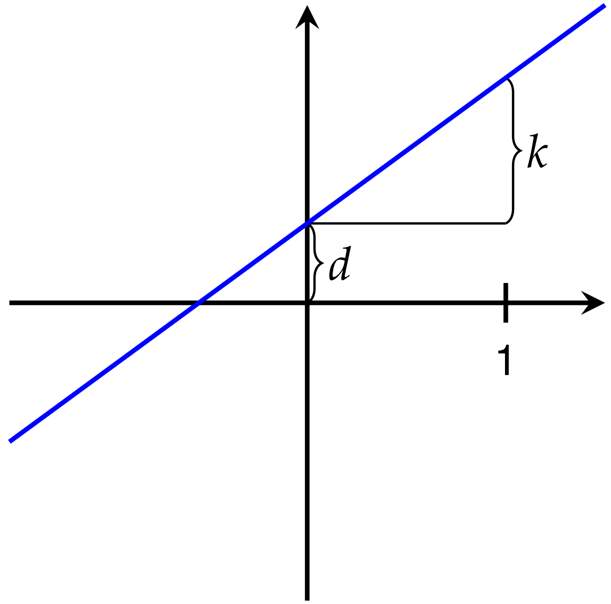
(Graph of function $f(x) = x^3$ and its inverse.)

Linear Function and Absolute Value*

- **Linear function**

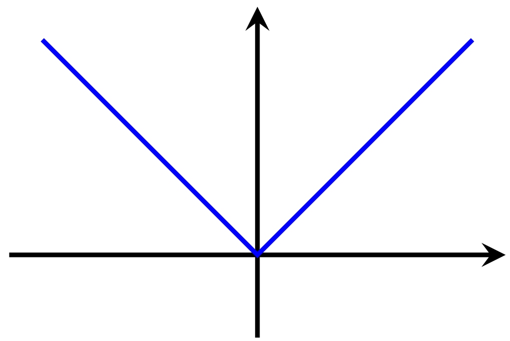
$$f(x) = k\,x + d$$

k … **slope**
d … **intercept**



- **Absolute value** (or *modulus*)

$$f(x) = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

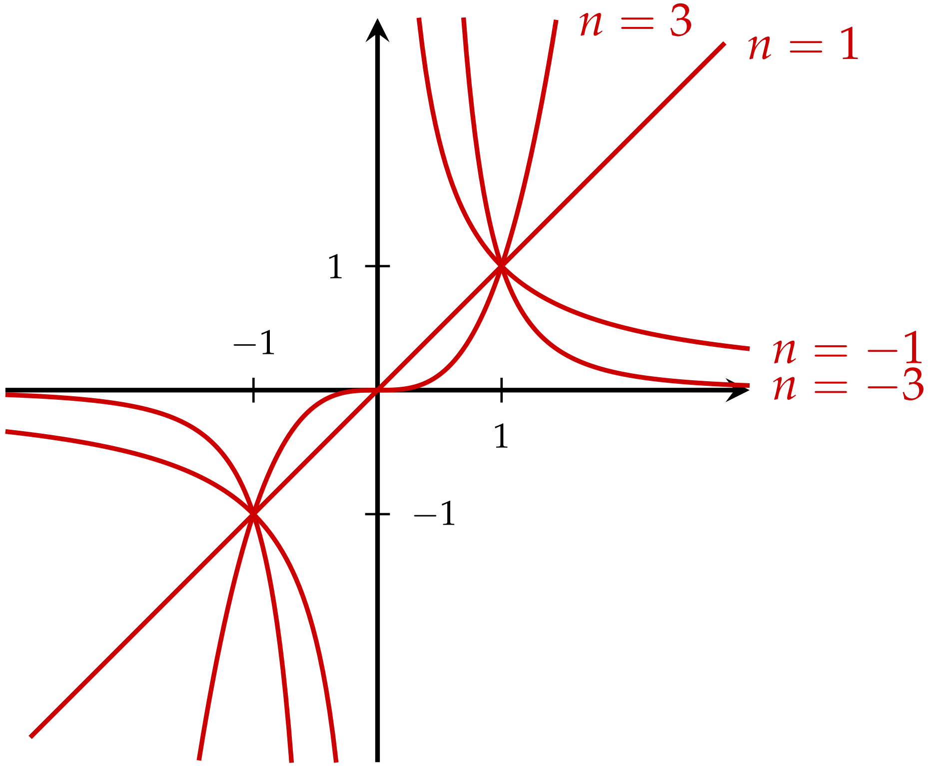


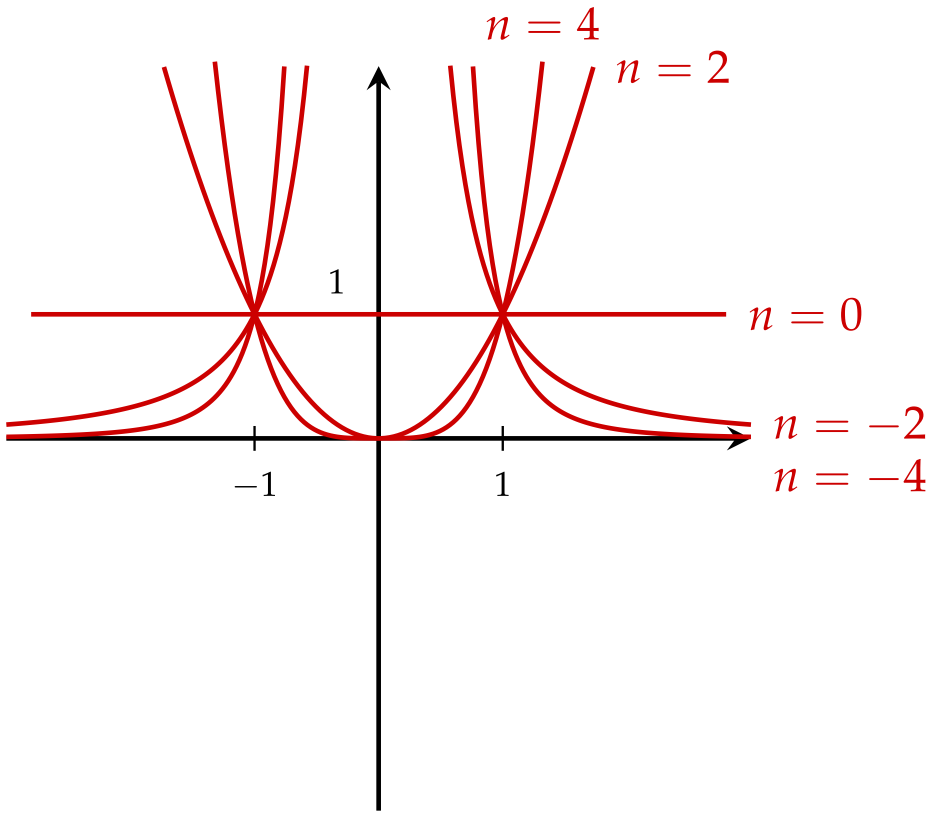
Power Function*

Power function with *integer* exponents:

$$f: x \mapsto x^n, \quad n \in \mathbb{Z}$$

$$D = \begin{cases} \mathbb{R} & \text{for } n \geq 0 \\ \mathbb{R} \setminus \{0\} & \text{for } n < 0 \end{cases}$$





Calculation Rule for Powers and Roots*

$$x^{-n} = \frac{1}{x^n} \qquad x^0 = 1 \qquad (x \neq 0)$$

$$x^{n+m} = x^n \cdot x^m \qquad x^{\frac{1}{m}} = \sqrt[m]{x} \qquad (x \geq 0)$$

$$x^{n-m} = \frac{x^n}{x^m} \qquad x^{\frac{n}{m}} = \sqrt[m]{x^n} \qquad (x \geq 0)$$

$$(x \cdot y)^n = x^n \cdot y^n \qquad x^{-\frac{n}{m}} = \frac{1}{\sqrt[m]{x^n}} \qquad (x \geq 0)$$

$$(x^n)^m = x^{n \cdot m}$$

Important!

0^0 is *not* defined!

Sources of Errors

Important!

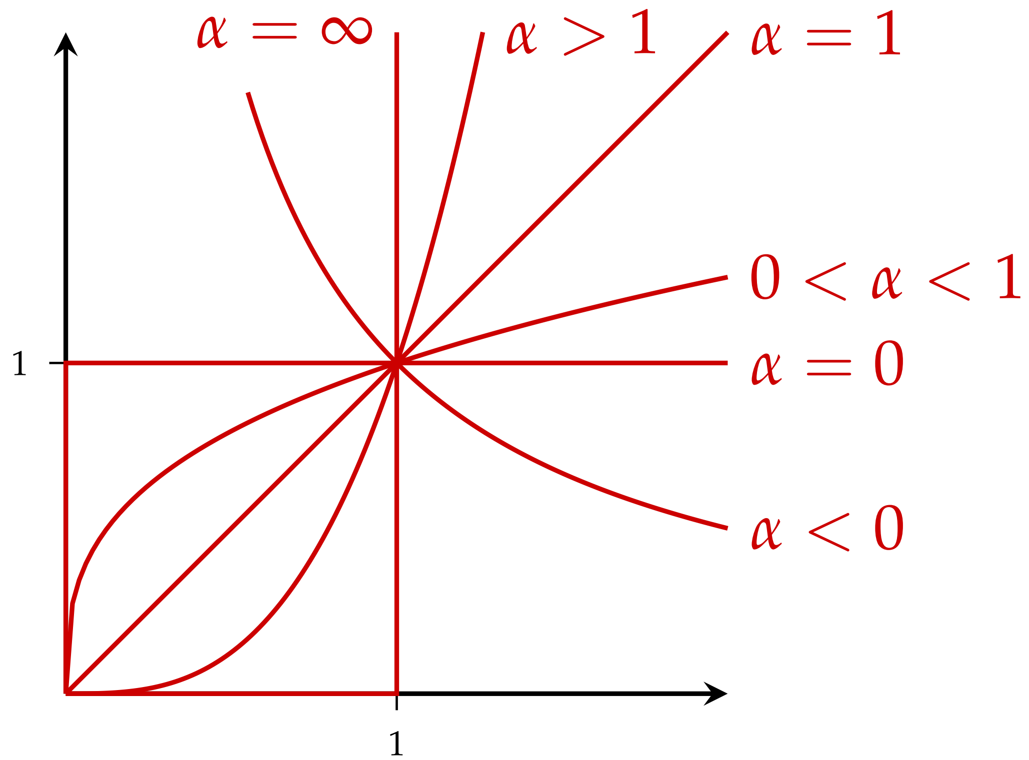
- $-x^2$ is **not** equal to $(-x)^2$!
- $(x+y)^n$ is **not** equal to $x^n + y^n$!
- $x^n + y^n$ can**not** be simplified (in general)!

Power Function*

Power function with *real* exponents:

$$f: x \mapsto x^\alpha \qquad \alpha \in \mathbb{R}$$

$$D = \begin{cases} [0,\infty) & \text{for } \alpha \geq 0 \\ (0,\infty) & \text{for } \alpha < 0 \end{cases}$$



Polynomial and Rational Functions*

- **Polynomial** of degree n:

$$f(x) = \sum_{k=0}^{n} a_k x^k$$

$a_i \in \mathbb{R}$, for $i = 1, \dots, n$, $\quad a_n \neq 0$.

- **Rational Function:**

$$D \to \mathbb{R}, \quad x \mapsto \frac{p(x)}{q(x)}$$

$p(x)$ and $q(x)$ are polynomials
$D = \mathbb{R} \setminus \{\text{roots of } q\}$

Calculation Rule for Fractions and Rational Terms*

Let $b, c, e \neq 0$.

$$\frac{c \cdot a}{c \cdot b} = \frac{a}{b}$$ *Reduce*

$$\frac{a}{b} = \frac{c \cdot a}{c \cdot b}$$ *Expand*

$$\frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$ *Multiplying*

$$\frac{a}{b} : \frac{e}{c} = \frac{a}{b} \cdot \frac{c}{e}$$ *Dividing*

$$\frac{\frac{a}{b}}{\frac{e}{c}} = \frac{a \cdot c}{b \cdot e}$$ *Compound fraction*

Calculation Rule for Fractions and Rational Terms*

Let $b, c \neq 0$.

$$\frac{a}{b} + \frac{d}{b} = \frac{a+d}{b}$$ *Addition with common denominator*

$$\frac{a}{b} + \frac{d}{c} = \frac{a \cdot c + d \cdot b}{b \cdot c}$$ *Addition*

Very important! *Really!*
You have to expand fractions such that they have a **common denominator** *before* you add them!

Sources of Errors

Very Important! *Really!*

$\dfrac{a+c}{b+c}$ is **not** equal to $\dfrac{a}{b}$

$\dfrac{x}{a}+\dfrac{y}{b}$ is **not** equal to $\dfrac{x+y}{a+b}$

$\dfrac{a}{b+c}$ is **not** equal to $\dfrac{a}{b}+\dfrac{a}{c}$

$$\frac{x+2}{y+2} \neq \frac{x}{y} \qquad \frac{1}{2}+\frac{1}{3} \neq \frac{1}{5}.$$

$$\frac{1}{x^2+y^2} \neq \frac{1}{x^2}+\frac{1}{y^2}$$

Exponential Function*

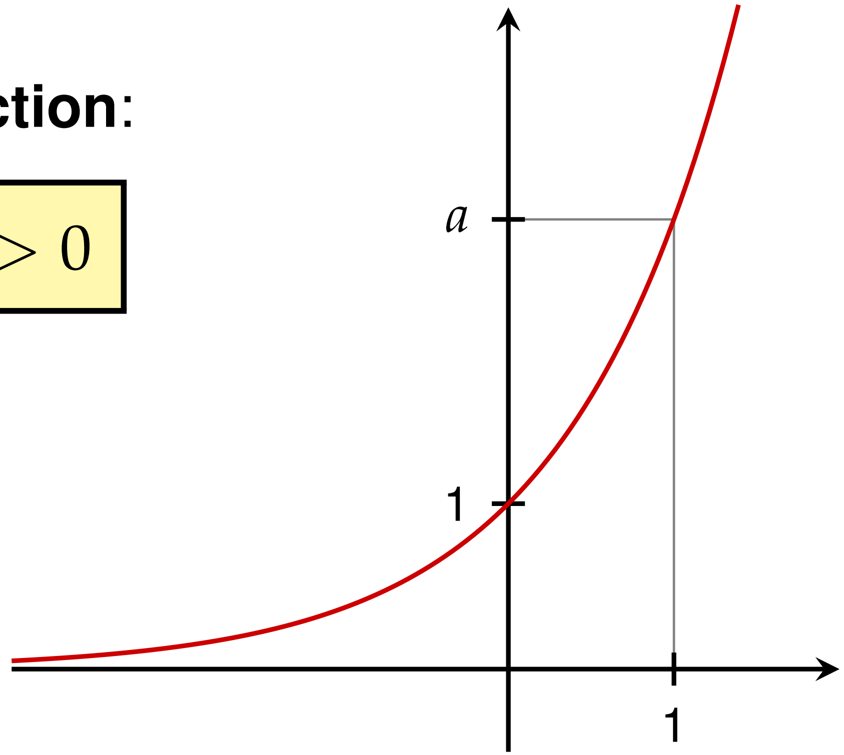
- **Exponential function**:

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto \exp(x) = e^x$$

$e = 2,7182818\ldots$ Euler's number

- **Generalized exponential function**:

$$\mathbb{R} \to \mathbb{R}^+, \quad x \mapsto a^x \qquad a > 0$$



Logarithm Function*

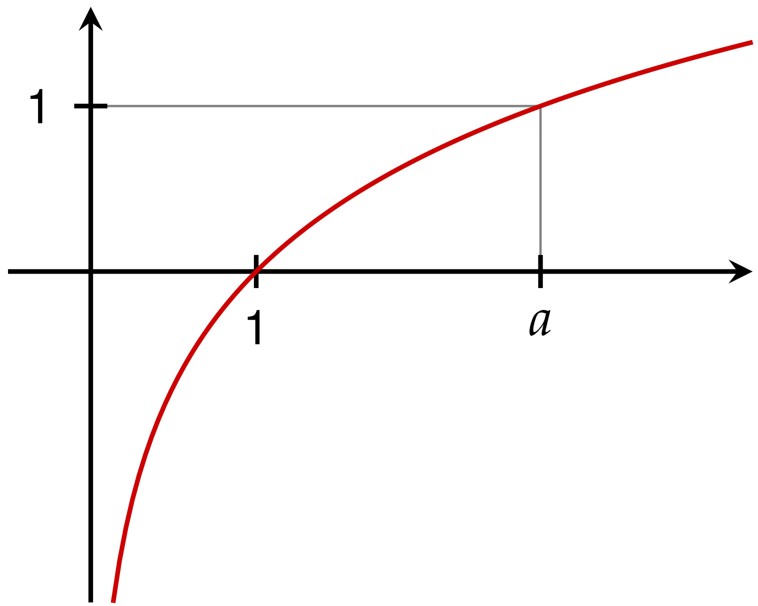
- **Logarithm**:

  Inverse function of *exponential function*.

$$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log(x) = \ln(x)$$

- **Generalized Logarithm** to basis a:

$$\mathbb{R}^+ \to \mathbb{R}, \quad x \mapsto \log_a(x)$$



Exponent and Logarithm*

A number y is called the **logarithm** to basis a, if $a^y = x$.
The logarithm is the *exponent of a number to basis* a.
We write

$$y = \log_a(x) \qquad \Leftrightarrow \qquad x = a^y$$

Important logarithms:

- **natural logarithm $\ln(x)$** with basis $e = 2.7182818\ldots$ (sometimes called *Euler's number*)
- **common logarithm $\lg(x)$** with basis 10 (sometimes called decadic or decimal logarithm)

Calculations with Exponent and Logarithm*

Conversation formula:

$$a^x = e^{x \ln(a)} \qquad \log_a(x) = \frac{\ln(x)}{\ln(a)}$$

Important:
Often one can see $\log(x)$ without a basis.

In this case the basis is (should be) implicitly given by the context of the book or article.

- In *mathematics*: *natural* logarithm
  financial mathematics, programs like R, *Mathematica*, Maxima, ...
- In *applied sciences*: *common* logarithm
  economics, pocket calculator, Excel, ...

Calculation Rules for Exponent and Logarithm*

$a^{x+y} = a^x \cdot a^y$	$\log_a(x \cdot y) = \log_a(x) + \log_a(y)$
$a^{x-y} = \frac{a^x}{a^y}$	$\log_a(\frac{x}{y}) = \log_a(x) - \log_a(y)$
$(a^x)^y = a^{x \cdot y}$	$\log_a(x^\beta) = \beta \cdot \log_a(x)$
$(a \cdot b)^x = a^x \cdot b^x$	
$a^{\log_a(x)} = x$	$\log_a(a^x) = x$
$a^0 = 1$	$\log_a(1) = 0$

$\log_a(x)$ has (as real-valued function) domain $x > 0$!

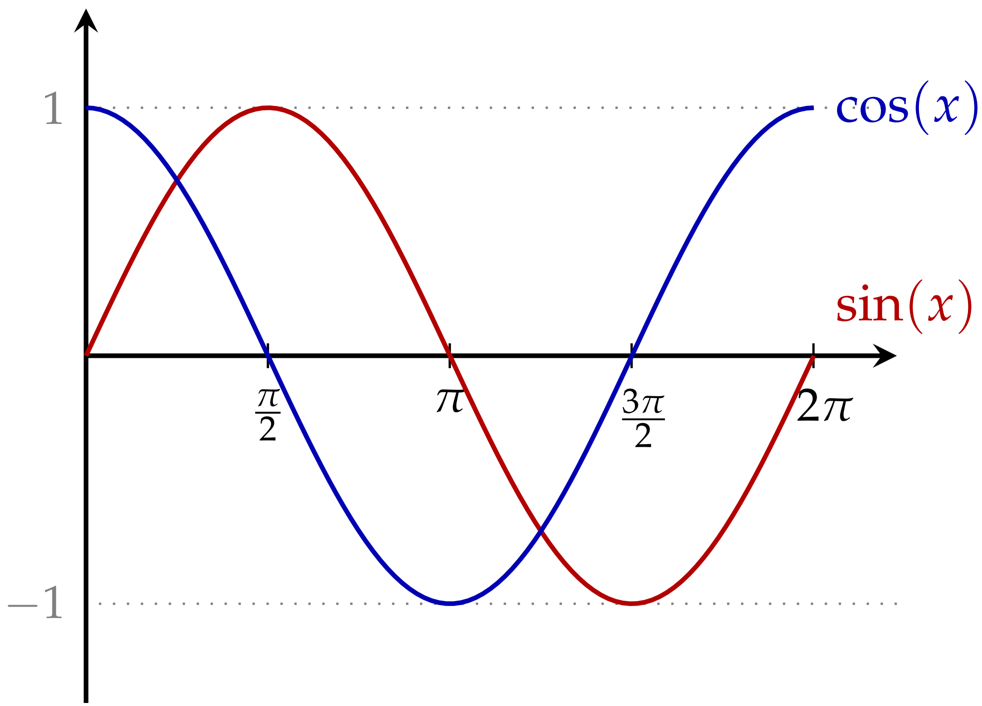
Trigonometric Functions*

- **Sine**:

$$\mathbb{R} \to [-1,1],\ x \mapsto \sin(x)$$

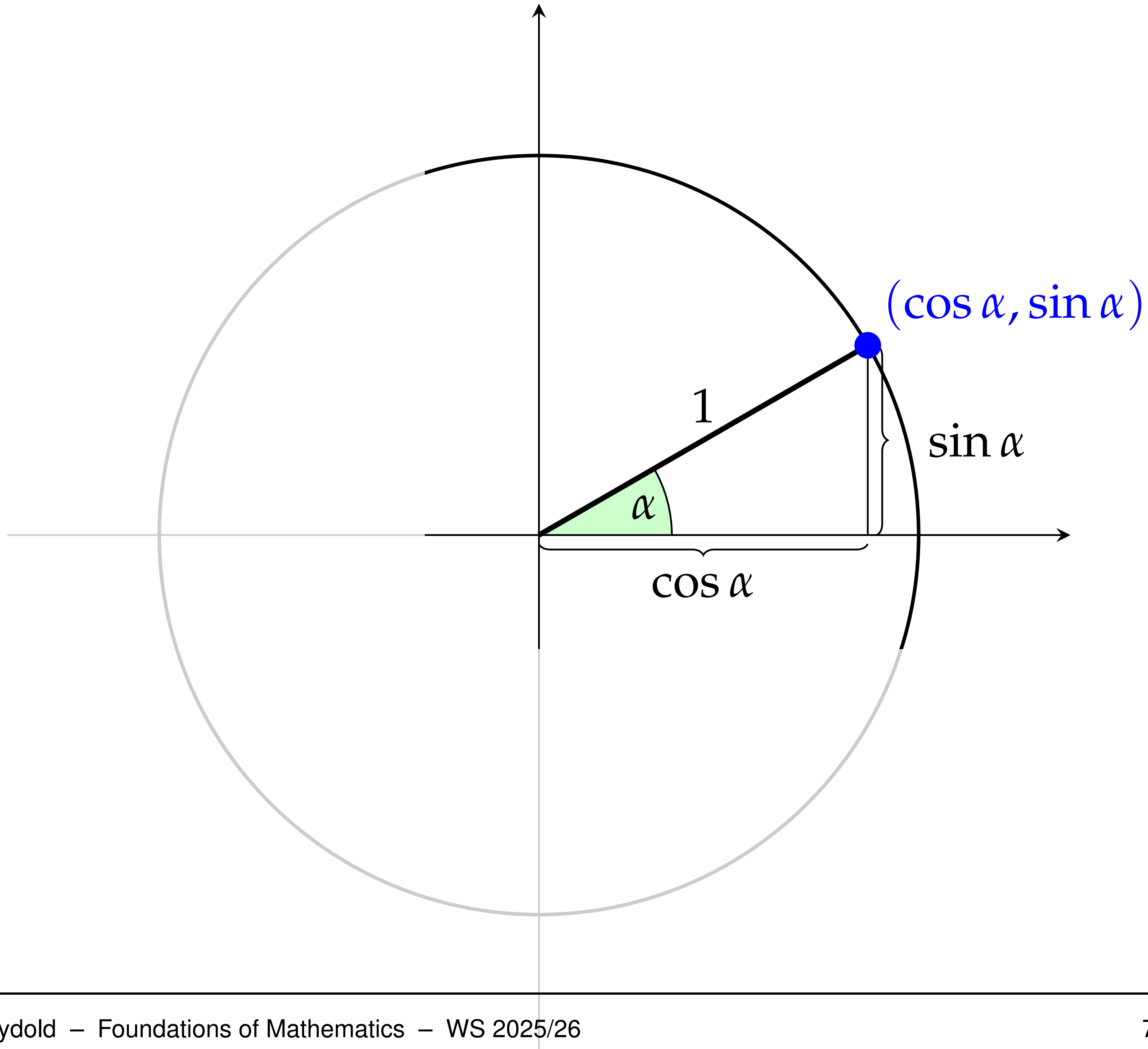
- **Cosine**:

$$\mathbb{R} \to [-1,1],\ x \mapsto \cos(x)$$



Beware!
These functions use **radian** for their arguments, i.e., angles are measured by means of the length of arcs on the unit circle and not by degrees. A right angle then corresponds to $x = \pi/2$.

Sine and Cosine*



Sine and Cosine*

Important formulas:

Periodic: For all $k \in \mathbb{Z}$,

$$\sin(x + 2k\pi) = \sin(x)$$
$$\cos(x + 2k\pi) = \cos(x)$$

Relation between $\sin$ and $\cos$:

$$\sin^2(x) + \cos^2(x) = 1$$

Multivariate Functions*

A **function of several variables** (or **multivariate function**)
is a function with more than one argument which evaluates to a real number.

$$f\colon \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$$

Arguments x_i are the **variables** of function f.

$$f(x, y) = \exp(-x^2 - 2y^2)$$

is a bivariate function in variables x and y.

$$p(x_1, x_2, x_3) = x_1^2 + x_1 x_2 - x_2^2 + 5x_1 x_3 - 2x_2 x_3$$

is a function in the three variables x_1, x_2, and x_3.

Graphs of Bivariate Functions*

Bivariate functions (i.e., of *two* variables) can be visualized by its graph:

$$\mathcal{G}_f = \{(x,y,z) \mid z = f(x,y) \text{ for } x,y \in \mathbb{R}\}$$

It can be seen as the two dimensional *surface* of a three dimensional landscape.

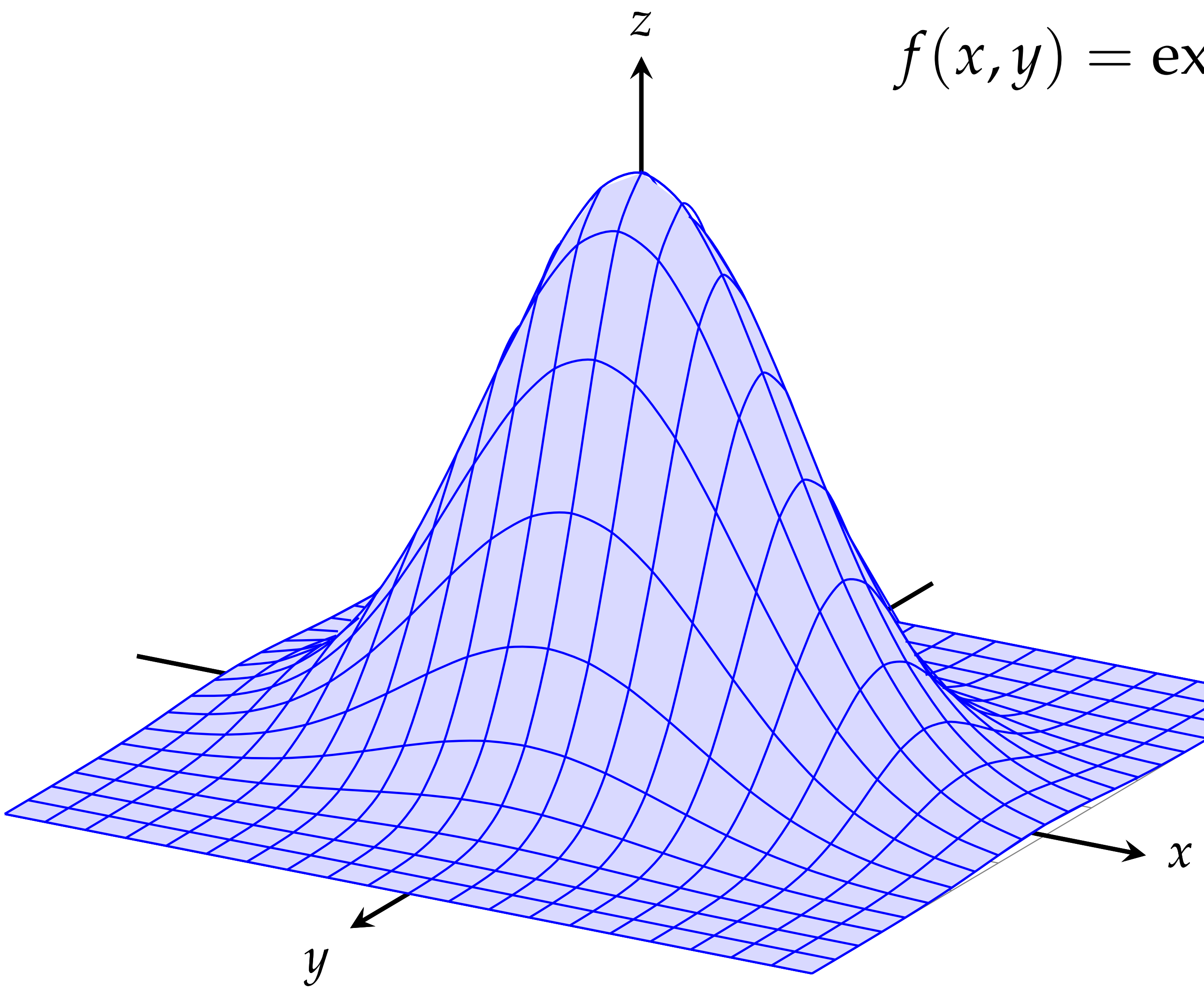
The notion of *graph* exists analogously for functions of three or more variables.

$$\mathcal{G}_f = \{(\mathbf{x},y) \mid y = f(\mathbf{x}) \text{ for an } \mathbf{x} \in \mathbb{R}^n\}$$

However, it can hardly be used to visualize such functions.

Graphs of Bivariate Functions

$$f(x,y) = \exp(-x^2 - 2y^2)$$



Contour Lines of Bivariate Functions*

Let $c \in \mathbb{R}$ be fixed. Then the set of all points (x, y) in the real plane with $f(x, y) = c$ is called **contour line** of function f.

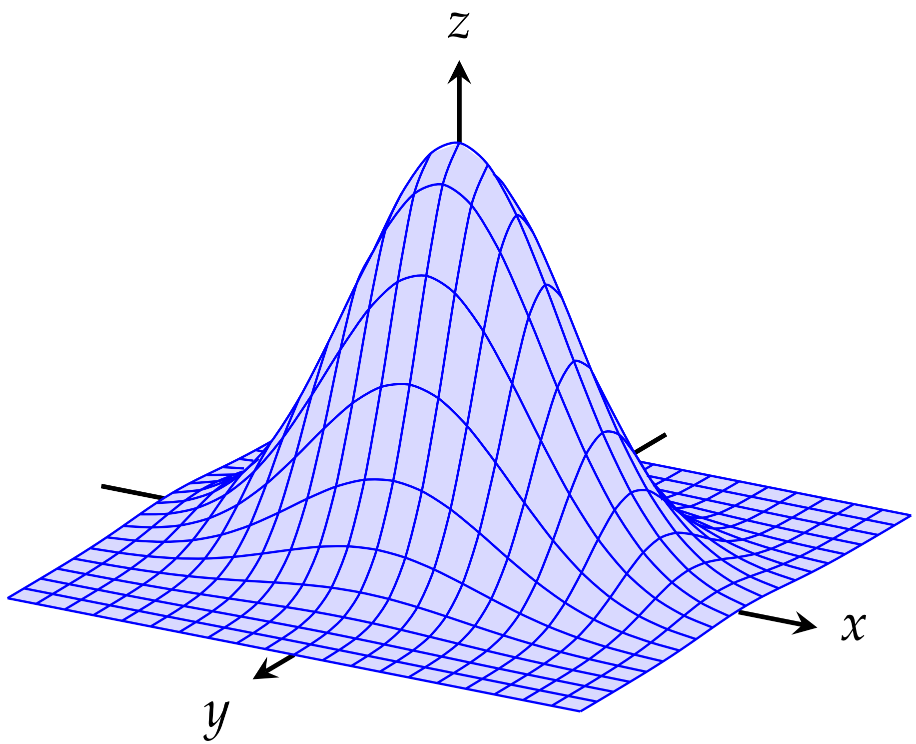
Obviously function f is constant on each of its contour lines.

Other names:

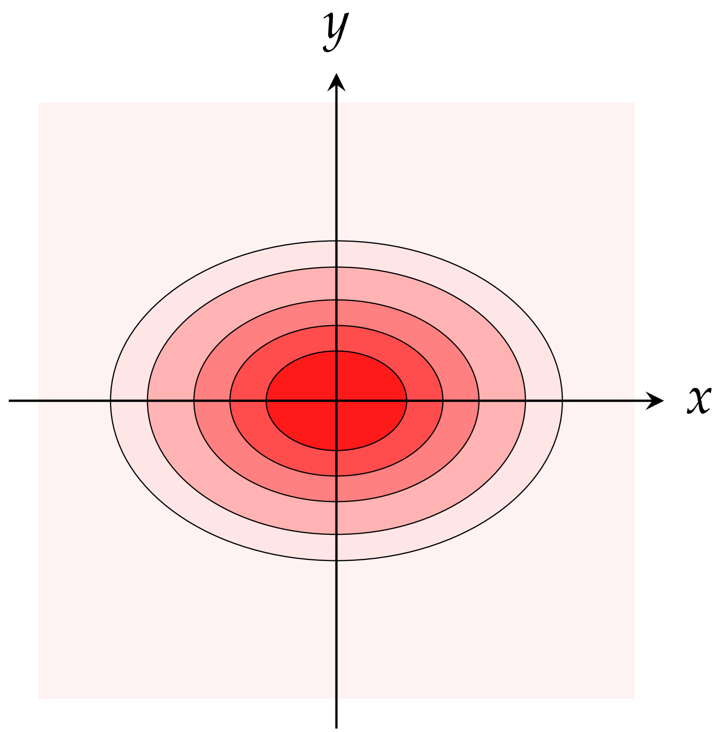
- *Indifference curve*
- *Isoquants*
- *Level set* (is a generalization of a contour line for functions of any number of variables.)

A collection of contour lines can be seen as a kind of "hiking map" for the "landscape" of the function.

Contour Lines of Bivariate Functions*



graph



contour lines

$$f(x,y) = \exp(-x^2 - 2y^2)$$

Indifference Curves*

Indifference curves are determined by an equation

$$F(x, y) = 0$$

We can (try to) draw such curves by expressing one of the variables as function of the other one
(i.e., solve the equation w.r.t. one of the two variables).

So we may get an univariate function. The graph of this function coincides with the indifference curve.

We then draw the graph of this univariate function by the method described above.

Example – Cobb-Douglas-Function*

We want to draw indifference curve

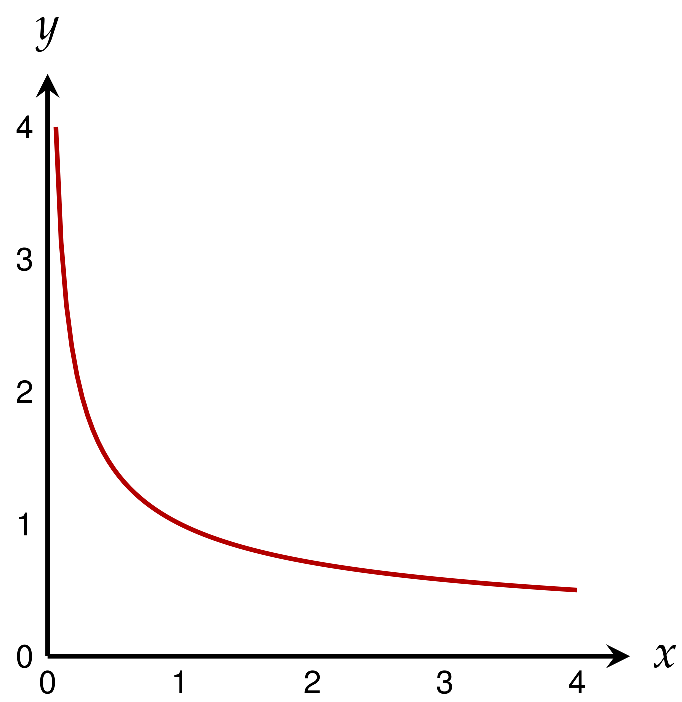
$$x^{\frac{1}{3}} y^{\frac{2}{3}} = 1\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$x = \frac{1}{y^2}$$

Alternatively we can express y by x:

$$y = \frac{1}{\sqrt{x}}$$



Example – CES-Function*

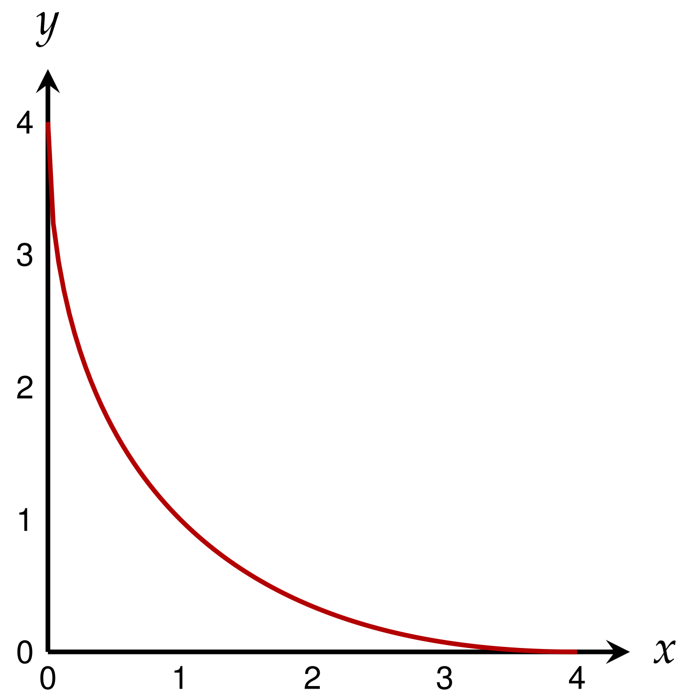
We want to draw indifference curve

$$\left(x^{\frac{1}{2}} + y^{\frac{1}{2}}\right)^2 = 4\,, \qquad x, y > 0.$$

Expressing x by y yields:

$$y = \left(2 - x^{\frac{1}{2}}\right)^2$$

(Take care about the domain of this curve!)



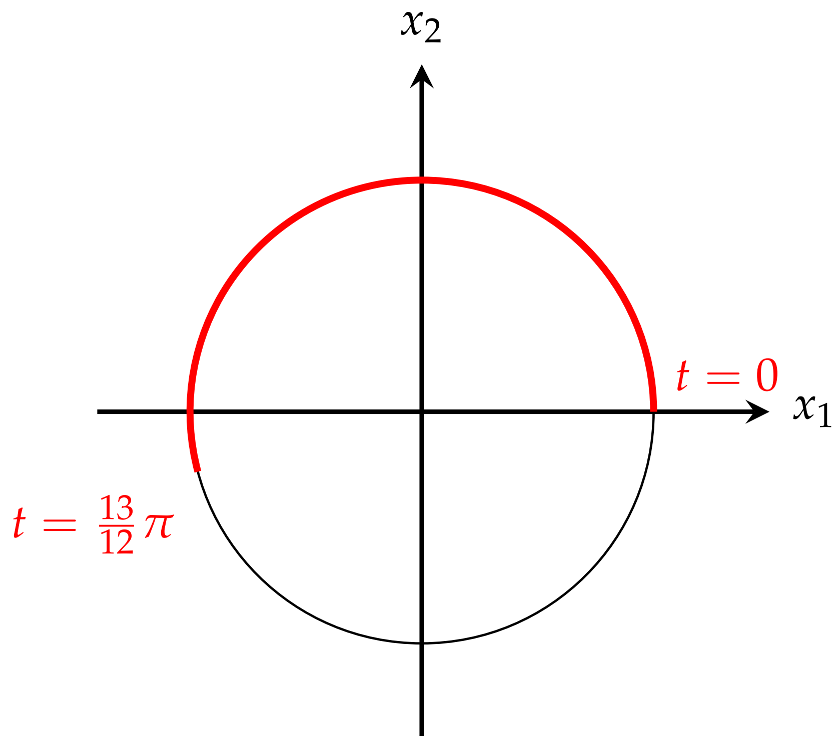
Paths*

A function

$$s\colon \mathbb{R} \to \mathbb{R}^n,\ t \mapsto s(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_n(t) \end{pmatrix}$$

is called a **path** in $\mathbb{R}^n$.
Variable t is often interpreted as *time*.

$$[0,\infty) \to \mathbb{R}^2,\ t \mapsto \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$



Vector-valued Functions*

Generalized vector-valued function:

$$\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ \vdots \\ f_m(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} f_1(x_1,\dots,x_n) \\ \vdots \\ f_m(x_1,\dots,x_n) \end{pmatrix}$$

- Univariate functions:
  $\mathbb{R} \to \mathbb{R},\ x \mapsto y = x^2$
- Multivariate functions:
  $\mathbb{R}^2 \to \mathbb{R},\ \mathbf{x} \mapsto y = x_1^2 + x_2^2$
- Paths:
  $[0,1) \to \mathbb{R}^n,\ s \mapsto (s, s^2)^\mathsf{T}$
- Linear maps:
  $\mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \mathbf{y} = \mathbf{A}\mathbf{x}$ $\qquad$ $\mathbf{A} \dots m \times n$-Matrix

Summary

- real functions
- implicit domain
- graph of a function
- sources of errors
- piece-wise defined functions
- one-to-one and onto
- function composition
- inverse function
- elementary functions
- multivariate functions
- paths
- vector-valued functions