Chapter 6

Eigenvalues

Closed Leontief Model

In a closed Leontief input-output-model consumption and production coincide, i.e.,

$$\mathbf{V} \cdot \mathbf{x} = \mathbf{x} = 1 \cdot \mathbf{x}$$

Is this possible for the given technology matrix $\mathbf{V}$?

This is a special case of a so called **eigenvalue problem**.

Eigenvalue and Eigenvector

A vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, is called **eigenvector** of an $n \times n$ matrix $\mathbf{A}$ corresponding to **eigenvalue** $\lambda \in \mathbb{R}$, if

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}}$$

The eigenvalues of matrix $\mathbf{A}$ are all numbers λ for which an eigenvector does exist.

Example – Eigenvalue and Eigenvector

For a 3×3 diagonal matrix we find

$$\mathbf{A} \cdot \mathbf{e}_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \\ 0 \end{pmatrix} = a_{11} \cdot \mathbf{e}_1$$

Thus $\mathbf{e}_1$ is a eigenvector corresponding to eigenvalue $\lambda = a_{11}$.

Analogously we find for an $n \times n$ diagonal matrix

$$\mathbf{A} \cdot \mathbf{e}_i = a_{ii} \cdot \mathbf{e}_i$$

So the eigenvalue of a diagonal matrix are its diagonal elements with unit vectors $\mathbf{e}_i$ as the corresponding eigenvectors.

Computation of Eigenvalues

In order to find eigenvectors of an $n \times n$ matrix $\mathbf{A}$ we have to solve equation

$$\mathbf{A}\,\mathbf{x} = \lambda \mathbf{x} = \lambda \mathbf{I} \mathbf{x} \quad \Leftrightarrow \quad (\mathbf{A} - \lambda \mathbf{I})\mathbf{x} = 0\,.$$

If $(\mathbf{A} - \lambda \mathbf{I})$ is invertible then we get

$$\mathbf{x} = (\mathbf{A} - \lambda \mathbf{I})^{-1} 0 = 0\,.$$

However, $\mathbf{x} = 0$ cannot be an eigenvector (by definition) and hence λ cannot be an eigenvalue.

Thus λ is an *eigenvalue* of $\mathbf{A}$ if and only if $(\mathbf{A} - \lambda \mathbf{I})$ is *not invertible*, i.e., if and only if

$$\boxed{\det(\mathbf{A} - \lambda \mathbf{I}) = 0}$$

Example – Eigenvalues

Compute the eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

We have to find all $\lambda \in \mathbb{R}$ where $|\mathbf{A} - \lambda \mathbf{I}|$ vanishes.

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| =$$
$$\left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0.$$

The roots of this quadratic equation are

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3.$$

Thus matrix $\mathbf{A}$ has eigenvalues 2 and 3.

Characteristic Polynomial

For an $n \times n$ matrix $\mathbf{A}$

$$\det(\mathbf{A} - \lambda \mathbf{I})$$

is a polynomial of degree n in λ.
It is called the **characteristic polynomial** of matrix $\mathbf{A}$.

The eigenvalues are then the roots of the characteristic polynomial.

For that reason eigenvalues and eigenvectors are sometimes called the *characteristic roots* and *characteristic vectors*, resp., of $\mathbf{A}$.

The set of all eigenvalues of $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$.
Spectral methods make use of eigenvalues.

Remark:
It may happen that characteristic roots are complex ($\lambda \in \mathbb{C}$).
These are then called *complex eigenvalues*.

Computation of Eigenvectors

Eigenvectors $\mathbf{x}$ corresponding to a *known* eigenvalue λ_0 can be computed by solving linear equation $(\mathbf{A} - \lambda_0 \mathbf{I})\mathbf{x} = 0$.

Eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ corresponding to $\lambda_1 = 2$:

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gaussian elimination yields: $x_2 = \alpha$ and $x_1 = -2\alpha$

$$\mathbf{v}_1 = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{for an } \alpha \in \mathbb{R} \setminus \{0\}.$$

Eigenspace

If $\mathbf{x}$ is an eigenvector corresponding to eigenvalue λ, then each multiple $\alpha\mathbf{x}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\alpha\mathbf{x}) = \alpha(\mathbf{A} \cdot \mathbf{x}) = \alpha\lambda \cdot \mathbf{x} = \lambda \cdot (\alpha\mathbf{x})$$

If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to the same eigenvalue λ, then $\mathbf{x} + \mathbf{y}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot \mathbf{y} = \lambda \cdot \mathbf{x} + \lambda \cdot \mathbf{y} = \lambda \cdot (\mathbf{x} + \mathbf{y})$$

The set of all eigenvectors corresponding to eigenvalue λ (including zero vector 0) is thus a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** corresponding to λ.

Computer programs return *bases of eigenspaces*.
(Beware: Bases are not uniquely determined!)

Example – Eigenspace

Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Eigenvector corresponding to eigenvalue $\lambda_2 = 3$: $\mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue λ_i are all non-vanishing (i.e., non-zero) multiples of $\mathbf{v}_i$.

Computer programs often return normalized eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Example

Eigenvalues and Eigenvectors of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}$.

Create the characteristic polynomial and compute its roots:

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 6 & 2-\lambda \end{vmatrix} = (2-\lambda) \cdot \lambda \cdot (\lambda - 5) = 0$$

Eigenvalues:

$$\lambda_1 = 2,\ \lambda_2 = 0,\ \text{and}\ \lambda_3 = 5\ .$$

Example

Eigenvector(s) corresponding to eigenvalue $\lambda_3 = 5$:

$$(\mathbf{A} - \lambda_3\mathbf{I})\mathbf{x} = \begin{pmatrix} (2-5) & 0 & 1 \\ 0 & (3-5) & 1 \\ 0 & 6 & (2-5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Gaussian elimination yields

$$\left(\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right) \rightsquigarrow \left(\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $x_3 = \alpha$, $x_2 = \frac{1}{2}\alpha$, and $x_1 = \frac{1}{3}\alpha$ for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$.
Eigenvector $\mathbf{v}_3 = (2, 3, 6)^\mathsf{T}$.

Example

Eigenvector corresponding to

- $\lambda_1 = 2$: $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\lambda_2 = 0$: $\mathbf{v}_2 = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix}$
- $\lambda_3 = 5$: $\mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue λ_i are all non-vanishing multiples of $\mathbf{v}_i$.

Properties of Eigenvalues

1. $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$ have the same eigenvalues.
2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$-matrices.
   Then $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ have the same eigenvalues.
3. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to λ,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue λ^k.
4. If $\mathbf{x}$ is an eigenvector of regular matrix $\mathbf{A}$ corresponding to λ,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue $\frac{1}{\lambda}$.

Properties of Eigenvalues

5. The product of all eigenvalues λ_i of an $n \times n$ matrix $\mathbf{A}$ is equal to the determinant of $\mathbf{A}$:

$$\boxed{\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i}$$

   This implies:
   $\mathbf{A}$ is regular if and only if all its eigenvalues are non-zero.

6. The sum of all eigenvalues λ_i of an $n \times n$ matrix $\mathbf{A}$ is equal to the sum of the diagonal elements of $\mathbf{A}$ (called the **trace** of $\mathbf{A}$).

$$\boxed{\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i}$$

Eigenvalues of Similar Matrices

Let $\mathbf{U}$ be a transformation matrix and $\mathbf{C} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}$.
If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ,
then $\mathbf{U}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{C}$ corresponding to λ:

$$\mathbf{C} \cdot (\mathbf{U}^{-1}\mathbf{x}) = (\mathbf{U}^{-1}\mathbf{A}\mathbf{U})\mathbf{U}^{-1}\mathbf{x} = \mathbf{U}^{-1}\mathbf{A}\mathbf{x} = \mathbf{U}^{-1}\lambda\mathbf{x} = \lambda \cdot (\mathbf{U}^{-1}\mathbf{x})$$

Similar matrices $\mathbf{A}$ and $\mathbf{C}$ have the same eigenvalues and (if we consider change of basis) the same eigenvectors.

We want to choose a basis such that the matrix that represents the given linear map becomes as simple as possible.
The simplest matrices are *diagonal matrices*.

Can we find a basis where the corresponding linear map is represented by a diagonal matrix?

Unfortunately not in the general case. But . . .

Symmetric Matrix

An $n \times n$ matrix $\mathbf{A}$ is called **symmetric**, if $\boxed{\mathbf{A}^\mathsf{T} = \mathbf{A}}$.

For a *symmetric* matrix $\mathbf{A}$ we find:

- All n eigenvalues are real.
- Eigenvectors $\mathbf{u}_i$ corresponding to distinct eigenvalues λ_i are *orthogonal* (i.e., $\mathbf{u}_i^\mathsf{T} \cdot \mathbf{u}_j = 0$ if $i \neq j$).
- There exists an **orthonormal basis** $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ (i.e. the vectors $\mathbf{u}_i$ are normalized and mutually orthogonal) that consists of eigenvectors of $\mathbf{A}$,

Matrix $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is then an **orthogonal matrix**:

$$\boxed{\mathbf{U}^\mathsf{T} \cdot \mathbf{U} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{U}^{-1} = \mathbf{U}^\mathsf{T}}$$

Diagonalization

For the i-th unit vector $\mathbf{e}_i$ we find

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} \cdot \mathbf{e}_i = \mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{u}_i = \mathbf{U}^\mathsf{T}\,\lambda_i\,\mathbf{u}_i = \lambda_i\,\mathbf{U}^\mathsf{T}\,\mathbf{u}_i = \lambda_i \cdot \mathbf{e}_i$$

and thus

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Every symmetric matrix $\mathbf{A}$ becomes a diagonal matrix with the eigenvalues of $\mathbf{A}$ as its entries if we use the orthonormal basis of eigenvectors.

This procedure is called **diagonalization** of matrix $\mathbf{A}$.

Example – Diagonalization

We want to diagonalize $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3$$

with respective normalized eigenvectors

$$\mathbf{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

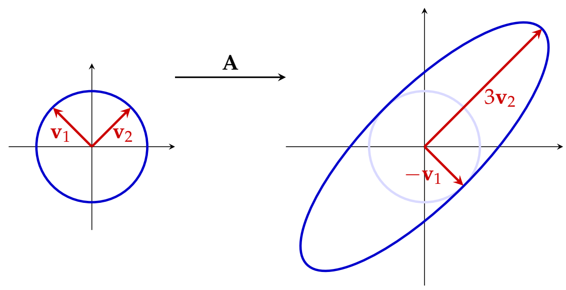
With respect to basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ matrix $\mathbf{A}$ becomes diagonal matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

A Geometric Interpretation I

Function $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ maps the unit circle in $\mathbb{R}^2$ into an ellipsis.

The two semi-axes of the ellipsis are given by $\lambda_1 \mathbf{v}_1$ and $\lambda_2 \mathbf{v}_2$, resp.



Quadratic Form

Let $\mathbf{A}$ be a *symmetric matrix*. Then function

$$\boxed{q_\mathbf{A}\colon \mathbb{R}^n \to \mathbb{R},\ \mathbf{x} \mapsto q_\mathbf{A}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{x}}$$

is called a **quadratic form**.

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then

$$q_\mathbf{A}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 2\,x_2^2 + 3\,x_3^2$$

Example – Quadratic Form

In general we find for $n \times n$ matrix $\mathbf{A} = (a_{ij})$:

$$q_\mathbf{A}(\mathbf{x}) = \sum_{i=1}^n \sum_{j=1}^n a_{ij}\, x_i x_j$$

$$\begin{aligned} q_\mathbf{A}(\mathbf{x}) &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} x_1 + x_2 - 2x_3 \\ x_1 + 2x_2 + 3x_3 \\ -2x_1 + 3x_2 + x_3 \end{pmatrix} \\ &= x_1^2 + 2x_1x_2 - 4x_1x_3 + 2x_2^2 + 6x_2x_3 + x_3^2 \end{aligned}$$

Definiteness

A quadratic form $q_\mathbf{A}$ is called

- **positive definite**, if for all $\mathbf{x} \neq 0$, $\quad q_\mathbf{A}(\mathbf{x}) > 0$.
- **positive semidefinite**, if for all $\mathbf{x}$, $\quad q_\mathbf{A}(\mathbf{x}) \geq 0$.
- **negative definite**, if for all $\mathbf{x} \neq 0$, $\quad q_\mathbf{A}(\mathbf{x}) < 0$.
- **negative semidefinite**, if for all $\mathbf{x}$, $\quad q_\mathbf{A}(\mathbf{x}) \leq 0$.
- **indefinite** in all other cases.

A matrix $\mathbf{A}$ is called *positive* (negative) *definite* (semidefinite), if the corresponding quadratic form is *positive* (negative) *definite* (semidefinite).

Definiteness

Every symmetric matrix is *diagonalizable*. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be the orthonormal basis of eigenvectors of $\mathbf{A}$. Then for every $\mathbf{x}$:

$$\mathbf{x} = \sum_{i=1}^n c_i(\mathbf{x})\mathbf{u}_i = \mathbf{U}\mathbf{c}(\mathbf{x})$$

$\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is the transformation matrix for the orthonormal basis, $\mathbf{c}$ the corresponding coefficient vector.

So if $\mathbf{D}$ is the diagonal matrix of eigenvalues λ_i of $\mathbf{A}$ we find

$$q_\mathbf{A}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{x} = (\mathbf{U}\mathbf{c})^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{U}\mathbf{c} = \mathbf{c}^\mathsf{T} \cdot \mathbf{U}^\mathsf{T}\mathbf{A}\mathbf{U} \cdot \mathbf{c} = \mathbf{c}^\mathsf{T} \cdot \mathbf{D} \cdot \mathbf{c}$$

and thus

$$q_\mathbf{A}(\mathbf{x}) = \sum_{i=1}^n c_i^2(\mathbf{x})\lambda_i$$

Definiteness and Eigenvalues

Equation $q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} c_i^2(\mathbf{x})\lambda_i$ immediately implies:

Let λ_i be the eigenvalues of symmetric matrix $\mathbf{A}$.
Then $\mathbf{A}$ (the quadratic form $q_{\mathbf{A}}$) is

- *positive definite*, if all $\lambda_i > 0$.
- *positive semidefinite*, if all $\lambda_i \geq 0$.
- *negative definite*, if all $\lambda_i < 0$.
- *negative semidefinite*, if all $\lambda_i \leq 0$.
- *indefinite*, if there exist $\lambda_i > 0$ and $\lambda_j < 0$.

Example – Definiteness and Eigenvalues

- The eigenvalues of $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ are $\lambda_1 = 6$ and $\lambda_2 = 1$.
  Thus the matrix is positive definite.
- The eigenvalues of $\begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$ are
  $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 9$. The matrix is positive semidefinite.
- The eigenvalues of $\begin{pmatrix} 7 & -5 & 4 \\ -5 & 7 & 4 \\ 4 & 4 & -2 \end{pmatrix}$ are
  $\lambda_1 = -6$, $\lambda_2 = 6$ and $\lambda_3 = 12$. Thus the matrix is indefinite.

Leading Principle Minors

The definiteness of a matrix can also be determined by means of minors.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$A_k = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$$

is called the k-th **leading principle minor** of $\mathbf{A}$.

Leading Principle Minors and Definiteness

A symmetric Matrix $\mathbf{A}$ is

- *positive definite*, if and only if all $A_k > 0$.
- *negative definite*, if and only if $(-1)^k A_k > 0$ for all k.
- *indefinite*, if $|\mathbf{A}| \neq 0$ and none of the two cases holds.

$(-1)^k A_k > 0$ means that

- $A_1, A_3, A_5, \ldots < 0$, and
- $A_2, A_4, A_6, \ldots > 0$.

Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 2 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 \quad > 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are positive definite.

Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 1 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = -28 \quad < 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are indefinite.

Principle Minors

Unfortunately the condition for semidefiniteness is more tedious.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$A_{i_1,\ldots,i_k} = \begin{vmatrix} a_{i_1,i_1} & \dots & a_{i_1,i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k,i_1} & \dots & a_{i_k,i_k} \end{vmatrix} \qquad 1 \leq i_1 < \ldots < i_k \leq n.$$

is called a **principle minor** of order k of $\mathbf{A}$.

Principle Minors and Semidefiniteness

A symmetric matrix $\mathbf{A}$ is

- *positive semidefinite*, if and only if all $A_{i_1,\ldots,i_k} \geq 0$.
- *negative semidefinite*, if and only if $(-1)^k A_{i_1,\ldots,i_k} \geq 0$ for all k.
- *indefinite* in all other cases.

$(-1)^k A_{i_1,\ldots,i_k} \geq 0$ means that

- $A_{i_1,\ldots,i_k} \geq 0$, if k is even, and
- $A_{i_1,\ldots,i_k} \leq 0$, if k is odd.

Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$$

principle minors of order 1:

$A_1 = 5$ ≥ 0 $\quad A_2 = 2$ ≥ 0

$A_3 = 5$ ≥ 0

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} 5 & -1 \\ -1 & 2 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{1,3} = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{2,3} = \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 9 \quad \geq 0$$

principle minors of order 3:

$A_{1,2,3} = |\mathbf{A}| = 0$ ≥ 0

The matrix is positive semidefinite.

(But not positive definite!)

Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} -5 & 1 & -4 \\ 1 & -2 & -1 \\ -4 & -1 & -5 \end{pmatrix}$$

principle minors of order 1:

$A_1 = -5$ ≤ 0 $\quad A_2 = -2$ ≤ 0

$A_3 = -5$ ≤ 0

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{1,3} = \begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix} = 9 \quad \geq 0$$

$$A_{2,3} = \begin{vmatrix} -2 & -1 \\ -1 & -5 \end{vmatrix} = 9 \quad \geq 0$$

principle minors of order 3:

$A_{1,2,3} = |\mathbf{A}| = 0$ ≤ 0

The matrix is negative semidefinite.

(But not negative definite!)

Leading Principle Minors and Semidefiniteness

Obviously every positive definite matrix is also positive semidefinite (but not necessarily the other way round).

Matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is positive definite as all leading principle minors are positive (see above).

Therefore $\mathbf{A}$ is also positive semidefinite.

In this case there is no need to compute the non-leading principle minors.

Recipe for Semidefiniteness

Recipe for finding semidefiniteness of matrix $\mathbf{A}$:

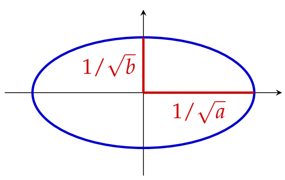
1. Compute all *leading principle minors*:
   - **If** the condition for positive definiteness holds, then $\mathbf{A}$ is *positive definite* and thus positive semidefinite.
   - Else **if** the condition for negative definiteness holds, then $\mathbf{A}$ is *negative definite* and thus negative semidefinite.
   - Else **if** $\det(\mathbf{A}) \neq 0$, then $\mathbf{A}$ is *indefinite*.
2. Else also compute all *non-leading principle minors*:
   - **If** the condition for positive semidefiniteness holds, then $\mathbf{A}$ is *positive semidefinite*.
   - Else **if** the condition for negative semidefiniteness holds, then $\mathbf{A}$ is *negative semidefinite*.
   - **Else** $\mathbf{A}$ is *indefinite*.

Ellipse

Equation

$$ax^2 + by^2 = 1\,, \qquad a, b > 0$$

describes an *ellipse* in **canonical form**.



The semi-axes have length $\frac{1}{\sqrt{a}}$ and $\frac{1}{\sqrt{b}}$, resp.

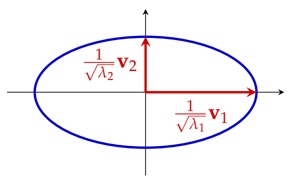
A Geometric Interpretation II

Term $ax^2 + by^2$ is a quadratic form with matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

It has eigenvalues and *normalized* eigenvectors

$$\lambda_1 = a \text{ with } \mathbf{v}_1 = \mathbf{e}_1 \quad \text{and} \quad \lambda_2 = b \text{ with } \mathbf{v}_2 = \mathbf{e}_2\,.$$



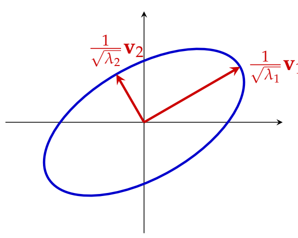
These eigenvectors coincide with the semi-axes of the ellipse.

A Geometric Interpretation II

Now let $\mathbf{A}$ be a symmetric 2×2 matrix with *positive* eigenvalues. Equation

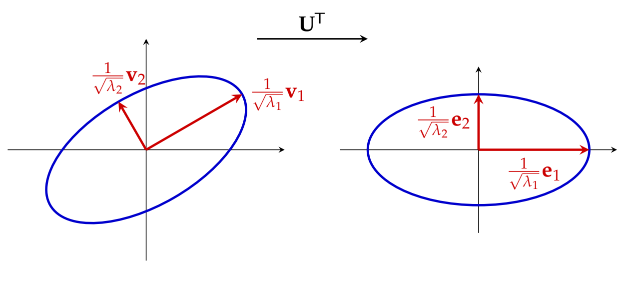
$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 1$$

describes an *ellipse* where the semi-axes (*principle axes*) coincide with the *normalized* eigenvectors of $\mathbf{A}$.



A Geometric Interpretation II

By a change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ using transformation $\mathbf{U} = (\mathbf{v}_1, \mathbf{v}_2)$ this ellipse is rotated into canonical form.



(That is, we diagonalize matrix $\mathbf{A}$.)

An Application in Statistics

Suppose we have n observations of k metric attributes $X_1, \dots, X_k$ which we combine into a vector:

$$\mathbf{x}_i = (x_{i1}, \dots, x_{ik}) \in \mathbb{R}^k$$

The arithmetic mean then is given by

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i = (\bar{x}_1, \dots, \bar{x}_k)$$

The *total sum of squares* is a measure for the statistical dispersion

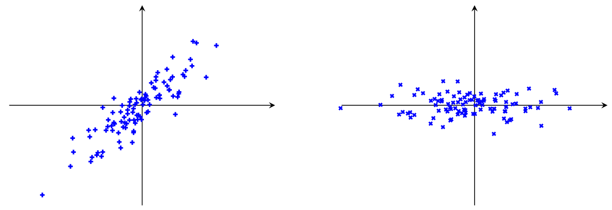
$$\mathrm{TSS} = \sum_{i=1}^{n} \|\mathbf{x}_i - \bar{\mathbf{x}}\|^2 = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} |x_{ij} - \bar{x}_j|^2 \right) = \sum_{j=1}^{k} \mathrm{TSS}_j$$

It can be computed component-wise.

An Application in Statistics

A change of basis by means of an *orthogonal* matrix does not change TSS.

However, it changes the contributions of each of the components.



Can we find a basis such that a few components contribute much more to the TSS than the remaining ones?

Principle Component Analysis (PCA)

Assumptions:

- The data are approximately *multinormal* distributed.

Procedure:

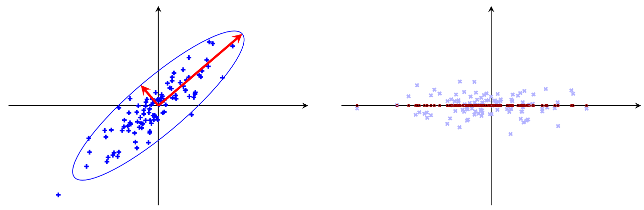
1. Compute the covariance matrix $\boldsymbol{\Sigma}$.
2. Compute all eigenvalues and normalized eigenvectors of $\boldsymbol{\Sigma}$.
3. Sort eigenvalues such that

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k \,.$$

4. Use corresponding eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ as new basis.
5. The contribution of the first m components in this basis to TSS is

$$\frac{\sum_{j=1}^{m} \mathrm{TSS}j}{\sum_{j=1}^{k} \mathrm{TSS}j} \approx \frac{\sum_{j=1}^{m} \lambda_j}{\sum_{j=1}^{k} \lambda_j} \,.$$

Principle Component Analysis (PCA)



By means of PCA it is possible to reduce the number of dimensions without reducing the TSS substantially.

Summary

- eigenvalues and eigenvectors
- characteristic polynomial
- eigenspace
- properties of eigenvalues
- symmetric matrices and diagonalization
- quadratic forms
- definitness
- principle minors
- principle component analysis