Chapter 6

Eigenvalues

Closed Leontief Model

In a closed Leontief input-output-model consumption and production coincide, i.e.,

$$\mathbf{V} \cdot \mathbf{x} = \mathbf{x} = 1 \cdot \mathbf{x}$$

Is this possible for the given technology matrix $\mathbf{V}$?

This is a special case of a so called **eigenvalue problem**.

Eigenvalue and Eigenvector

A vector $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq 0$, is called **eigenvector** of an $n \times n$ matrix $\mathbf{A}$ corresponding to **eigenvalue** $\lambda \in \mathbb{R}$, if

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \lambda \cdot \mathbf{x}}$$

The eigenvalues of matrix $\mathbf{A}$ are all numbers λ for which an eigenvector does exist.

Example – Eigenvalue and Eigenvector

For a 3×3 diagonal matrix we find

$$\mathbf{A} \cdot \mathbf{e}_1 = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a_{11} \\ 0 \\ 0 \end{pmatrix} = a_{11} \cdot \mathbf{e}_1$$

Thus $\mathbf{e}_1$ is a eigenvector corresponding to eigenvalue $\lambda = a_{11}$.

Analogously we find for an $n \times n$ diagonal matrix

$$\mathbf{A} \cdot \mathbf{e}_i = a_{ii} \cdot \mathbf{e}_i$$

So the eigenvalue of a diagonal matrix are its diagonal elements with unit vectors $\mathbf{e}_i$ as the corresponding eigenvectors.

Computation of Eigenvalues

In order to find eigenvectors of an $n \times n$ matrix $\mathbf{A}$ we have to solve equation

$$\mathbf{A}\,\mathbf{x} = \lambda\mathbf{x} = \lambda\mathbf{I}\mathbf{x} \quad \Leftrightarrow \quad (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0\,.$$

If $(\mathbf{A} - \lambda\mathbf{I})$ is invertible then we get

$$\mathbf{x} = (\mathbf{A} - \lambda\mathbf{I})^{-1} 0 = 0\,.$$

However, $\mathbf{x} = 0$ cannot be an eigenvector (by definition) and hence λ cannot be an eigenvalue.

Thus λ is an *eigenvalue* of $\mathbf{A}$ if and only if $(\mathbf{A} - \lambda\mathbf{I})$ is *not invertible*, i.e., if and only if

$$\boxed{\det(\mathbf{A} - \lambda\mathbf{I}) = 0}$$

Example – Eigenvalues

Compute the eigenvalues of matrix $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

We have to find all $\lambda \in \mathbb{R}$ where $|\mathbf{A} - \lambda\mathbf{I}|$ vanishes.

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| =$$

$$\left| \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right| = \begin{vmatrix} 1-\lambda & -2 \\ 1 & 4-\lambda \end{vmatrix} = \lambda^2 - 5\lambda + 6 = 0.$$

The roots of this quadratic equation are

$$\lambda_1 = 2 \quad \text{and} \quad \lambda_2 = 3.$$

Thus matrix $\mathbf{A}$ has eigenvalues 2 and 3.

Characteristic Polynomial

For an $n \times n$ matrix $\mathbf{A}$

$$\det(\mathbf{A} - \lambda \mathbf{I})$$

is a polynomial of degree n in λ.
It is called the **characteristic polynomial** of matrix $\mathbf{A}$.

The eigenvalues are then the roots of the characteristic polynomial.

For that reason eigenvalues and eigenvectors are sometimes called the *characteristic roots* and *characteristic vectors*, resp., of $\mathbf{A}$.

The set of all eigenvalues of $\mathbf{A}$ is called the *spectrum* of $\mathbf{A}$.
Spectral methods make use of eigenvalues.

Remark:
It may happen that characteristic roots are complex ($\lambda \in \mathbb{C}$).
These are then called *complex eigenvalues*.

Computation of Eigenvectors

Eigenvectors $\mathbf{x}$ corresponding to a *known* eigenvalue λ_0 can be computed by solving linear equation $(\mathbf{A} - \lambda_0 \mathbf{I})\mathbf{x} = 0$.

Eigenvectors of $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ corresponding to $\lambda_1 = 2$:

$$(\mathbf{A} - \lambda_1 \mathbf{I})\mathbf{x} = \begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Gaussian elimination yields: $x_2 = \alpha$ and $x_1 = -2\alpha$

$$\mathbf{v}_1 = \alpha \begin{pmatrix} -2 \\ 1 \end{pmatrix} \qquad \text{for an } \alpha \in \mathbb{R} \setminus \{0\}.$$

Eigenspace

If $\mathbf{x}$ is an eigenvector corresponding to eigenvalue λ, then each multiple $\alpha\mathbf{x}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\alpha \mathbf{x}) = \alpha(\mathbf{A} \cdot \mathbf{x}) = \alpha\lambda \cdot \mathbf{x} = \lambda \cdot (\alpha \mathbf{x})$$

If $\mathbf{x}$ and $\mathbf{y}$ are eigenvectors corresponding to the same eigenvalue λ, then $\mathbf{x} + \mathbf{y}$ is an eigenvector, too:

$$\mathbf{A} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot \mathbf{y} = \lambda \cdot \mathbf{x} + \lambda \cdot \mathbf{y} = \lambda \cdot (\mathbf{x} + \mathbf{y})$$

The set of all eigenvectors corresponding to eigenvalue λ (including zero vector 0) is thus a *subspace* of $\mathbb{R}^n$ and is called the **eigenspace** corresponding to λ.

Computer programs return *bases of eigenspaces*.
(Beware: Bases are not uniquely determined!)

Example – Eigenspace

Let $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$.

Eigenvector corresponding to eigenvalue $\lambda_1 = 2$: $\quad \mathbf{v}_1 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$

Eigenvector corresponding to eigenvalue $\lambda_2 = 3$: $\quad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue λ_i are all non-vanishing (i.e., non-zero) multiples of $\mathbf{v}_i$.

Computer programs often return normalized eigenvectors:

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

Example

Eigenvalues and Eigenvectors of $\mathbf{A} = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 3 & 1 \\ 0 & 6 & 2 \end{pmatrix}$.

Create the characteristic polynomial and compute its roots:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 3-\lambda & 1 \\ 0 & 6 & 2-\lambda \end{vmatrix} = (2-\lambda) \cdot \lambda \cdot (\lambda - 5) = 0$$

Eigenvalues:

$$\lambda_1 = 2,\ \lambda_2 = 0,\ \text{and } \lambda_3 = 5\ .$$

Example

Eigenvector(s) corresponding to eigenvalue $\lambda_3 = 5$:

$$(\mathbf{A} - \lambda_3 \mathbf{I})\mathbf{x} = \begin{pmatrix} (2-5) & 0 & 1 \\ 0 & (3-5) & 1 \\ 0 & 6 & (2-5) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Gaussian elimination yields

$$\left(\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 6 & -3 & 0 \end{array}\right) \quad \rightsquigarrow \quad \left(\begin{array}{rrr|r} -3 & 0 & 1 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Thus $x_3 = \alpha$, $x_2 = \frac{1}{2}\alpha$, and $x_1 = \frac{1}{3}\alpha$ for arbitrary $\alpha \in \mathbb{R} \setminus \{0\}$.

Eigenvector $\mathbf{v}_3 = (2, 3, 6)^\mathsf{T}$.

Example

Eigenvector corresponding to

- $\lambda_1 = 2$: $\quad \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$
- $\lambda_2 = 0$: $\quad \mathbf{v}_2 = \begin{pmatrix} -3 \\ -2 \\ 6 \end{pmatrix}$
- $\lambda_3 = 5$: $\quad \mathbf{v}_3 = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$

Eigenvectors corresponding to eigenvalue λ_i are all non-vanishing multiples of $\mathbf{v}_i$.

Properties of Eigenvalues

1. $\mathbf{A}$ and $\mathbf{A}^\mathsf{T}$ have the same eigenvalues.
2. Let $\mathbf{A}$ and $\mathbf{B}$ be $n \times n$-matrices.
   Then $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ have the same eigenvalues.
3. If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to λ,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^k$ corresponding to eigenvalue λ^k.
4. If $\mathbf{x}$ is an eigenvector of regular matrix $\mathbf{A}$ corresponding to λ,
   then $\mathbf{x}$ is an eigenvector of $\mathbf{A}^{-1}$ corresponding to eigenvalue $\frac{1}{\lambda}$.

Properties of Eigenvalues

5. The product of all eigenvalues λ_i of an $n \times n$ matrix $\mathbf{A}$ is equal to the determinant of $\mathbf{A}$:

   $$\boxed{\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i}$$

   This implies:
   $\mathbf{A}$ is regular if and only if all its eigenvalues are non-zero.
6. The sum of all eigenvalues λ_i of an $n \times n$ matrix $\mathbf{A}$ is equal to the sum of the diagonal elements of $\mathbf{A}$ (called the **trace** of $\mathbf{A}$).

   $$\boxed{\operatorname{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i}$$

Eigenvalues of Similar Matrices

Let $\mathbf{U}$ be a transformation matrix and $\mathbf{C} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}$.

If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to eigenvalue λ,
then $\mathbf{U}^{-1}\mathbf{x}$ is an eigenvector of $\mathbf{C}$ corresponding to λ:

$$\mathbf{C} \cdot (\mathbf{U}^{-1}\mathbf{x}) = (\mathbf{U}^{-1}\mathbf{A}\mathbf{U})\mathbf{U}^{-1}\mathbf{x} = \mathbf{U}^{-1}\mathbf{A}\mathbf{x} = \mathbf{U}^{-1}\lambda\mathbf{x} = \lambda \cdot (\mathbf{U}^{-1}\mathbf{x})$$

Similar matrices $\mathbf{A}$ and $\mathbf{C}$ have the same eigenvalues and (if we consider change of basis) the same eigenvectors.

We want to choose a basis such that the matrix that represents the given linear map becomes as simple as possible.
The simplest matrices are *diagonal matrices*.

Can we find a basis where the corresponding linear map is represented by a diagonal matrix?

Unfortunately not in the general case. But ...

Symmetric Matrix

An $n \times n$ matrix $\mathbf{A}$ is called **symmetric**, if $\boxed{\mathbf{A}^\mathsf{T} = \mathbf{A}}$.

For a *symmetric* matrix $\mathbf{A}$ we find:

- All n eigenvalues are real.
- Eigenvectors $\mathbf{u}_i$ corresponding to distinct eigenvalues λ_i are *orthogonal* (i.e., $\mathbf{u}_i^\mathsf{T} \cdot \mathbf{u}_j = 0$ if $i \neq j$).
- There exists an **orthonormal basis** $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ (i.e. the vectors $\mathbf{u}_i$ are normalized and mutually orthogonal) that consists of eigenvectors of $\mathbf{A}$,

Matrix $\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is then an **orthogonal matrix**:

$$\boxed{\mathbf{U}^\mathsf{T} \cdot \mathbf{U} = \mathbf{I} \quad \Leftrightarrow \quad \mathbf{U}^{-1} = \mathbf{U}^\mathsf{T}}$$

Diagonalization

For the i-th unit vector $\mathbf{e}_i$ we find

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} \cdot \mathbf{e}_i = \mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{u}_i = \mathbf{U}^\mathsf{T}\,\lambda_i\,\mathbf{u}_i = \lambda_i\,\mathbf{U}^\mathsf{T}\,\mathbf{u}_i = \lambda_i \cdot \mathbf{e}_i$$

and thus

$$\mathbf{U}^\mathsf{T}\,\mathbf{A}\,\mathbf{U} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Every symmetric matrix $\mathbf{A}$ becomes a diagonal matrix with the eigenvalues of $\mathbf{A}$ as its entries if we use the orthonormal basis of eigenvectors.

This procedure is called **diagonalization** of matrix $\mathbf{A}$.

Example – Diagonalization

We want to diagonalize $\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

Eigenvalues

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 3$$

with respective normalized eigenvectors

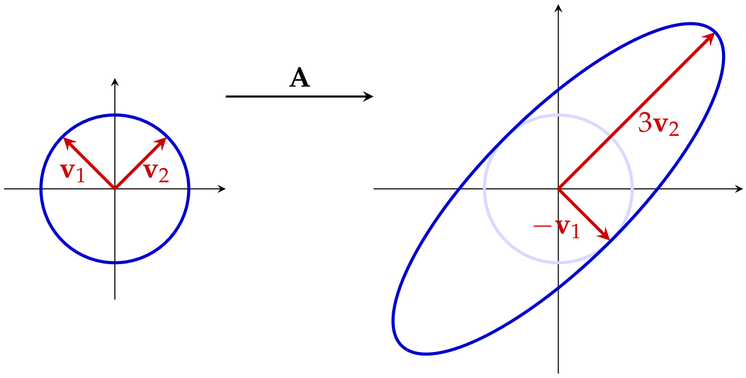
$$\mathbf{u}_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

With respect to basis $\{\mathbf{u}_1, \mathbf{u}_2\}$ matrix $\mathbf{A}$ becomes diagonal matrix

$$\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}$$

A Geometric Interpretation I

Function $\mathbf{x} \mapsto \mathbf{A}\mathbf{x} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \mathbf{x}$ maps the unit circle in $\mathbb{R}^2$ into an ellipsis.
The two semi-axes of the ellipsis are given by $\lambda_1 \mathbf{v}_1$ and $\lambda_2 \mathbf{v}_2$, resp.



Quadratic Form

Let $\mathbf{A}$ be a *symmetric matrix*. Then function

$$\boxed{q_{\mathbf{A}} \colon \mathbb{R}^n \to \mathbb{R}, \mathbf{x} \mapsto q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^\mathsf{T} \cdot \mathbf{A} \cdot \mathbf{x}}$$

is called a **quadratic form**.

Let $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Then

$$q_{\mathbf{A}}(\mathbf{x}) = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^\mathsf{T} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1^2 + 2\,x_2^2 + 3\,x_3^2$$

Example – Quadratic Form

In general we find for $n \times n$ matrix $\mathbf{A} = (a_{ij})$:

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}\, x_i x_j$$

$$\begin{aligned}
q_{\mathbf{A}}(\mathbf{x}) &= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{\mathsf{T}} \cdot \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}^{\mathsf{T}} \cdot \begin{pmatrix} x_1 + x_2 - 2x_3 \\ x_1 + 2x_2 + 3x_3 \\ -2x_1 + 3x_2 + x_3 \end{pmatrix} \\
&= x_1^2 + 2x_1x_2 - 4x_1x_3 + 2x_2^2 + 6x_2x_3 + x_3^2
\end{aligned}$$

Definiteness

A quadratic form $q_{\mathbf{A}}$ is called

- **positive definite**, if for all $\mathbf{x} \neq 0$, $q_{\mathbf{A}}(\mathbf{x}) > 0$.
- **positive semidefinite**, if for all $\mathbf{x}$, $q_{\mathbf{A}}(\mathbf{x}) \geq 0$.
- **negative definite**, if for all $\mathbf{x} \neq 0$, $q_{\mathbf{A}}(\mathbf{x}) < 0$.
- **negative semidefinite**, if for all $\mathbf{x}$, $q_{\mathbf{A}}(\mathbf{x}) \leq 0$.
- **indefinite** in all other cases.

A matrix $\mathbf{A}$ is called *positive* (negative) *definite* (semidefinite), if the corresponding quadratic form is *positive* (negative) *definite* (semidefinite).

Definiteness

Every symmetric matrix is *diagonalizable*. Let $\{\mathbf{u}_1, \dots, \mathbf{u}_n\}$ be the orthonormal basis of eigenvectors of $\mathbf{A}$. Then for every $\mathbf{x}$:

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x})\mathbf{u}_i = \mathbf{U}\mathbf{c}(\mathbf{x})$$

$\mathbf{U} = (\mathbf{u}_1, \dots, \mathbf{u}_n)$ is the transformation matrix for the orthonormal basis, $\mathbf{c}$ the corresponding coefficient vector.

So if $\mathbf{D}$ is the diagonal matrix of eigenvalues λ_i of $\mathbf{A}$ we find

$$q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^{\mathsf{T}} \cdot \mathbf{A} \cdot \mathbf{x} = (\mathbf{U}\mathbf{c})^{\mathsf{T}} \cdot \mathbf{A} \cdot \mathbf{U}\mathbf{c} = \mathbf{c}^{\mathsf{T}} \cdot \mathbf{U}^{\mathsf{T}}\mathbf{A}\mathbf{U} \cdot \mathbf{c} = \mathbf{c}^{\mathsf{T}} \cdot \mathbf{D} \cdot \mathbf{c}$$

and thus

$$q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} c_i^2(\mathbf{x})\lambda_i$$

Definiteness and Eigenvalues

Equation $q_{\mathbf{A}}(\mathbf{x}) = \sum_{i=1}^{n} c_i^2(\mathbf{x})\lambda_i$ immediately implies:

Let λ_i be the eigenvalues of symmetric matrix $\mathbf{A}$.
Then $\mathbf{A}$ (the quadratic form $q_{\mathbf{A}}$) is

- *positive definite*, if all $\lambda_i > 0$.
- *positive semidefinite*, if all $\lambda_i \geq 0$.
- *negative definite*, if all $\lambda_i < 0$.
- *negative semidefinite*, if all $\lambda_i \leq 0$.
- *indefinite*, if there exist $\lambda_i > 0$ and $\lambda_j < 0$.

Example – Definiteness and Eigenvalues

- The eigenvalues of $\begin{pmatrix} 2 & -2 \\ -2 & 5 \end{pmatrix}$ are $\lambda_1 = 6$ and $\lambda_2 = 1$.

  Thus the matrix is positive definite.

- The eigenvalues of $\begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$ are

  $\lambda_1 = 0$, $\lambda_2 = 3$, and $\lambda_3 = 9$. The matrix is positive semidefinite.

- The eigenvalues of $\begin{pmatrix} 7 & -5 & 4 \\ -5 & 7 & 4 \\ 4 & 4 & -2 \end{pmatrix}$ are

  $\lambda_1 = -6$, $\lambda_2 = 6$ and $\lambda_3 = 12$. Thus the matrix is indefinite.

Leading Principle Minors

The definiteness of a matrix can also be determined by means of minors.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$A_k = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$$

is called the k-th **leading principle minor** of $\mathbf{A}$.

Leading Principle Minors and Definiteness

A symmetric Matrix $\mathbf{A}$ is

- *positive definite*, if and only if all $A_k > 0$.
- *negative definite*, if and only if $(-1)^k A_k > 0$ for all k.
- *indefinite*, if $|\mathbf{A}| \neq 0$ and none of the two cases holds.

$(-1)^k A_k > 0$ means that

- $A_1, A_3, A_5, \ldots < 0$, and
- $A_2, A_4, A_6, \ldots > 0$.

Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 2 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 \quad > 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are positive definite.

Example – Leading Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{pmatrix}$$

$$A_1 = \det(a_{11}) = a_{11} = 1 \quad > 0$$

$$A_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 1 & 1 & -2 \\ 1 & 2 & 3 \\ -2 & 3 & 1 \end{vmatrix} = -28 \quad < 0$$

$\mathbf{A}$ and $q_{\mathbf{A}}$ are indefinite.

Principle Minors

Unfortunately the condition for semidefiniteness is more tedious.

Let $\mathbf{A} = (a_{ij})$ be a symmetric $n \times n$ matrix.
Then the determinant of submatrix

$$A_{i_1,\ldots,i_k} = \begin{vmatrix} a_{i_1,i_1} & \ldots & a_{i_1,i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k,i_1} & \ldots & a_{i_k,i_k} \end{vmatrix} \qquad 1 \le i_1 < \ldots < i_k \le n.$$

is called a **principle minor** of order k of $\mathbf{A}$.

Principle Minors and Semidefiniteness

A symmetric matrix $\mathbf{A}$ is

- *positive semidefinite*, if and only if all $A_{i_1,\ldots,i_k} \ge 0$.
- *negative semidefinite*, if and only if $(-1)^k A_{i_1,\ldots,i_k} \ge 0$ for all k.
- *indefinite* in all other cases.

$(-1)^k A_{i_1,\ldots,i_k} \ge 0$ means that

- $A_{i_1,\ldots,i_k} \ge 0$, if k is even, and
- $A_{i_1,\ldots,i_k} \le 0$, if k is odd.

Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} 5 & -1 & 4 \\ -1 & 2 & 1 \\ 4 & 1 & 5 \end{pmatrix}$$

The matrix is positive semidefinite.
(But not positive definite!)

principle minors of order 1:
$A_1 = 5 \quad \ge 0 \qquad A_2 = 2 \quad \ge 0$
$A_3 = 5 \quad \ge 0$

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} 5 & -1 \\ -1 & 2 \end{vmatrix} = 9 \quad \ge 0$$

$$A_{1,3} = \begin{vmatrix} 5 & 4 \\ 4 & 5 \end{vmatrix} = 9 \quad \ge 0$$

$$A_{2,3} = \begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} = 9 \quad \ge 0$$

principle minors of order 3:
$A_{1,2,3} = |\mathbf{A}| = 0 \quad \ge 0$

Example – Principle Minors

Definiteness of matrix

$$\mathbf{A} = \begin{pmatrix} -5 & 1 & -4 \\ 1 & -2 & -1 \\ -4 & -1 & -5 \end{pmatrix}$$

The matrix is
negative semidefinite.

(But not negative definite!)

principle minors of order 1:

$A_1 = -5$ $\quad \le 0$ $\qquad A_2 = -2$ $\quad \le 0$

$A_3 = -5$ $\quad \le 0$

principle minors of order 2:

$$A_{1,2} = \begin{vmatrix} -5 & 1 \\ 1 & -2 \end{vmatrix} = 9 \quad \ge 0$$

$$A_{1,3} = \begin{vmatrix} -5 & -4 \\ -4 & -5 \end{vmatrix} = 9 \quad \ge 0$$

$$A_{2,3} = \begin{vmatrix} -2 & -1 \\ -1 & -5 \end{vmatrix} = 9 \quad \ge 0$$

principle minors of order 3:

$A_{1,2,3} = |\mathbf{A}| = 0 \quad \le 0$

Leading Principle Minors and Semidefiniteness

Obviously every positive definite matrix is also positive semidefinite (but not necessarily the other way round).

Matrix

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

is positive definite as all leading principle minors are positive (see above).

Therefore $\mathbf{A}$ is also positive semidefinite.

In this case there is no need to compute the non-leading principle minors.

Recipe for Semidefiniteness

Recipe for finding semidefiniteness of matrix $\mathbf{A}$:

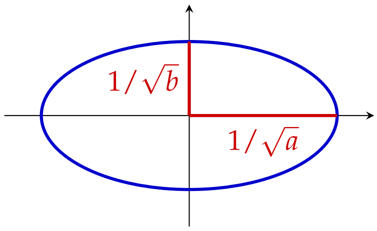
1. Compute all *leading principle minors*:
   - **If** the condition for positive definiteness holds, then
     $\mathbf{A}$ is *positive definite* and thus positive semidefinite.
   - Else **if** the condition for negative definiteness holds, then
     $\mathbf{A}$ is *negative definite* and thus negative semidefinite.
   - Else **if** $\det(\mathbf{A}) \neq 0$, then
     $\mathbf{A}$ is *indefinite*.
2. Else also compute all *non-leading principle minors*:
   - **If** the condition for positive semidefiniteness holds, then
     $\mathbf{A}$ is *positive semidefinite*.
   - Else **if** the condition for negative semidefiniteness holds, then
     $\mathbf{A}$ is *negative semidefinite*.
   - **Else**
     $\mathbf{A}$ is *indefinite*.

Ellipse

Equation

$$ax^2 + by^2 = 1\,, \qquad a, b > 0$$

describes an *ellipse* in **canonical form**.



The semi-axes have length $\frac{1}{\sqrt{a}}$ and $\frac{1}{\sqrt{b}}$, resp.

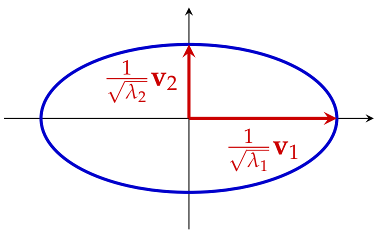
A Geometric Interpretation II

Term $ax^2 + by^2$ is a quadratic form with matrix

$$\mathbf{A} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$$

It has eigenvalues and *normalized* eigenvectors

$$\lambda_1 = a \text{ with } \mathbf{v}_1 = \mathbf{e}_1 \qquad \text{and} \qquad \lambda_2 = b \text{ with } \mathbf{v}_2 = \mathbf{e}_2\,.$$



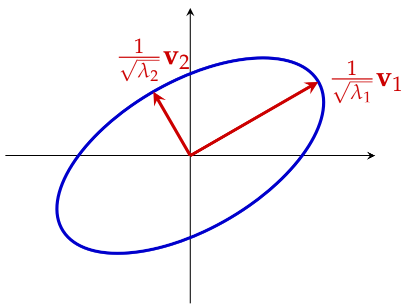
These eigenvectors coincide with the semi-axes of the ellipse.

A Geometric Interpretation II

Now let $\mathbf{A}$ be a symmetric 2×2 matrix with *positive* eigenvalues.
Equation

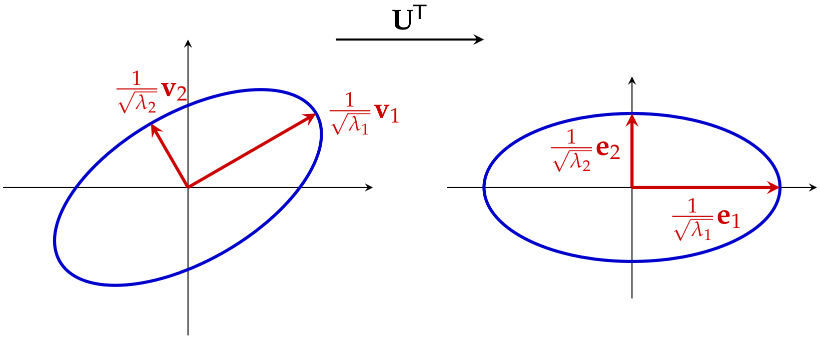
$$\mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x} = 1$$

describes an *ellipse* where the semi-axes (*principle axes*) coincide with the *normalized* eigenvectors of $\mathbf{A}$.



A Geometric Interpretation II

By a change of basis from $\{\mathbf{e}_1, \mathbf{e}_2\}$ to $\{\mathbf{v}_1, \mathbf{v}_2\}$ using transformation $\mathbf{U} = (\mathbf{v}_1, \mathbf{v}_2)$ this ellipse is rotated into canonical form.



(That is, we diagonalize matrix $\mathbf{A}$.)

An Application in Statistics

Suppose we have n observations of k metric attributes $X_1, \ldots, X_k$ which we combine into a vector:

$$\mathbf{x}_i = (x_{i1}, \ldots, x_{ik}) \in \mathbb{R}^k$$

The arithmetic mean then is given by

$$\bar{\mathbf{x}} = \frac{1}{n} \sum_{i=1}^{n} \mathbf{x}_i = (\bar{x}_1, \ldots, \bar{x}_k)$$

The *total sum of squares* is a measure for the statistical dispersion

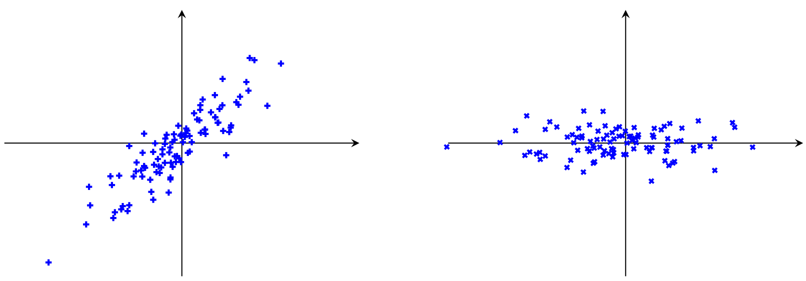
$$\mathrm{TSS} = \sum_{i=1}^{n} \|\mathbf{x}_i - \bar{\mathbf{x}}\|^2 = \sum_{j=1}^{k} \left(\sum_{i=1}^{n} |x_{ij} - \bar{x}_j|^2 \right) = \sum_{j=1}^{k} \mathrm{TSS}_j$$

It can be computed component-wise.

An Application in Statistics

A change of basis by means of an *orthogonal* matrix does not change TSS.

However, it changes the contributions of each of the components.



Can we find a basis such that a few components contribute much more to the TSS than the remaining ones?

Principle Component Analysis (PCA)

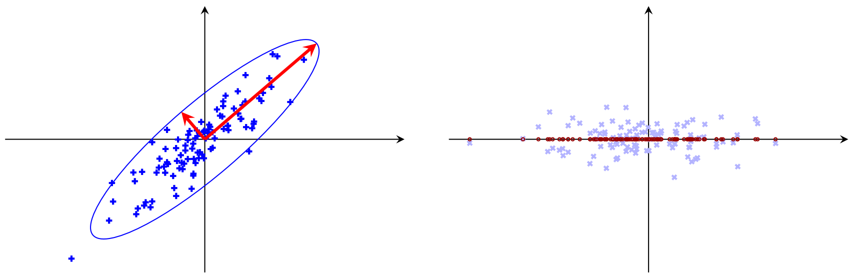
Assumptions:

- The data are approximately *multinormal* distributed.

Procedure:

1. Compute the covariance matrix $\mathbf{\Sigma}$.
2. Compute all eigenvalues and normalized eigenvectors of $\mathbf{\Sigma}$.
3. Sort eigenvalues such that
$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \, .$$
4. Use corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ as new basis.
5. The contribution of the first m components in this basis to TSS is
$$\frac{\sum_{j=1}^{m} \mathrm{TSS}j}{\sum_{j=1}^{k} \mathrm{TSS}j} \approx \frac{\sum_{j=1}^{m} \lambda_j}{\sum_{j=1}^{k} \lambda_j} \, .$$

Principle Component Analysis (PCA)



By means of PCA it is possible to reduce the number of dimensions without reducing the TSS substantially.

Summary

- eigenvalues and eigenvectors
- characteristic polynomial
- eigenspace
- properties of eigenvalues
- symmetric matrices and diagonalization
- quadratic forms
- definitness
- principle minors
- principle component analysis