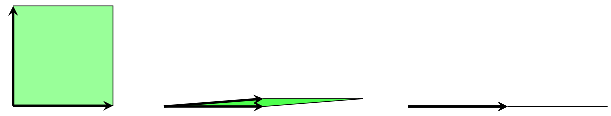
Chapter 5

Determinant

What is a Determinant?

We want to *"compute"* whether n vectors in $\mathbb{R}^n$ are linearly dependent and *measure* "how far" they are from being linearly dependent, resp.

Idea:
Two vectors in $\mathbb{R}^2$ span a parallelogram:



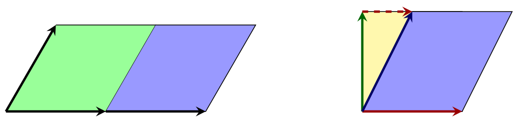
vectors are linearly *dependent* $\Leftrightarrow$ area is zero

We use the n-*dimensional volume* of the created parallelepiped for our function that "measures" linear dependency.

Properties of a Volume

We define our function indirectly by the properties of this volume.

- Multiplication of a vector by a scalar α yields the α-fold volume.
- Adding some vector to another one does not change the volume.
- If two vectors coincide, then the volume is zero.
- The volume of a unit cube is one.



Determinant

The **determinant** is a function which maps an $n \times n$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ into a real number $\det(\mathbf{A})$ with the following properties:

(D1) The determinant is **linear** in each column:

$$\det(\dots, \mathbf{a}_i + \mathbf{b}_i, \dots) = \det(\dots, \mathbf{a}_i, \dots) + \det(\dots, \mathbf{b}_i, \dots)$$
$$\det(\dots, \alpha\, \mathbf{a}_i, \dots) = \alpha \det(\dots, \mathbf{a}_i, \dots)$$

(D2) The determinant is zero, if two columns coincide:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_i, \dots) = 0$$

(D3) The determinant is **normalized**:

$$\det(\mathbf{I}) = 1$$

Notations: $\boxed{\det(\mathbf{A}) = |\mathbf{A}|}$

Example – Properties

(D1)

$$\begin{vmatrix} 1 & 2+10 & 3 \\ 4 & 5+11 & 6 \\ 7 & 8+12 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 10 & 3 \\ 4 & 11 & 6 \\ 7 & 12 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 \cdot 2 & 3 \\ 4 & 3 \cdot 5 & 6 \\ 7 & 3 \cdot 8 & 9 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(D2)

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 7 & 8 & 7 \end{vmatrix} = 0$$

Determinant – Remarks

- Properties (D1)–(D3) define a function uniquely.
  (I.e., such a function does exist and two functions with these properties are identical.)
- The determinant as defined above can be negative.
  So it can be seen as "signed volume".
- We derive more properties of the determinant below.
- Take care about the notation:
  Do not mix up $|\mathbf{A}|$ with the absolute value of a number $|x|$.
- The determinant is a so called *normalized alternating multi-linear form*.

Further Properties

(D4) The determinant is **alternating**:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_k, \dots) = -\det(\dots, \mathbf{a}_k, \dots, \mathbf{a}_i, \dots)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix}$$

Further Properties

(D5) The determinant does not change if we add some multiple of a column to another column:

$$\det(\dots, \mathbf{a}_i + \alpha\, \mathbf{a}_k, \dots, \mathbf{a}_k, \dots) = \det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_k, \dots)$$

$$\begin{vmatrix} 1 & 2+2\cdot 1 & 3 \\ 4 & 5+2\cdot 4 & 6 \\ 7 & 8+2\cdot 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Further Properties

(D6) The determinant does not change if we *transpose* a matrix:

$$\boxed{\det(\mathbf{A}^\mathsf{T}) = \det(\mathbf{A})}$$

Consequently,
all statements about columns hold analogously for rows.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

Further Properties

(D7) $\det(\mathbf{A}) \neq 0$ $\Leftrightarrow$ columns (rows) of $\mathbf{A}$ are linearly independent
$\Leftrightarrow$ $\mathbf{A}$ ist regular
$\Leftrightarrow$ $\mathbf{A}$ ist invertible

(D8) The determinant of the product of two matrices is equal to the product of their determinants:

$$\boxed{\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})}$$

(D9) The determinant if the inverse matrix is equal to the reciprocal of the determinant of the matrix:

$$\boxed{\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}}$$

Further Properties

(10) The determinant of a triangular matrix is the product of its diagonal elements:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \cdot \ldots \cdot a_{nn}$$

(11) The absolute value of the determinant $|\det(\mathbf{a}_1, \dots, \mathbf{a}_n)|$ is the volume of the parallelepiped spanned by the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$.

2×2 Matrix

$$\boxed{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

3×3 Matrix: Sarrus' Rule

$$\left|\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}\right| \begin{matrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{matrix} = \begin{matrix} 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 \\ -7 \cdot 5 \cdot 3 - 8 \cdot 6 \cdot 1 - 9 \cdot 4 \cdot 2 = 0 \end{matrix}$$

Source of Error

Determinants of 4×4 matrices must be computed by means of transformation into a triangular matrix or by Laplace expansion.

There is no such thing like Sarrus' rule for 4×4 matrices.

Transform into Triangular Matrix

(1) Transform into upper triangular matrix similar to Gaussian elimination.

- Add a multiple of a row to another row. (D5)
- Multiply a row by some scalar $\alpha \neq 0$ *and* the determinant by the reciprocal $\frac{1}{\alpha}$. (D1)
- Exchange two rows *and* switch the *sign* of the determinant. (D4)

(2) Compute the determinant as the product of its diagonal elements. (Property D10)

Example – Transform into Triangular Matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot (-3) \cdot 0 = 0$$

$$\begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{vmatrix}$$

$$= -\frac{1}{2} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{vmatrix} = -\frac{1}{2} \cdot 1 \cdot 2 \cdot (-4) = 4$$

Laplace Expansion

Laplace expansion along the k-th column and i-th row, resp.:

$$\boxed{\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik} = \sum_{k=1}^{n} a_{ik} \cdot (-1)^{i+k} M_{ik}}$$

where M_{ik} is the *determinant* of the $(n-1) \times (n-1)$ submatrix which we obtain by *deleting* the i-th row and the k-th column of $\mathbf{A}$.
It is called a **minor** of $\mathbf{A}$.

We get the signs $(-1)^{i+k}$ by means of a chessboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Expansion along the First Row

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \\
&\quad + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \\
&\quad + 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) \\
&= 0
\end{aligned}$$

Expansion along the Second Column

$$\begin{aligned}
\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \\
&\quad + 5 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} \\
&\quad + 8 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\
&= -2 \cdot (-6) + 5 \cdot (-12) - 8 \cdot (-6) \\
&= 0
\end{aligned}$$

Laplace and Leibzig Formula

Laplace expansion allows to compute the determinant *recursively*:

The deterimant of a $k \times k$ matrix is expanded into a sum of k determinants of $(k-1) \times (k-1)$ matrices.

For an $n \times n$ matrices we can repeat this recursion step n times and yield a summation of $n!$ products of n numbers each:

$$\det(\mathbf{A}) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{\sigma(i),i}$$

where $\mathcal{S}_n$ is the permutation group of order n.

This formula is shown here just for completeness.
Its explanation is out of the scope of this course.

Adjugate Matrix

In Laplace expansion the factors $\boxed{A_{ik} = (-1)^{i+k} M_{ik}}$ are called the **cofactors** of a_{ik}:

$$\det(\mathbf{A}) = \sum_{i=1}^{n} a_{ik} \cdot A_{ik}$$

The matrix formed by these cofactors is called the **cofactor matrix** $\mathbf{A}^*$.

Its transpose $\mathbf{A}^{*\mathsf{T}}$ is called the **adjugate** of $\mathbf{A}$.

$$\operatorname{adj}(\mathbf{A}) = \mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

Product $\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}}$

$$\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = |\mathbf{A}| \cdot \mathbf{I}$$

Product $\mathbf{A} \cdot \mathbf{A}^{*\mathsf{T}}$

Product of the k-th row of $\mathbf{A}^{*\mathsf{T}}$ by the j-th column of $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$:

$$\begin{aligned}
\left[\mathbf{A}^{*\mathsf{T}} \cdot \mathbf{A}\right]_{kj} = \sum_{i=1}^{n} A_{ik} \cdot a_{ij} &= \sum_{i=1}^{n} a_{ij} \cdot (-1)^{i+k} M_{ik} \\
\text{[expansion along } k\text{-th column]} \quad &= \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{a}_j}_{k\text{-th column}}, \dots, \mathbf{a}_n) \\
&= \begin{cases} |\mathbf{A}| & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \\
&= |\mathbf{A}| \delta_{kj}
\end{aligned}$$

Hence $\quad \mathbf{A}^{*\mathsf{T}} \cdot \mathbf{A} = |\mathbf{A}| \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathsf{T}}$

Cramer's Rule for the Inverse Matrix

We get a formula for the inverse of Matrix $\mathbf{A}$:

$$\boxed{\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathsf{T}}}$$

This formula is not practical for inverting a matrix . . .

. . . except for 2×2 matrices where it is very convenient:

$$\boxed{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}}$$

Example – Inverse Matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix}^{-1} = \frac{1}{4} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & 2 & -\frac{1}{2} \\ 0 & -2 & 1 \\ \frac{1}{4} & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Cramer's Rule for Linear Equations

We want to solve linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

If $\mathbf{A}$ is regular (i.e., $|\mathbf{A}| \neq 0$), then we find

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{|\mathbf{A}|} \mathbf{A}^{*\mathsf{T}} \cdot \mathbf{b}$$

So we get for x_k

$$\begin{aligned} x_k &= \frac{1}{|\mathbf{A}|} \sum_{i=1}^{n} A_{ik} \cdot b_i = \frac{1}{|\mathbf{A}|} \sum_{i=1}^{n} b_i \cdot (-1)^{i+k} M_{ik} \\ &= \frac{1}{|\mathbf{A}|} \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{b}}_{k\text{-th column}}, \dots, \mathbf{a}_n) \end{aligned}$$

Cramer's Rule for Linear Equations

Let $\mathbf{A}_k$ be the matrix where the k-th column of $\mathbf{A}$ is replaced by $\mathbf{b}$.

If $\mathbf{A}$ is an invertible matrix, then the solution of

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

is given by

$$\boxed{\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ \vdots \\ |\mathbf{A}_n| \end{pmatrix}}$$

This procedure does not work if $\mathbf{A}$ is not regular.

Example – Cramer's Rule

Compute the solution of equation

$$\begin{pmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution:

$$\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ |\mathbf{A}_2| \\ |\mathbf{A}_3| \end{pmatrix} = \begin{pmatrix} 12 \\ -22 \\ 45 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 1$$

$$|\mathbf{A}_1| = \begin{vmatrix} 1 & 11 & 3 \\ 2 & 13 & 4 \\ 3 & 3 & 1 \end{vmatrix} = 12$$

$$|\mathbf{A}_2| = \begin{vmatrix} 9 & 1 & 3 \\ 9 & 2 & 4 \\ 2 & 3 & 1 \end{vmatrix} = -22$$

$$|\mathbf{A}_3| = \begin{vmatrix} 9 & 11 & 1 \\ 9 & 13 & 2 \\ 2 & 3 & 3 \end{vmatrix} = 45$$

Summary

- definition of determinant
- properties
- relation between determinant and regularity
- volume of a parallelepiped
- compuation of the determinant
  (Sarrus' rule, transformation into triangular matrix)
- Laplace expansion
- Cramer's rule