

Chapter 5

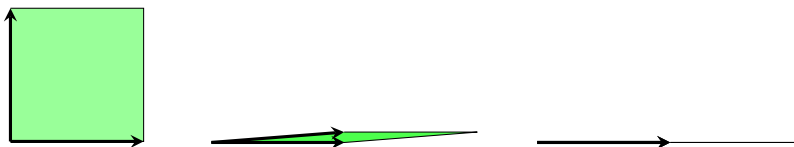
Determinant

What is a Determinant?

We want to “compute” whether n vectors in \mathbb{R}^n are linearly dependent and *measure* “how far” they are from being linearly dependent, resp.

Idea:

Two vectors in \mathbb{R}^2 span a parallelogram:



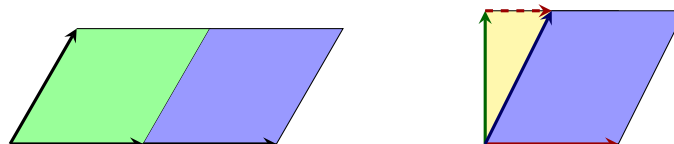
vectors are linearly *dependent* \Leftrightarrow area is zero

We use the n -dimensional volume of the created parallelepiped for our function that “measures” linear dependency.

Properties of a Volume

We define our function indirectly by the properties of this volume.

- ▶ Multiplication of a vector by a scalar α yields the α -fold volume.
- ▶ Adding some vector to another one does not change the volume.
- ▶ If two vectors coincide, then the volume is zero.
- ▶ The volume of a unit cube is one.



Determinant

The **determinant** is a function which maps an $n \times n$ matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ into a real number $\det(\mathbf{A})$ with the following properties:

(D1) The determinant is **linear** in each column:

$$\det(\dots, \mathbf{a}_i + \mathbf{b}_i, \dots) = \det(\dots, \mathbf{a}_i, \dots) + \det(\dots, \mathbf{b}_i, \dots)$$
$$\det(\dots, \alpha \mathbf{a}_i, \dots) = \alpha \det(\dots, \mathbf{a}_i, \dots)$$

(D2) The determinant is zero, if two columns coincide:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_i, \dots) = 0$$

(D3) The determinant is **normalized**:

$$\det(\mathbf{I}) = 1$$

Notations:

$$\det(\mathbf{A}) = |\mathbf{A}|$$

Example – Properties

(D1)

$$\begin{vmatrix} 1 & 2+10 & 3 \\ 4 & 5+11 & 6 \\ 7 & 8+12 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} + \begin{vmatrix} 1 & 10 & 3 \\ 4 & 11 & 6 \\ 7 & 12 & 9 \end{vmatrix}$$

$$\begin{vmatrix} 1 & 3 \cdot 2 & 3 \\ 4 & 3 \cdot 5 & 6 \\ 7 & 3 \cdot 8 & 9 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

(D2)

$$\begin{vmatrix} 1 & 2 & 1 \\ 4 & 5 & 4 \\ 7 & 8 & 7 \end{vmatrix} = 0$$

Determinant – Remarks

- ▶ Properties (D1)–(D3) define a function uniquely. (I.e., such a function does exist and two functions with these properties are identical.)
- ▶ The determinant as defined above can be negative. So it can be seen as “signed volume”.
- ▶ We derive more properties of the determinant below.
- ▶ Take care about the notation: Do not mix up $|\mathbf{A}|$ with the absolute value of a number $|x|$.
- ▶ The determinant is a so called *normalized alternating multi-linear form*.

Further Properties

(D4) The determinant is **alternating**:

$$\det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_k, \dots) = -\det(\dots, \mathbf{a}_k, \dots, \mathbf{a}_i, \dots)$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = - \begin{vmatrix} 1 & 3 & 2 \\ 4 & 6 & 5 \\ 7 & 9 & 8 \end{vmatrix}$$

Further Properties

(D5) The determinant does not change if we add some multiple of a column to another column:

$$\det(\dots, \mathbf{a}_i + \alpha \mathbf{a}_k, \dots, \mathbf{a}_k, \dots) = \det(\dots, \mathbf{a}_i, \dots, \mathbf{a}_k, \dots)$$

$$\begin{vmatrix} 1 & 2 + 2 \cdot 1 & 3 \\ 4 & 5 + 2 \cdot 4 & 6 \\ 7 & 8 + 2 \cdot 7 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Further Properties

(D6) The determinant does not change if we *transpose* a matrix:

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

Consequently,
all statements about columns hold analogously for rows.

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

Further Properties

- (D7) $\det(\mathbf{A}) \neq 0 \Leftrightarrow$ columns (rows) of \mathbf{A} are linearly independent
 $\Leftrightarrow \mathbf{A}$ ist regular
 $\Leftrightarrow \mathbf{A}$ ist invertible

- (D8) The determinant of the product of two matrices is equal to the product of their determinants:

$$\det(\mathbf{A} \cdot \mathbf{B}) = \det(\mathbf{A}) \cdot \det(\mathbf{B})$$

- (D9) The determinant of the inverse matrix is equal to the reciprocal of the determinant of the matrix:

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

Further Properties

- (10) The determinant of a triangular matrix is the product of its diagonal elements:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22} & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33} & \dots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{nn} \end{vmatrix} = a_{11} \cdot a_{22} \cdot a_{33} \cdot \dots \cdot a_{nn}$$

- (11) The absolute value of the determinant $|\det(\mathbf{a}_1, \dots, \mathbf{a}_n)|$ is the volume of the parallelepiped spanned by the column vectors $\mathbf{a}_1, \dots, \mathbf{a}_n$.

2 × 2 Matrix

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1 \cdot 4 - 2 \cdot 3 = -2$$

3 × 3 Matrix: Sarrus' Rule

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{vmatrix} = 1 \cdot 5 \cdot 9 + 2 \cdot 6 \cdot 7 + 3 \cdot 4 \cdot 8 - 7 \cdot 5 \cdot 3 - 8 \cdot 6 \cdot 1 - 9 \cdot 4 \cdot 2 = 0$$

Source of Error

Determinants of 4×4 matrices must be computed by means of transformation into a triangular matrix or by Laplace expansion.

There is no such thing like Sarrus' rule for 4×4 matrices.

Transform into Triangular Matrix

(1) Transform into upper triangular matrix similar to Gaussian elimination.

- ▶ Add a multiple of a row to another row. (D5)
- ▶ Multiply a row by some scalar $\alpha \neq 0$ and the determinant by the reciprocal $\frac{1}{\alpha}$. (D1)
- ▶ Exchange two rows and switch the *sign* of the determinant. (D4)

(2) Compute the determinant as the product of its diagonal elements. (Property D10)

Example – Transform into Triangular Matrix

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix} \\ = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix} = 1 \cdot (-3) \cdot 0 = 0$$

$$\begin{vmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 2 & 5 & 6 \end{vmatrix} = - \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 1 & 0 \end{vmatrix} \\ = -\frac{1}{2} \cdot \begin{vmatrix} 1 & 2 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & -4 \end{vmatrix} = -\frac{1}{2} \cdot 1 \cdot 2 \cdot (-4) = 4$$

Laplace Expansion

Laplace expansion along the k -th column and i -th row, resp.:

$$\det(\mathbf{A}) = \sum_{i=1}^n a_{ik} \cdot (-1)^{i+k} M_{ik} = \sum_{k=1}^n a_{ik} \cdot (-1)^{i+k} M_{ik}$$

where M_{ik} is the *determinant* of the $(n-1) \times (n-1)$ submatrix which we obtain by *deleting* the i -th row and the k -th column of \mathbf{A} . It is called a **minor** of \mathbf{A} .

We get the signs $(-1)^{i+k}$ by means of a chessboard pattern:

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Expansion along the First Row

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} \\ + 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \\ + 3 \cdot (-1)^{1+3} \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ = 1 \cdot (-3) - 2 \cdot (-6) + 3 \cdot (-3) \\ = 0$$

Expansion along the Second Column

$$\begin{aligned} \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} &= 2 \cdot (-1)^{1+2} \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} \\ &+ 5 \cdot (-1)^{2+2} \cdot \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} \\ &+ 8 \cdot (-1)^{3+2} \cdot \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2 \cdot (-6) + 5 \cdot (-12) - 8 \cdot (-6) \\ &= 0 \end{aligned}$$

Laplace and Leibzig Formula

Laplace expansion allows to compute the determinant *recursively*:

The determinant of a $k \times k$ matrix is expanded into a sum of k determinants of $(k-1) \times (k-1)$ matrices.

For an $n \times n$ matrices we can repeat this recursion step n times and yield a summation of $n!$ products of n numbers each:

$$\det(\mathbf{A}) = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n a_{\sigma(i),i}$$

where \mathcal{S}_n is the permutation group of order n .

This formula is shown here just for completeness.
Its explanation is out of the scope of this course.

Adjugate Matrix

In Laplace expansion the factors $A_{ik} = (-1)^{i+k} M_{ik}$ are called the **cofactors** of a_{ik} :

$$\det(\mathbf{A}) = \sum_{i=1}^n a_{ik} \cdot A_{ik}$$

The matrix formed by these cofactors is called the **cofactor matrix** \mathbf{A}^* .

Its transpose $\mathbf{A}^{*\top}$ is called the **adjugate** of \mathbf{A} .

$$\operatorname{adj}(\mathbf{A}) = \mathbf{A}^{*\top} = \begin{pmatrix} A_{11} & A_{21} & \dots & A_{n1} \\ A_{12} & A_{22} & \dots & A_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ A_{1n} & A_{2n} & \dots & A_{nn} \end{pmatrix}$$

Product $\mathbf{A} \cdot \mathbf{A}^{*\mathbf{T}}$

$$\mathbf{A} \cdot \mathbf{A}^{*\mathbf{T}} = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 5 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 4 \\ 2 & 3 \end{vmatrix} \\ -\begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} & \begin{vmatrix} 0 & 4 \\ 2 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ 1 & 3 \end{vmatrix} \\ \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 0 & 2 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} = |\mathbf{A}| \cdot \mathbf{I}$$

Product $\mathbf{A} \cdot \mathbf{A}^{*\mathbf{T}}$

Product of the k -th row of $\mathbf{A}^{*\mathbf{T}}$ by the j -th column of $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$:

$$\begin{aligned} [\mathbf{A}^{*\mathbf{T}} \cdot \mathbf{A}]_{kj} &= \sum_{i=1}^n A_{ik} \cdot a_{ij} = \sum_{i=1}^n a_{ij} \cdot (-1)^{i+k} M_{ik} \\ \text{[expansion along } k\text{-th column]} &= \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{a}_j, \dots, \mathbf{a}_n}_{k\text{-th column}}) \\ &= \begin{cases} |\mathbf{A}| & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \\ &= |\mathbf{A}| \delta_{kj} \end{aligned}$$

$$\text{Hence } \mathbf{A}^{*\mathbf{T}} \cdot \mathbf{A} = |\mathbf{A}| \cdot \mathbf{I} \quad \Rightarrow \quad \mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathbf{T}}$$

Cramer's Rule for the Inverse Matrix

We get a formula for the inverse of Matrix \mathbf{A} :

$$\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \mathbf{A}^{*\mathbf{T}}$$

This formula is not practical for inverting a matrix ...

... except for 2×2 matrices where it is very convenient:

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}^{-1} = \frac{1}{|\mathbf{A}|} \cdot \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}$$

Example – Inverse Matrix

$$\begin{pmatrix} 0 & 2 & 4 \\ 1 & 2 & 3 \\ 2 & 5 & 6 \end{pmatrix}^{-1} = \frac{1}{4} \cdot \begin{pmatrix} -3 & 8 & -2 \\ 0 & -8 & 4 \\ 1 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} & 2 & -\frac{1}{2} \\ 0 & -2 & 1 \\ \frac{1}{4} & 1 & -\frac{1}{2} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^{-1} = \frac{1}{-2} \cdot \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Cramer's Rule for Linear Equations

We want to solve linear equation

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

If \mathbf{A} is regular (i.e., $|\mathbf{A}| \neq 0$), then we find

$$\mathbf{x} = \mathbf{A}^{-1} \cdot \mathbf{b} = \frac{1}{|\mathbf{A}|} \mathbf{A}^{*\top} \cdot \mathbf{b}$$

So we get for x_k

$$\begin{aligned} x_k &= \frac{1}{|\mathbf{A}|} \sum_{i=1}^n A_{ik} \cdot b_i = \frac{1}{|\mathbf{A}|} \sum_{i=1}^n b_i \cdot (-1)^{i+k} M_{ik} \\ &= \frac{1}{|\mathbf{A}|} \det(\mathbf{a}_1, \dots, \underbrace{\mathbf{b}}_{k\text{-th column}}, \dots, \mathbf{a}_n) \end{aligned}$$

Cramer's Rule for Linear Equations

Let \mathbf{A}_k be the matrix where the k -th column of \mathbf{A} is replaced by \mathbf{b} .

If \mathbf{A} is an invertible matrix, then the solution of

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

is given by

$$\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ \vdots \\ |\mathbf{A}_n| \end{pmatrix}$$

This procedure does not work if \mathbf{A} is not regular.

Example – Cramer's Rule

Compute the solution of equation

$$\begin{pmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Solution:

$$\mathbf{x} = \frac{1}{|\mathbf{A}|} \begin{pmatrix} |\mathbf{A}_1| \\ |\mathbf{A}_2| \\ |\mathbf{A}_3| \end{pmatrix} = \begin{pmatrix} 12 \\ -22 \\ 45 \end{pmatrix}$$

$$|\mathbf{A}| = \begin{vmatrix} 9 & 11 & 3 \\ 9 & 13 & 4 \\ 2 & 3 & 1 \end{vmatrix} = 1$$

$$|\mathbf{A}_1| = \begin{vmatrix} 1 & 11 & 3 \\ 2 & 13 & 4 \\ 3 & 3 & 1 \end{vmatrix} = 12$$

$$|\mathbf{A}_2| = \begin{vmatrix} 9 & 1 & 3 \\ 9 & 2 & 4 \\ 2 & 3 & 1 \end{vmatrix} = -22$$

$$|\mathbf{A}_3| = \begin{vmatrix} 9 & 11 & 1 \\ 9 & 13 & 2 \\ 2 & 3 & 3 \end{vmatrix} = 45$$

Summary

- ▶ definition of determinant
- ▶ properties
- ▶ relation between determinant and regularity
- ▶ volume of a parallelepiped
- ▶ computation of the determinant
(Sarrus' rule, transformation into triangular matrix)
- ▶ Laplace expansion
- ▶ Cramer's rule