Chapter 4

Vector Space

Real Vector Space

The set of all vectors $\mathbf{x}$ with n components is denoted by

$$\boxed{\mathbb{R}^n = \left\{ \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \,\middle|\, x_i \in \mathbb{R},\ 1 \le i \le n \right\}}$$

It is *the* prototype example of an n*-dimensional (real) vector space*.

Definition:
A **vector space** $\mathcal{V}$ is a *set* of objects which may be *added* together and *multiplied* by numbers, called *scalars*.
Elements of a vector space are called **vectors**.

For details see course “Mathematics 1”.

Example – Vector Space

The set of all 2×2 matrices

$$\mathbb{R}^{2\times 2} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R},\ i,j \in \{1,2\} \right\}$$

together with matrix addition and scalar multiplication forms a vector space.

Similarly the set of all $m \times n$ matrices

$$\mathbb{R}^{m\times n} = \left\{ \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} \middle| \, a_{ij} \in \mathbb{R},\ i = 1,\dots,m,\ j = 1,\dots n \right\}$$

forms a vector space.

A More Abstract Example

Let $\mathcal{P}_n = \{\sum_{i=0}^n a_i x^i | a_i \in \mathbb{R}\}$ be the set of all polynomials in x of degree less than or equal to n.

Obviously we can multiply a polynomial by a scalar:

$$3 \cdot (4x^2 - 2x + 5) = 12x^2 - 6x + 15 \in \mathcal{P}_2$$

and add them point-wise:

$$(4x^2 - 2x + 5) + (-4x^2 + 5x - 2) = 3x + 3 \in \mathcal{P}_2$$

So for every $p(x), q(x) \in \mathcal{P}_n$ and $\alpha \in \mathbb{R}$ we find

$$\alpha p(x) \in \mathcal{P}_n \quad \text{and} \quad p(x) + q(x) \in \mathcal{P}_n \ .$$

Thus $\mathcal{P}_n$ together with point-wise addition and scalar multiplication forms a vector space.

Linear Combination

Let $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathbb{R}^n$ be vectors and $c_1, \dots, c_k \in \mathbb{R}$ arbitrary numbers.
Then we get a new vector by a **linear combination** of these vectors:

$$\boxed{\mathbf{x} = c_1\,\mathbf{v}_1 + \cdots + c_k\,\mathbf{v}_k = \sum_{i=1}^{k} c_i\,\mathbf{v}_i}$$

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}.$

Then the following are linear combinations of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$:

$$\mathbf{x} = 1\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + 3\,\mathbf{v}_3 - 2\,\mathbf{v}_4 = (-3, -4, 3)^\mathsf{T},$$

$$\mathbf{y} = -\mathbf{v}_1 + \mathbf{v}_2 - 2\,\mathbf{v}_3 + 3\,\mathbf{v}_4 = (4, 7, -2)^\mathsf{T}, \quad \text{and}$$

$$\mathbf{z} = 2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = (0, 0, 0)^\mathsf{T} = 0$$

Subspace

A **Subspace** $\mathcal{S}$ of a vector space $\mathcal{V}$ is a *subset* of $\mathcal{V}$ which itself forms a *vector space* (with the same rules for addition and scalar multiplication).

In order to verify that a *subset* $\mathcal{S} \subseteq \mathcal{V}$ is a *subspace* of $\mathcal{V}$ we have to verify that for all $\mathbf{x}, \mathbf{y} \in \mathcal{S}$ and all $\alpha, \beta \in \mathbb{R}$

$$\boxed{\alpha\mathbf{x} + \beta\mathbf{y} \in \mathcal{S}}$$

We say that $\mathcal{S}$ is closed under linear combinations.

Equivalently: We have to verify that

(i) if $\mathbf{x}, \mathbf{y} \in \mathcal{S}$, then $\mathbf{x} + \mathbf{y} \in \mathcal{S}$; and

(ii) if $\mathbf{x} \in \mathcal{S}$ and $\alpha \in \mathbb{R}$, then $\alpha\mathbf{x} \in \mathcal{S}$.

Example – Subspace

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix} : x_i \in \mathbb{R}, 1 \le i \le 2 \right\} \subset \mathbb{R}^3$$ is a subspace of $\mathbb{R}^3$.

$$\left\{ \mathbf{x} = \alpha \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} : \alpha \in \mathbb{R} \right\} \subset \mathbb{R}^3$$ is a subspace of $\mathbb{R}^3$.

$$\left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_i \ge 0, 1 \le i \le 3 \right\} \subset \mathbb{R}^3$$ is **not** a subspace of $\mathbb{R}^3$.

Example – Homogeneous Linear Equation

Let $\mathbf{A}$ be an $m \times n$ matrix.
The solution set $\mathcal{L}$ of the *homogeneous* linear equation

$$\mathbf{A}\mathbf{x} = 0$$

forms a subspace of $\mathbb{R}^n$:

Let $\mathbf{x}, \mathbf{y} \in \mathcal{L} \subseteq \mathbb{R}^n$, i.e., $\mathbf{A}\mathbf{x} = 0$ and $\mathbf{A}\mathbf{y} = 0$, and $\alpha, \beta \in \mathbb{R}$.

Then a straightforward computation yields

$$\mathbf{A}(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha\mathbf{A}\mathbf{x} + \beta\mathbf{A}\mathbf{y} = \alpha 0 + \beta 0 = 0$$

i.e., $\alpha\mathbf{x} + \beta\mathbf{y}$ solves the linear equation and hence $\alpha\mathbf{x} + \beta\mathbf{y} \in \mathcal{L}$.

Therefore $\mathcal{L}$ is a subspace of $\mathbb{R}^n$.

Example – Subspace

$$\left\{ \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix} \middle| \, a_{ii} \in \mathbb{R},\ i \in \{1,2\} \right\}$$ is a subspace of $\mathbb{R}^{2\times 2}$.

$$\left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \, a, b \in \mathbb{R} \right\}$$ is a subspace of $\mathbb{R}^{2\times 2}$.

$\left\{ \mathbf{A} \in \mathbb{R}^{2\times 2} \,\middle|\, \mathbf{A} \text{ is invertible} \right\}$ is **not** a subspace of $\mathbb{R}^{2\times 2}$.

Linear Span

The set of all *linear combinations* of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathcal{V}$

$$\boxed{\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k) = \left\{ \sum_{i=1}^{k} c_i \mathbf{v}_i \,\middle|\, c_i \in \mathbb{R},\ i = 1, \dots, k \right\}}$$

forms a subspace of $\mathcal{V}$ and is called the **linear span** of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Linear Span

Let $\mathbf{x}, \mathbf{y} \in \mathcal{S} = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ and $\alpha, \beta \in \mathbb{R}$.

Then there exist $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, k$, such that

$$\mathbf{x} = \sum_{i=1}^{k} a_i \mathbf{v}_i \quad \text{and} \quad \mathbf{y} = \sum_{i=1}^{k} b_i \mathbf{v}_i \ .$$

But then

$$\alpha\mathbf{x} + \beta\mathbf{y} = \alpha \sum_{i=1}^{k} a_i \mathbf{v}_i + \beta \sum_{i=1}^{k} b_i \mathbf{v}_i = \sum_{i=1}^{k} \underbrace{(\alpha a_i + \beta b_i)}_{\in\mathbb{R}} \mathbf{v}_i \in \mathcal{S}$$

as the last summation is a linear combination of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Hence $\mathcal{S} = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k)$ is a subspace of $\mathcal{V}$.

Example – Linear Span

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$.

span $(\mathbf{v}_1) = \{c\,\mathbf{v}_1 \colon c \in \mathbb{R}\}$ is a straight line in $\mathbb{R}^3$ through the origin.

span $(\mathbf{v}_1, \mathbf{v}_2)$ is a plane in $\mathbb{R}^3$ through the origin.

span $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) =$ span $(\mathbf{v}_1, \mathbf{v}_2)$

span$(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \mathbb{R}^3$.

Linear Independency

Every vector $\mathbf{x} \in \text{span}(\mathbf{v}_1, \dots, \mathbf{v}_k)$ can be written as a linear combination of $\mathbf{v}_1, \dots, \mathbf{v}_k$.

Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $\mathbf{v}_3 = \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}$, $\mathbf{v}_4 = \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix}$.

$$\mathbf{x} = \begin{pmatrix} -3 \\ -4 \\ 3 \end{pmatrix} = 1\,\mathbf{v}_1 + 0\,\mathbf{v}_2 + 3\,\mathbf{v}_3 - 2\,\mathbf{v}_4 = -1\,\mathbf{v}_1 + 2\,\mathbf{v}_2 + 6\,\mathbf{v}_3 - 2\,\mathbf{v}_4$$

The representation in this example is not unique!

Reason: $2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = 0$

One of the vectors seems to be needless:

$$\text{span}\,(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{span}\,(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$$

Linear Independency

Vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are called **linearly independent** if the homogeneous system of equations

$$c_1\,\mathbf{v}_1 + c_2\,\mathbf{v}_2 + \dots + c_k\,\mathbf{v}_k = 0$$

has the unique solution $c_1 = c_2 = \dots = c_k = 0$. They are called **linearly dependent** if these equations have other (non-zero) solutions.

If vectors are linearly dependent then *some* vector (but *not* necessarily *each* of these!) can be written as a linear combination of the other vectors.

$$2\,\mathbf{v}_1 - 2\,\mathbf{v}_2 - 3\,\mathbf{v}_3 + 0\,\mathbf{v}_4 = 0 \quad \Leftrightarrow \quad \mathbf{v}_3 = \tfrac{2}{3}\,\mathbf{v}_1 - \tfrac{2}{3}\,\mathbf{v}_2$$

Hence $\quad \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$.

Linear Independency

Determine linear (in)dependency

(1) Create matrix $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_k)$.

(2) Transform $\mathbf{V}$ into row echelon form by means of Gaussian elimination.

(3) Count the number of non-zero rows.

(4) If this is equal to k (the number of vectors), then these vectors are linearly Independence.

If it is smaller, then the vectors or linearly dependent.

This procedure checks whether the linear equation $\mathbf{V} \cdot \mathbf{c} = 0$ has a unique solution.

Example – Linearly Independent

Are the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

linearly independent?

(1) Create a matrix:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

Example – Linearly Independent

(2) Transform:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 1 & -3 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & -27 \end{pmatrix}$$

(3) We count 3 non-zero rows.

(4) The number of non-zero rows coincides with the number of vectors ($= 3$).

Thus the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly independent*.

Example – Linearly Dependent

Are vectors $\mathbf{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 4 \\ 5 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$

linearly independent?

(1) Create a matrix . . . (2) and transform:

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix} \leadsto \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 10 & -3 \end{pmatrix} \leadsto \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

(3) We count 2 non-zero rows.

(4) The number of non-zero rows ($= 2$) is less than the number of vectors ($= 3$).

Thus the three vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are *linearly dependent*.

Rank of a Matrix

The **rank** of matrix $\mathbf{A}$ is the maximal number of linearly independent columns.

We have: $\boxed{\text{rank}(\mathbf{A}^\mathsf{T}) = \text{rank}(\mathbf{A})}$

The rank of an $n \times k$ matrix is at most $\min(n, k)$.

An $n \times n$ matrix is called **regular**, if it has **full rank**,
i.e. if $\text{rank}(\mathbf{A}) = n$.

Rank of a Matrix

Computation of the rank:

(1) Transform matrix $\mathbf{A}$ into row echelon form by means of Gaussian elimination.

(2) Then $\text{rank}(\mathbf{A})$ is given by the number of non-zero rows.

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & -27 \end{pmatrix} \Rightarrow \quad \text{rank}(\mathbf{A}) = 3$$

$$\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 3 & 1 & 3 \\ 0 & 10 & -3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \quad \text{rank}(\mathbf{A}) = 2$$

Invertible and Regular

An $n \times n$ matrix $\mathbf{A}$ is *invertible*, if and only if it is *regular*.

The following 3×3 matrix $\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 2 & 1 & 1 \end{pmatrix}$ has full rank (3).

Thus it is regular and hence invertible.

The following 3×3 matrix $\begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix}$ has only rank 2.

Thus it is *not* regular and hence singular (i.e., not invertible).

Basis

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ **spans** (or *generates*) a vector space $\mathcal{V}$, if

$$\operatorname{span}(\mathbf{v}_1, \dots, \mathbf{v}_d) = \mathcal{V}$$

This set is thus called a **generating set** for the vector space.

If these vectors are *linearly independent*, then this set is called a **basis** of the vector space.

The basis of a vector space is not uniquely determined!

However, the number of vectors in a basis is uniquely determined.
It is called the **dimension** of the vector space.

$$\dim(\mathcal{V}) = d$$

Characterizations of a Basis

There are several *equivalent* characterizations of a basis.

A basis B of vector space $\mathcal{V}$ is a

- linearly independent generating set of $\mathcal{V}$
- minimal generating set of $\mathcal{V}$
  (i.e., every proper subset of B does not span $\mathcal{V}$)
- maximal linearly independent set
  (i.e., every proper superset of B is linearly dependent)

Example – Basis

The so called **canonical basis** of the $\mathbb{R}^n$ consists of the n unit vectors:

$$\boxed{B_0 = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \subset \mathbb{R}^n}$$

Thus we can conclude that

$$\dim(\mathbb{R}^n) = n$$

an that every basis of $\mathbb{R}^n$ consists of n (linearly independent) vectors.

Another basis of the $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} \right\}$$

Non-Example – Basis

The following are *not* bases of the $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \end{pmatrix} \right\}$$

is not linearly independent (because it has too many vectors).

$$\left\{ \begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} \right\}$$

does not span $\mathbb{R}^3$ (because it has too few vectors).

Beware: Three vectors need not necessarily form a basis of $\mathbb{R}^3$.
They might be linearly dependent.

Example – Basis

The *canonical basis* of $\mathbb{R}^{2\times 2}$ consists of the four matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence

$$\dim(\mathbb{R}^{2\times 2}) = 4 \,.$$

Example – Basis

The simplest basis of vector space $\mathcal{P}_2 = \{\sum_{i=0}^{2} a_i x^i | a_i \in \mathbb{R}\}$ is given by

$$\{1, x, x^2\}$$

and hence

$$\dim(\mathcal{P}_2) = 3\ .$$

Coordinates of a Vector

Let $B = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of vector space $\mathcal{V}$.
Then for every $c_i \in \mathbb{R}$ we get a vector

$$\mathbf{x} = \sum_{i=1}^{n} c_i \mathbf{v}_i$$

On the other hand for a given vector $\mathbf{x}$ we can find (unique) numbers $c_i(\mathbf{x}) \in \mathbb{R}$ such that

$$\mathbf{x} = \sum_{i=1}^{n} c_i(\mathbf{x}) \mathbf{v}_i$$

The numbers $c_i(\mathbf{x})$ are called the **coefficients** of $\mathbf{x}$ w.r.t. basis B.
The vector

$$\mathbf{c}(\mathbf{x}) = (c_1(\mathbf{x}, \ldots, c_n(\mathbf{x}))$$

is called the **coefficient vector** of $\mathbf{x}$ w.r.t. basis B.

Space of Coordinate Vectors

For a fixed basis B the coefficient vector $\mathbf{c}(\mathbf{x})$ of $\mathbf{x}$ is unique and

$$\mathbf{c}(\mathbf{x}) \in \mathbb{R}^n = \mathbb{R}^{\dim(\mathcal{V})} .$$

So we have a bijection

$$\mathcal{V} \to \mathbb{R}^n, \ \mathbf{x} \mapsto \mathbf{c}(\mathbf{x})$$

with the nice (structure preserving) property

- $\mathbf{c}(\alpha\mathbf{x}) = \alpha\mathbf{c}(\mathbf{x})$
- $\mathbf{c}(\mathbf{x}+\mathbf{y}) = \mathbf{c}(\mathbf{x}) + \mathbf{c}(\mathbf{y})$

for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and all $\alpha \in \mathbb{R}$.

That is, instead of dealing with vectors in $\mathcal{V}$ we can fix a basis B and do all computations with coefficient vectors in $\mathbb{R}^n$.

Thus every n-dimensional vector space $\mathcal{V}$ is *isomorphic* to (i.e., looks like) an $\mathbb{R}^n$.

Coordinates of Vectors in $\mathbb{R}^n$

Let $B = \{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ be a basis of $\mathbb{R}^n$.
We obtain the coordinate vector $\mathbf{c}(\mathbf{x})$ of $\mathbf{x} \in \mathbb{R}^n$ w.r.t. B by solving the linear equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n = \mathbf{x}\ .$$

In matrix notation with $\mathbf{V} = (\mathbf{v}_1, \dots, \mathbf{v}_n)$:

$$\mathbf{V} \cdot \mathbf{c} = \mathbf{x} \qquad \Rightarrow \qquad \boxed{\mathbf{c} = \mathbf{V}^{-1}\mathbf{x}}$$

By construction $\mathbf{V}$ has full rank.

Observe that components $x_1, \dots, x_n$ of vector $\mathbf{x}$ can be seen as its coordinate w.r.t. the canonical basis.

Example – Coordinate Vector

Compute the coordinates $\mathbf{c}$ of $\mathbf{x} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

w.r.t. basis $B = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$

We have to solve equation $\mathbf{V}\mathbf{c} = \mathbf{x}$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|r} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & -1 \\ 3 & 5 & 6 & 2 \end{array} \right)$$

Example – Coordinate Vector

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 3 & 3 & -1 \\ 3 & 5 & 6 & 2 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 3 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 5 \end{array}\right)$$

Back substitution yields $c_1 = 4$, $c_2 = -8$ and $c_3 = 5$.

The coordinate vector of $\mathbf{x}$ w.r.t. basis B is thus

$$\mathbf{c}(\mathbf{x}) = \begin{pmatrix} 4 \\ -8 \\ 5 \end{pmatrix}$$

Alternatively we could compute $\mathbf{V}^{-1}$ and get as $\mathbf{c} = \mathbf{V}^{-1}\mathbf{x}$.

Change of Basis

Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be the coordinate vectors of $\mathbf{x} \in \mathcal{V}$ w.r.t. bases $B_1 = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ and $B_2 = \{\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n\}$, resp.

Consequently $\quad \mathbf{c}_2(\mathbf{x}) = \mathbf{W}^{-1}\mathbf{x} = \mathbf{W}^{-1}\mathbf{V}\mathbf{c_1}(\mathbf{x})$.

Such a transformation of a coordinate vector w.r.t. one basis into that of another one is called a **change of basis**.

Matrix

$$\boxed{\mathbf{U} = \mathbf{W}^{-1}\mathbf{V}}$$

is called the **transformation matrix** for this change from basis $\mathcal{B}_1$ to $\mathcal{B}_2$.

Example – Change of Basis

Let

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \\ 6 \end{pmatrix} \right\} \text{ and } B_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \right\}$$

two bases of $\mathbb{R}^3$.

Transformation matrix for the change of basis from B_1 to B_2:
$\mathbf{U} = \mathbf{W}^{-1} \cdot \mathbf{V}$.

$$\mathbf{W} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 3 \\ 3 & 5 & 6 \end{pmatrix} \qquad \Rightarrow \qquad \mathbf{W}^{-1} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\mathbf{V} = \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ 1 & 1 & 6 \end{pmatrix}$$

Example – Change of Basis

Transformation matrix for the change of basis from B_1 to B_2:

$$\mathbf{U} = \mathbf{W}^{-1} \cdot \mathbf{V} = \begin{pmatrix} 3 & -1 & 0 \\ -3 & 3 & -1 \\ 1 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 & 3 \\ 1 & 1 & 5 \\ 1 & 1 & 6 \end{pmatrix} = \begin{pmatrix} 2 & -7 & 4 \\ -1 & 8 & 0 \\ 0 & -3 & -1 \end{pmatrix}$$

Let $\mathbf{c}_1 = (3, 2, 1)^\mathsf{T}$ be the coordinate vector of $\mathbf{x}$ w.r.t. basis B_1.
Then the coordinate vector $\mathbf{c}_2$ w.r.t. basis B_2 is given by

$$\mathbf{c}_2 = \mathbf{U}\mathbf{c}_1 = \begin{pmatrix} 2 & -7 & 4 \\ -1 & 8 & 0 \\ 0 & -3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 13 \\ -7 \end{pmatrix}$$

Linear Map

A map φ from vector space $\mathcal{V}$ into $\mathcal{W}$

$$\varphi\colon \mathcal{V} \to \mathcal{W},\ \mathbf{x} \mapsto \mathbf{y} = \varphi(\mathbf{x})$$

is called **linear**, if for all $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ and $\alpha \in \mathbb{R}$

(i) $\varphi(\mathbf{x} + \mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y})$

(ii) $\varphi(\alpha\,\mathbf{x}) = \alpha\,\varphi(\mathbf{x})$

We already have seen such a map: $\mathcal{V} \to \mathbb{R}^n,\ \mathbf{x} \mapsto \mathbf{c}(\mathbf{x})$

Linear Map

Let $\mathbf{A}$ be an $m \times n$ matrix. Then map
$\varphi\colon \mathbb{R}^n \to \mathbb{R}^m,\ \mathbf{x} \mapsto \varphi_{\mathbf{A}}(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$ is linear:

$$\varphi_{\mathbf{A}}(\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{A} \cdot \mathbf{x} + \mathbf{A} \cdot \mathbf{y} = \varphi_{\mathbf{A}}(\mathbf{x}) + \varphi_{\mathbf{A}}(\mathbf{y})$$

$$\varphi_{\mathbf{A}}(\alpha\, \mathbf{x}) = \mathbf{A} \cdot (\alpha\, \mathbf{x}) = \alpha\, (\mathbf{A} \cdot \mathbf{x}) = \alpha\, \varphi_{\mathbf{A}}(\mathbf{x})$$

Vice versa every linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$ can be represented by an appropriate $m \times n$ matrix $\mathbf{A}_\varphi$: $\varphi(\mathbf{x}) = \mathbf{A}_\varphi\, \mathbf{x}$.

Matrices represent all possible linear maps $\mathbb{R}^n \to \mathbb{R}^m$.

More generally they represent linear maps between any vector space once we have bases for these and do all computations with their coordinate vectors.

In this sense, *matrices "are" linear maps*.

Geometric Interpretation of Linear Maps

We have the following “elementary” maps:

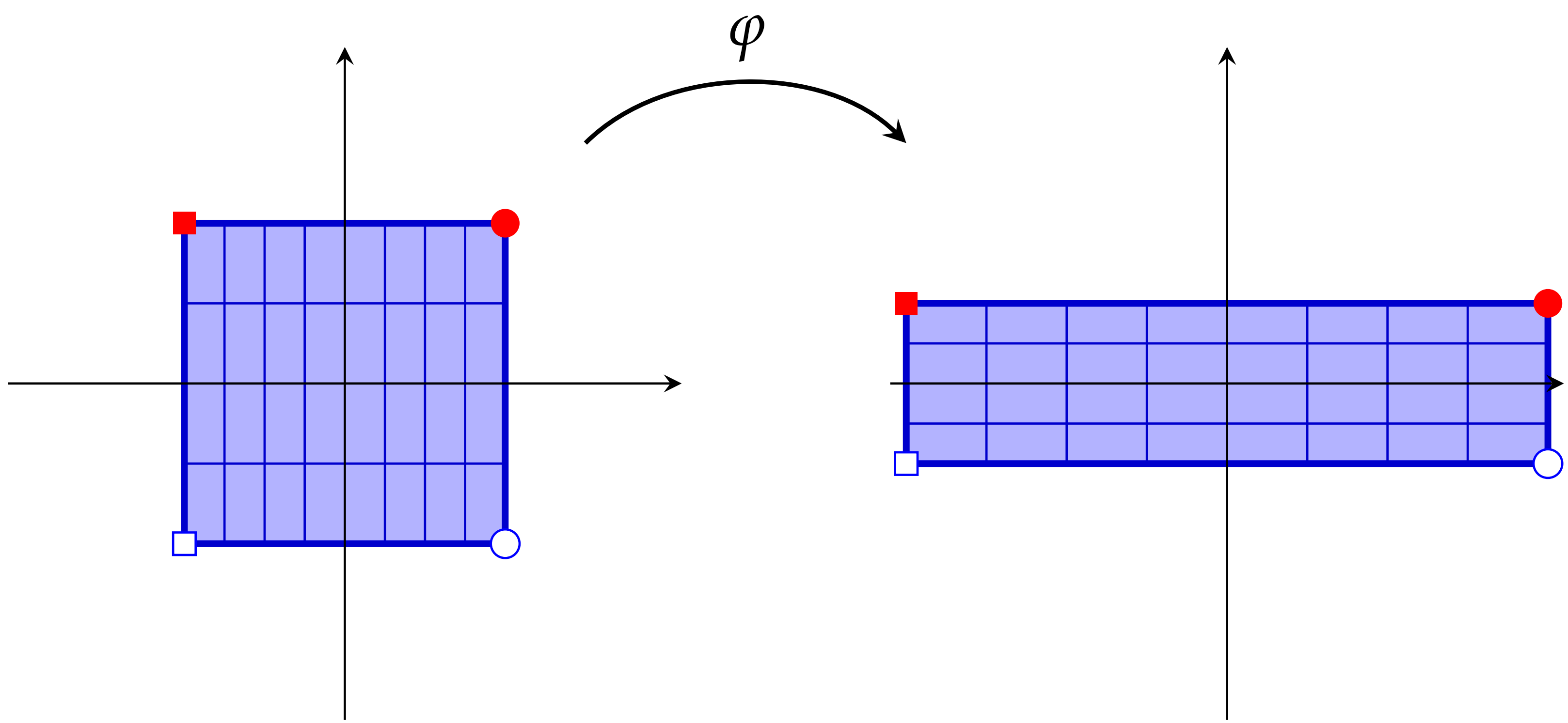
- *lengthening* / *shortening* in some direction
- *shear* in some direction
- *projection* into a subspace
- *rotation*
- *reflection* at a subspace

These maps can be combined into more complex ones.

Lengthening / Shortening

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} 2 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \mathbf{x}$

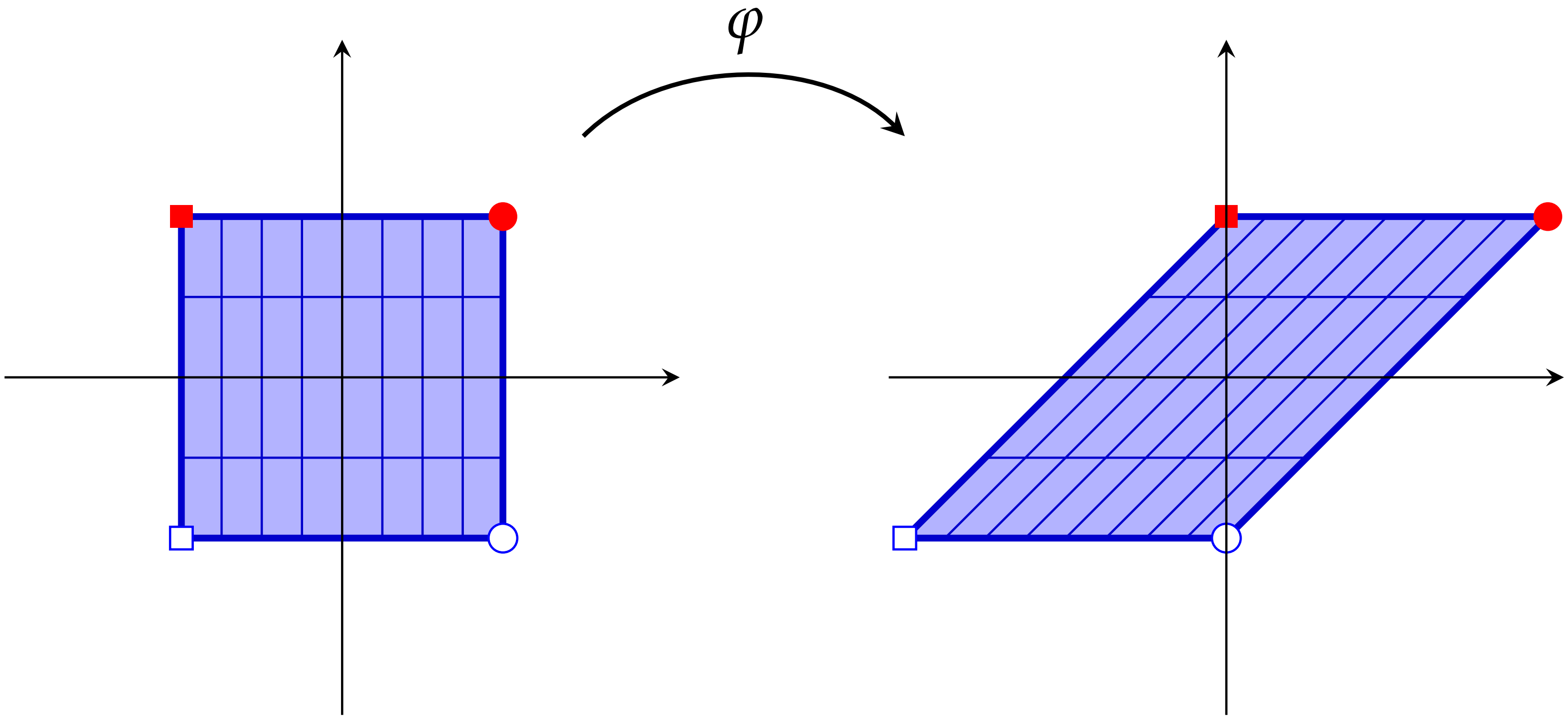
lengthens the x-coordinate by factor 2 and shortens the y-coordinate by factor $\frac{1}{2}$.



Shear

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

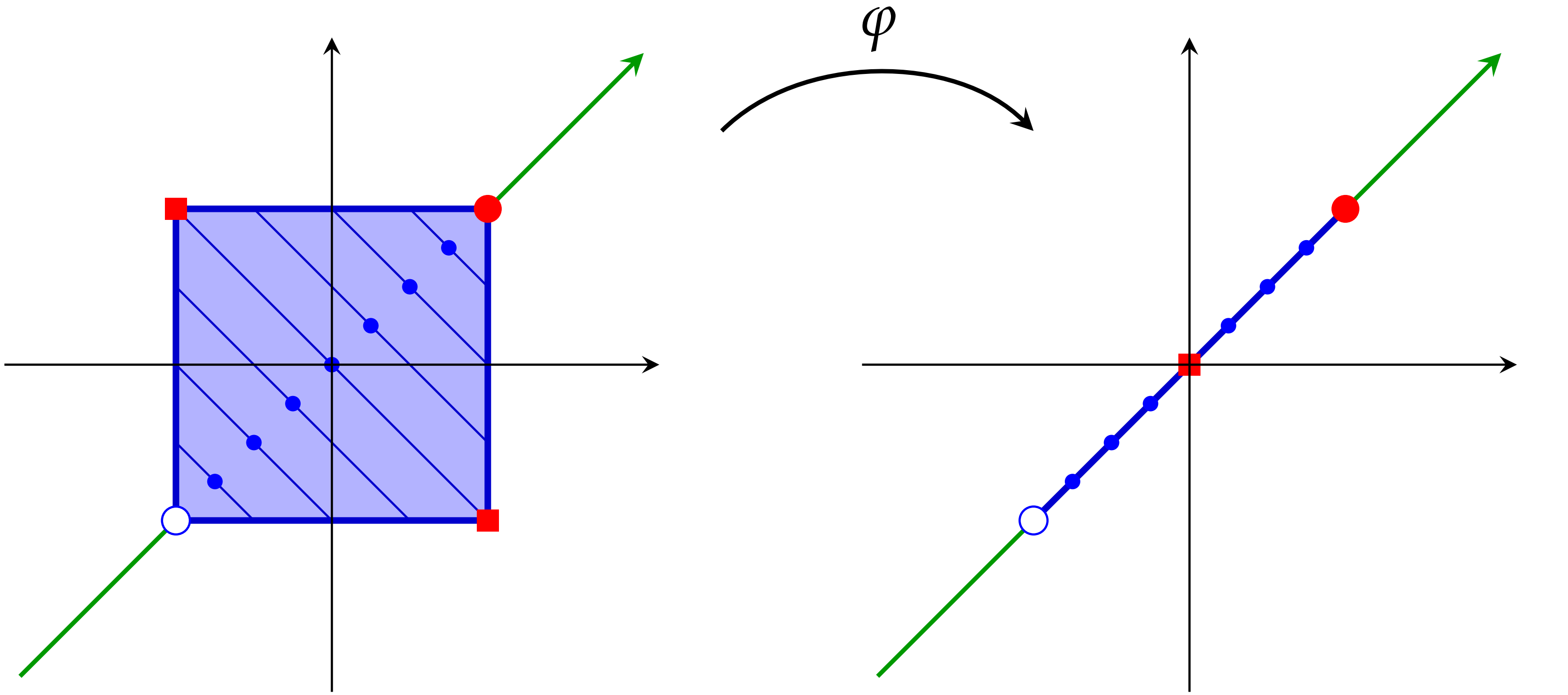
shears the rectangle into the x-coordinate.



Projection

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \mathbf{x}$

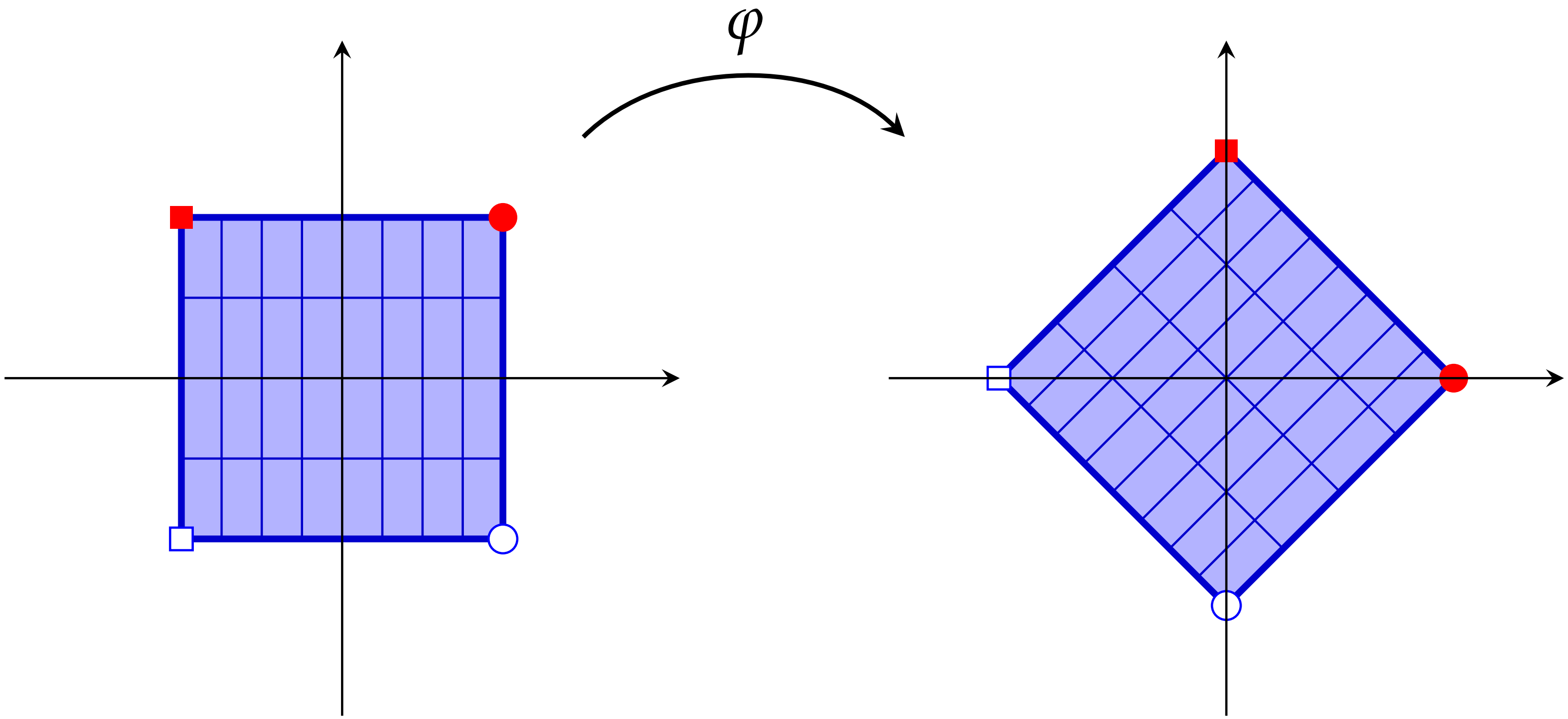
projects a point $\mathbf{x}$ orthogonally into the subspace generated by vector $(1,1)^\mathsf{T}$, i.e., $\operatorname{span}\left((1,1)^\mathsf{T}\right)$.



Rotation

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} \mathbf{x}$

rotates a point $\mathbf{x}$ clock-wise by 45° around the origin.



Reflection

Map $\varphi\colon \mathbf{x} \mapsto \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{x}$

reflects a point $\mathbf{x}$ at the y-axis.

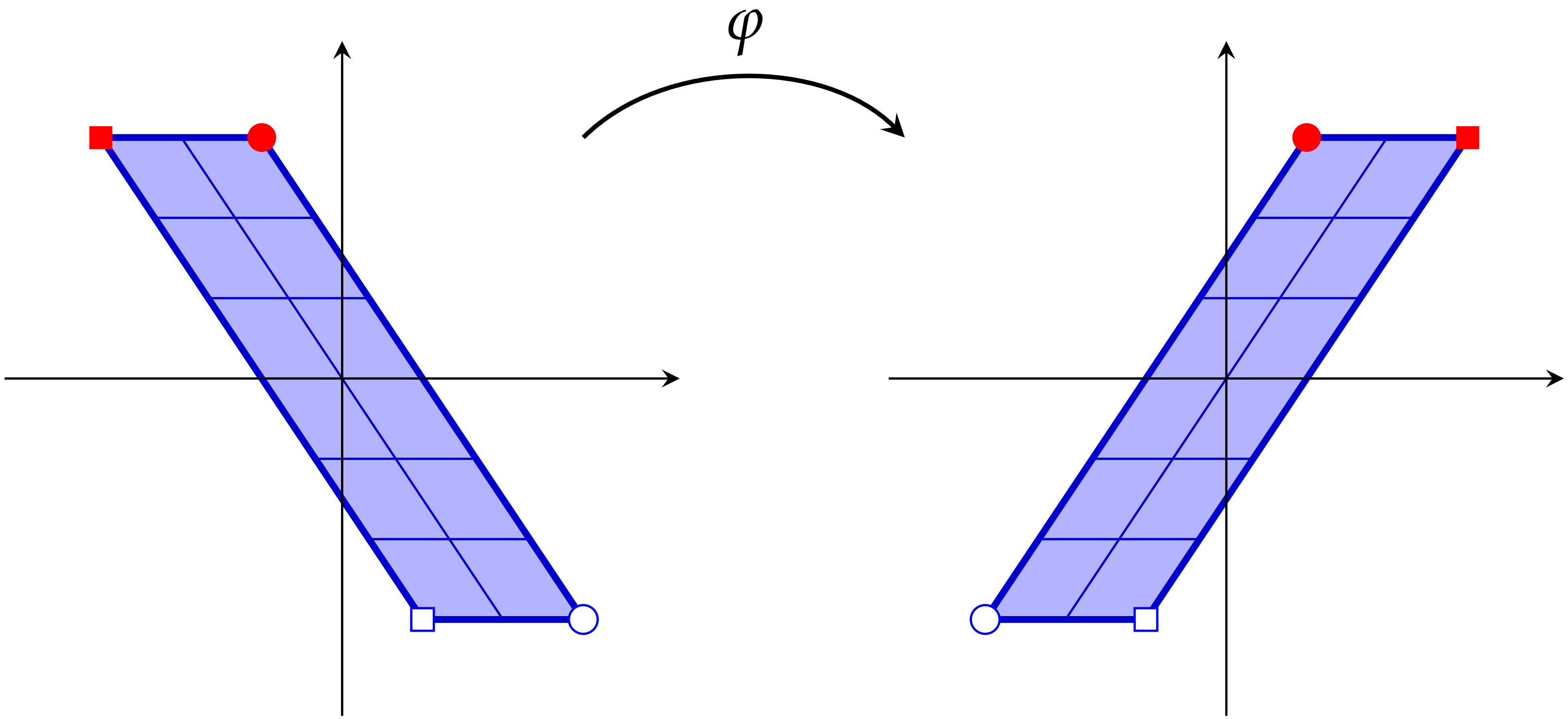


Image and Kernel

Let $\varphi\colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \varphi(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x}$ be a linear map.

The **image** of φ is a subspace of $\mathbb{R}^m$.

$$\boxed{\mathrm{Im}(\varphi) = \{\varphi(\mathbf{v})\colon \mathbf{v} \in \mathbb{R}^n\} \subseteq \mathbb{R}^m}$$

The **kernel** (or **null space**) of φ is a subspace of $\mathbb{R}^n$.

$$\boxed{\mathrm{Ker}(\varphi) = \{\mathbf{v} \in \mathbb{R}^n\colon \varphi(\mathbf{v}) = 0\} \subseteq \mathbb{R}^n}$$

The kernel is the preimage of 0.

Image $\mathrm{Im}(\mathbf{A})$ and *kernel* $\mathrm{Ker}(\mathbf{A})$ of a matrix $\mathbf{A}$ are the respective image and kernel of the corresponding linear map.

Generating Set of the Image

Let $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$, $\mathbf{x} \in \mathbb{R}^n$ an arbitrary vector, and $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

We can write $\mathbf{x}$ as a linear combination of the canonical basis:

$$\mathbf{x} = \sum_{i=1}^{n} x_i \, \mathbf{e}_i$$

Recall that $\mathbf{A}\mathbf{e}_i = \mathbf{a}_i$.
So we can write $\varphi(\mathbf{x})$ as a linear combination of the columns of $\mathbf{A}$:

$$\varphi(\mathbf{x}) = \mathbf{A} \cdot \mathbf{x} = \mathbf{A} \cdot \sum_{i=1}^{n} x_i \, \mathbf{e}_i = \sum_{i=1}^{n} x_i \, \mathbf{A}\mathbf{e}_i = \sum_{i=1}^{n} x_i \, \mathbf{a}_i$$

That is, the columns $\mathbf{a}_i$ of $\mathbf{A}$ span (generate) $\mathrm{Im}(\varphi)$.

Basis of the Kernel

Let $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ and $\varphi(\mathbf{x}) = \mathbf{A}\mathbf{x}$.

If $\mathbf{y}, \mathbf{z} \in \text{Ker}(\varphi)$ and $\alpha, \beta \in \mathbb{R}$, then

$$\varphi(\alpha\mathbf{y} + \beta\mathbf{z}) = \alpha\varphi(\mathbf{y}) + \beta\varphi(\mathbf{z}) = \alpha 0 + \beta 0 = 0$$

Thus $\text{Ker}(\varphi)$ is closed under linear combination,
i.e., $\text{Ker}(\varphi)$ is a subspace.

We obtain a basis of $\text{Ker}(\varphi)$ by solving the homogeneous linear equation $\mathbf{A} \cdot \mathbf{x} = 0$ by means of Gaussian elimination.

Dimension of Image and Kernel

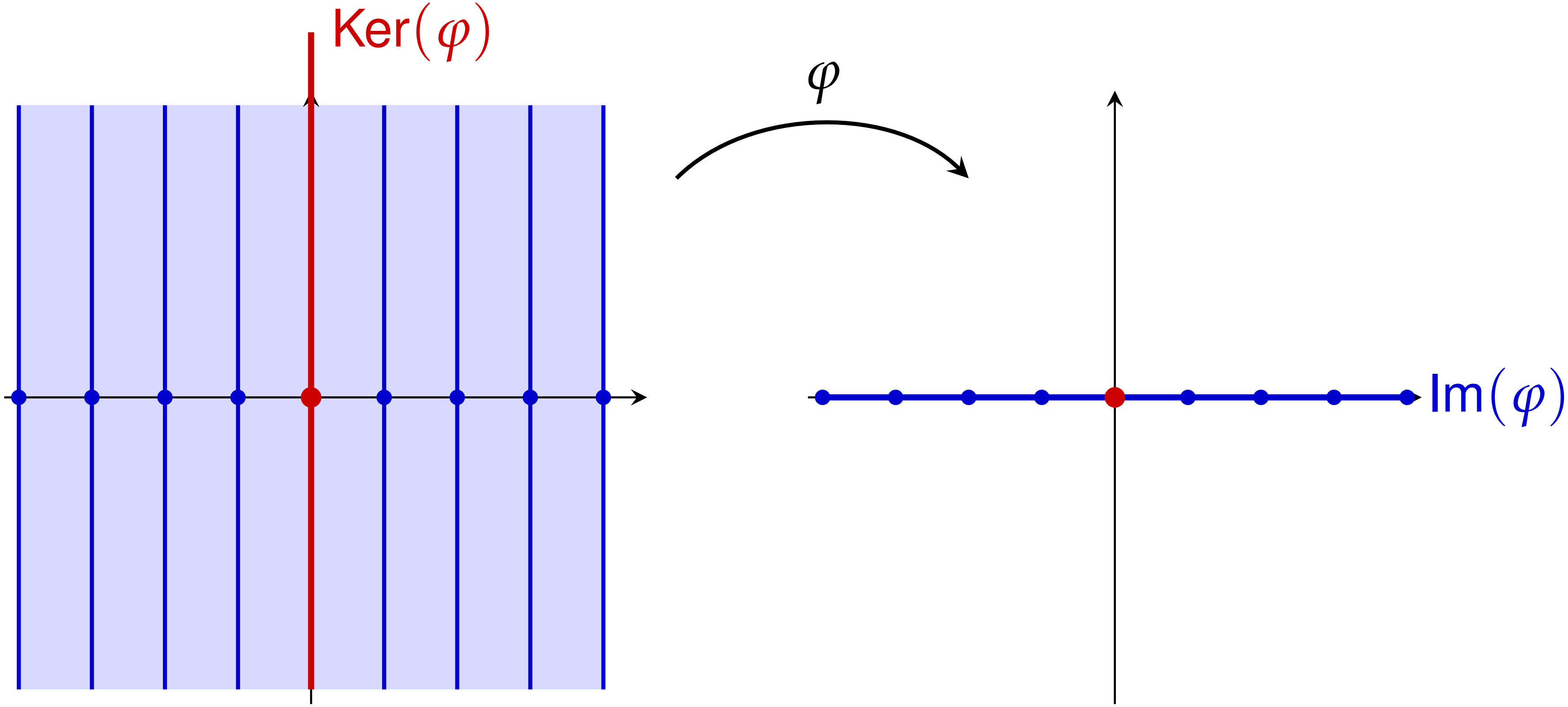
Rank-nullity theorem:

$$\boxed{\dim \mathcal{V} = \dim \mathsf{Im}(\varphi) + \dim \mathsf{Ker}(\varphi)}$$

Example – Dimension of Image and Kern

Map $\varphi\colon \mathbb{R}^2 \to \mathbb{R}^2,\ \mathbf{x} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbf{x}$

projects a point $\mathbf{x}$ orthogonally onto the x axis.



$\dim \mathbb{R}^2 = 2,\ \dim \mathsf{Ker}(\varphi) = 1$ $\qquad$ $\dim \mathsf{Im}(\varphi) = 1$

Linear Map and Rank

The *rank* of matrix $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ is (per definition) the dimension of $\text{span}(\mathbf{a}_1, \dots, \mathbf{a}_n)$.

Hence it is the dimension of the image of the corresponding linear map.

$$\boxed{\dim \text{Im}(\varphi_{\mathbf{A}}) = \text{rank}(\mathbf{A})}$$

The dimension of the solution set $\mathcal{L}$ of a homogeneous linear equation $\mathbf{A}\,\mathbf{x} = 0$ is then the kernel of this map.

$$\dim \mathcal{L} = \dim \text{Ker}(\varphi_{\mathbf{A}}) = \dim \mathbb{R}^n - \dim \text{Im}(\varphi_{\mathbf{A}}) = n - \text{rank}(\mathbf{A})$$

Matrix Multiplication

By *multiplying* two matrices $\mathbf{A}$ and $\mathbf{B}$ we obtain the matrix of a *compound* linear map:

$$(\varphi_{\mathbf{A}} \circ \varphi_{\mathbf{B}})(\mathbf{x}) = \varphi_{\mathbf{A}}(\varphi_{\mathbf{B}}(\mathbf{x})) = \mathbf{A}\,(\mathbf{B}\,\mathbf{x}) = (\mathbf{A}\cdot\mathbf{B})\,\mathbf{x}$$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\mathbf{B}"] \arrow[rr, bend left, "\mathbf{AB}"] & \mathbb{R}^m \arrow[r, "\mathbf{A}"] & \mathbb{R}^k \\
\mathbf{x} \arrow[r, mapsto] & \mathbf{Bx} \arrow[r, mapsto] & \mathbf{ABx}
\end{tikzcd}$$

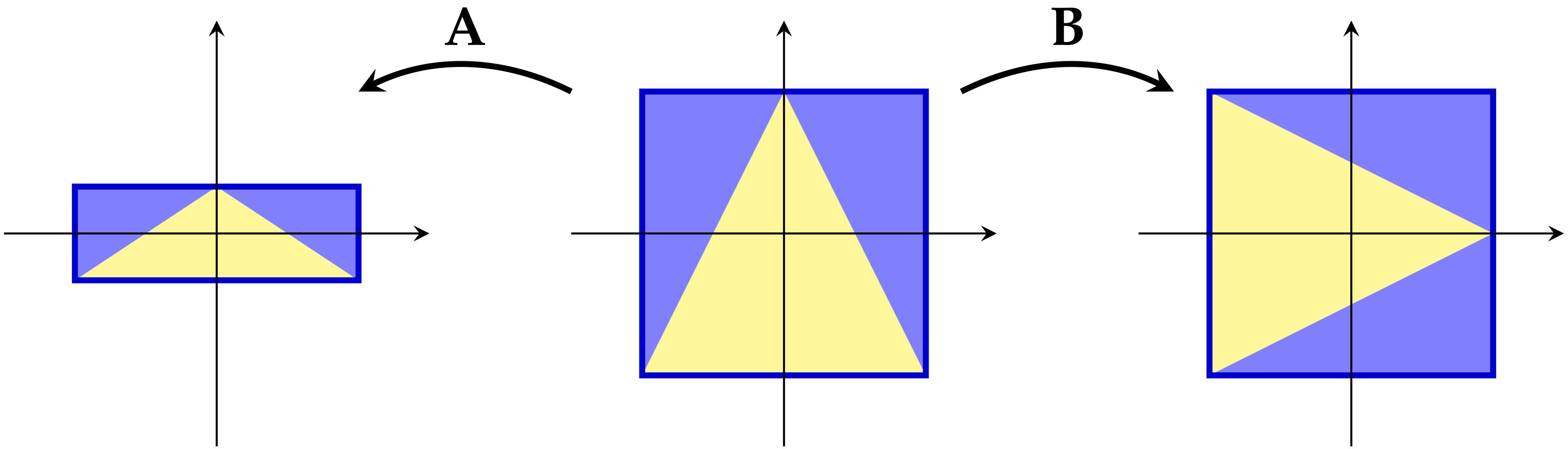
This point of view implies:

$$\boxed{\text{rank}(\mathbf{A}\cdot\mathbf{B}) \leq \min\{\text{rank}(\mathbf{A}), \text{rank}(\mathbf{B})\}}$$

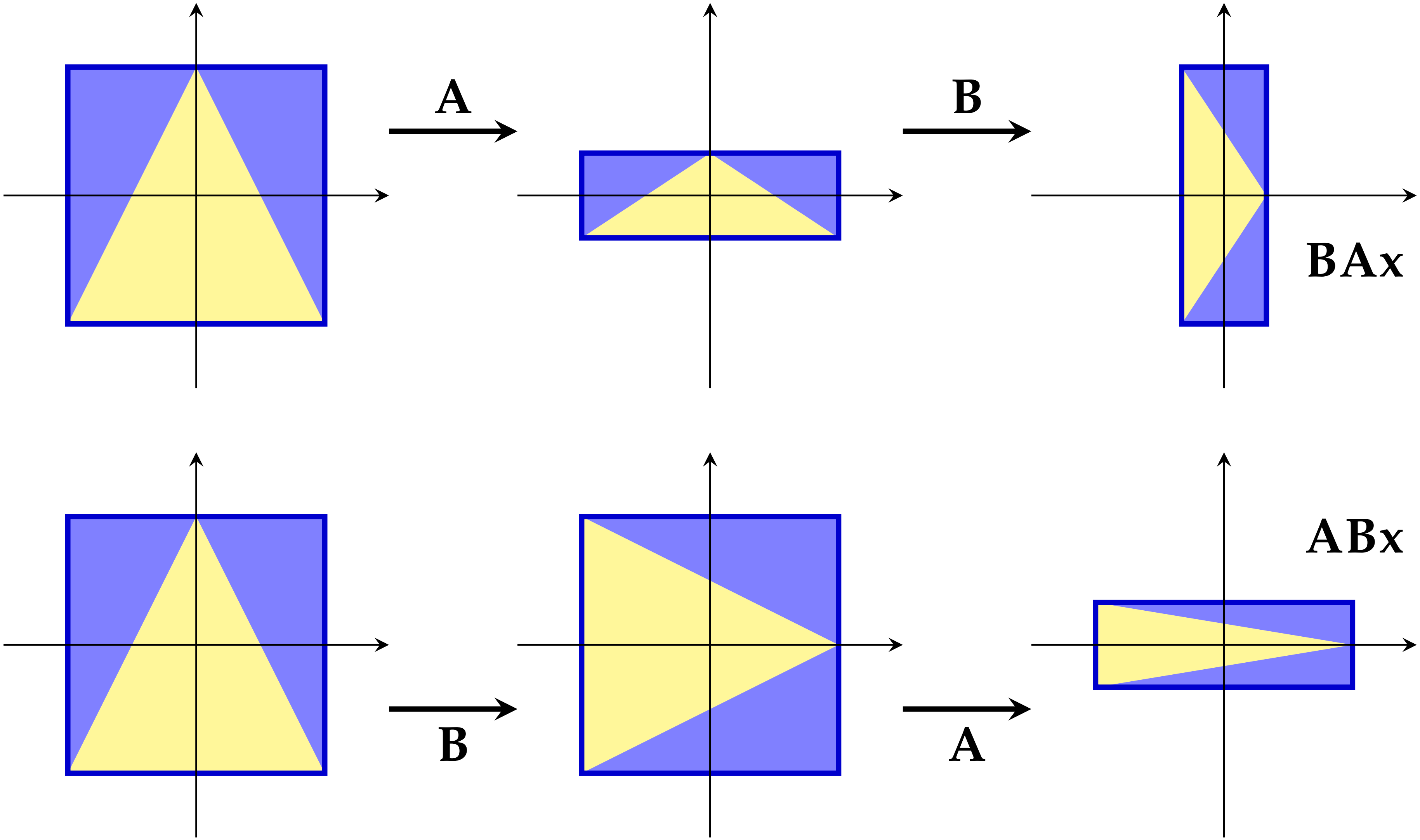
Non-Commutative Matrix Multiplication

$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ represents a shortening of the y-coordinate.

$\mathbf{B} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ represents a clock-wise rotation about 90°.



Non-Commutative Matrix Multiplication



Inverse Matrix

The *inverse matrix* $\mathbf{A}^{-1}$ of $\mathbf{A}$ exists if and only if map $\varphi_{\mathbf{A}}(\mathbf{x}) = \mathbf{A}\,\mathbf{x}$ is one-to-one and onto, i.e., if and only if

$$\varphi_{\mathbf{A}}(\mathbf{x}) = x_1\,\mathbf{a}_1 + \cdots + x_n\,\mathbf{a}_n = 0 \quad \Leftrightarrow \quad \mathbf{x} = 0$$

i.e., if and only if$\mathbf{A}$ is *regular*.

From this point of view implies $(\mathbf{A}\cdot\mathbf{B})^{-1} = \mathbf{B}^{-1}\cdot\mathbf{A}^{-1}$

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "\mathbf{B}"] \arrow[rr, bend left, "\mathbf{AB}"] & \mathbb{R}^m \arrow[r, "\mathbf{A}"] & \mathbb{R}^k \\
\mathbf{x} \arrow[r, mapsto] & \mathbf{Bx} \arrow[r, mapsto] & \mathbf{ABx} \\
\mathbf{B}^{-1}\mathbf{A}^{-1}\mathbf{z} & \mathbf{A}^{-1}\mathbf{z} \arrow[l, mapsto] & \mathbf{z} \arrow[l, mapsto] \arrow[ll, bend left, "\mathbf{B}^{-1}\mathbf{A}^{-1}"]
\end{tikzcd}$$

Similar Matrices

The basis of a vector space and thus the coordinates of a vector are not uniquely determined. Matrix $\mathbf{A}_\varphi$ of a linear map $\varphi\colon \mathbb{R}^n \to \mathbb{R}^n$ also depends on the chosen bases.
Let $\mathbf{A}$ be the matrix w.r.t. basis B_1.
Which matrix represents linear map φ if we use basis B_2 instead?

$$\begin{array}{lccc}
\text{basis } B_1 & \mathbf{U}\,\mathbf{x} & \xrightarrow{\mathbf{A}} & \mathbf{A}\,\mathbf{U}\,\mathbf{x} \\
& \mathbf{U}\uparrow & & \downarrow\mathbf{U}^{-1} \\
\text{basis } B_2 & \mathbf{x} & \xrightarrow{\mathbf{C}} & \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}\,\mathbf{x}
\end{array}
\qquad \text{and thus} \quad \mathbf{C}\,\mathbf{x} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}\,\mathbf{x}$$

Two $n \times n$ matrices $\mathbf{A}$ and $\mathbf{C}$ are called **similar**, if there exists a regular matrix $\mathbf{U}$ such that

$$\boxed{\mathbf{C} = \mathbf{U}^{-1}\,\mathbf{A}\,\mathbf{U}}$$

Summary

- vector space and subspace
- linear independency and rank
- basis and dimension
- coordinate vector
- change of basis
- linear map
- image and kernel
- similar matrices