Chapter 3

Linear Equations

System of Linear Equations

System of m linear equations in n unknowns:

$$\begin{array}{ccccccccc} a_{11}\,x_1 & + & a_{12}\,x_2 & + & \cdots & + & a_{1n}\,x_n & = & b_1 \\ a_{21}\,x_1 & + & a_{22}\,x_2 & + & \cdots & + & a_{2n}\,x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}\,x_1 & + & a_{m2}\,x_2 & + & \cdots & + & a_{mn}\,x_n & = & b_m \end{array}$$

$$\underbrace{\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}}_{\text{coefficient matrix}} \cdot \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{\text{variables}} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}}_{\text{vector of constants}}$$

$$\boxed{\mathbf{A} \cdot \mathbf{x} = \mathbf{b}}$$

Matrix Representation

Advantages of matrix representation:

- Short and compact notation.

  Compare
  $$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$
  to
  $$\sum_{j=1}^{n} a_{ij} x_j = b_i\,, \quad \text{for } i = 1, \dots, m$$

- We can transform equations by means of matrix algebra.
- We can use names for parts of the equation, like PRODUCTION VECTOR, DEMAND VECTOR, TECHNOLOGY MATRIX, etc. in the case of a Leontief model.

Leontief Model

Input-output model with

$\mathbf{A}$ ... technology matrix
$\mathbf{x}$ ... production vector
$\mathbf{b}$ ... demand vector

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

For a given output $\mathbf{b}$ we get the corresponding input $\mathbf{x}$ by

$$\begin{aligned} \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad & \mid \; -\mathbf{A}\mathbf{x} \\ \mathbf{x} - \mathbf{A}\mathbf{x} = \mathbf{b} \qquad & \\ (\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{b} \qquad & \mid \; (\mathbf{I} - \mathbf{A})^{-1}\cdot \\ \mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\mathbf{b} & \end{aligned}$$

Solutions of a System of Linear Equations

Three possibilities:

- The system of equations has *exactly one* solution.
- The system of equations is *inconsistent*(not solvable).
- The system of equations has *infinitely many* solutions.

In **Gaussian elimination** the augmented coefficient matrix $(\mathbf{A}, \mathbf{b})$ is transformed into **row echelon form**.

Then the solution set is obtained by **back substitution**.

It is not possible to determine the number of solutions from the numbers of equations and unknowns. We have to transform the system first.

Row Echelon Form

In **row echelon form** the number of *leading zeros* strictly increases from one row to the row below.

$$\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & 1 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & 0 & 1 & a_{35} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For our purposes it is not required that the first nonzero entries are equal to 1.

Steps in Gaussian Elimination

We (have to) obtain the row echelon form (only) by means of following transformations *which do not change the set of solutions*:

- Multiplication of a row by some *nonzero* constant.
- Addition of the multiple of some row to another row.
- Exchange of two rows.

Example – Gaussian Elimination

$$\begin{array}{rrr|r} 1.0 & -0.2 & -0.2 & 7.0 \\ -0.4 & 0.8 & -0.1 & 12.5 \\ 0.0 & -0.5 & 0.9 & 16.5 \end{array}$$

We first add 0.4 times the first row to the second row.
We denote this operation by

$$R_2 \leftarrow R_2 + 0.4 \times R_1$$

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & -0.50 & 0.90 & 16.5 \end{array}$$

Example – Gaussian Elimination

$$R_3 \leftarrow R_3 + \frac{0.5}{0.72} \times R_2$$

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & 0 & 0.775 & 27.125 \end{array}$$

Example – Back Substitution

$$\begin{array}{rrr|r} 1 & -0.20 & -0.20 & 7.0 \\ 0 & 0.72 & -0.18 & 15.3 \\ 0 & 0 & 0.775 & 27.125 \end{array}$$

From the third row we immediately get:

$$0.775 \cdot x_3 = 27.125 \quad \Rightarrow \quad x_3 = 35$$

We obtain the remaining variables x_2 and x_1 by **back substitution**:

$$0.72 \cdot x_2 - 0.18 \cdot 35 = 15.3 \quad \Rightarrow \quad x_2 = 30$$

$$x_1 - 0.2 \cdot 30 - 0.2 \cdot 35 = 7 \quad \Rightarrow \quad x_1 = 20$$

The solution is unique: $\mathbf{x} = (20, 30, 35)^\mathsf{T}$

Example 2

Find the solution of equation

$$\begin{aligned} 3\,x_1 + 4\,x_2 + 5\,x_3 &= 1 \\ x_1 + x_2 - x_3 &= 2 \\ 5\,x_1 + 6\,x_2 + 3\,x_3 &= 4 \end{aligned}$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 1 & 1 & -1 & 2 \\ 5 & 6 & 3 & 4 \end{array}$$

$$R_2 \leftarrow 3 \times R_2 - R_1, \quad R_3 \leftarrow 3 \times R_3 - 5 \times R_1$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 0 & -1 & -8 & 5 \\ 0 & -2 & -16 & 7 \end{array}$$

Example 2

$$R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrr|r} 3 & 4 & 5 & 1 \\ 0 & -1 & -8 & 5 \\ 0 & 0 & 0 & -3 \end{array}$$

The third row implies $0 = -3$, a **contradiction**.

This system of equations is **inconsistent**; solution set $L = \emptyset$.

Example 3

Find the solution of equation

$$\begin{aligned} 2\,x_1 + 8\,x_2 + 10\,x_3 + 10\,x_4 &= 0 \\ x_1 + 5\,x_2 + 2\,x_3 + 9\,x_4 &= 1 \\ -3\,x_1 - 10\,x_2 - 21\,x_3 - 6\,x_4 &= -4 \end{aligned}$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 1 & 5 & 2 & 9 & 1 \\ -3 & -10 & -21 & -6 & -4 \end{array}$$

$$R_2 \leftarrow 2 \times R_2 - R_1, \quad R_3 \leftarrow 2 \times R_3 + 3 \times R_1$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 4 & -12 & 18 & -8 \end{array}$$

Example 3

$$R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 0 & 0 & 2 & -12 \end{array}$$

This equation has **infinitely many** solutions.
This can be seen from the *row echelon form* as there are *more* variables than nonzero rows.

Example 3

The third row immediately implies

$$2 \cdot x_4 = -12 \quad \Rightarrow \quad x_4 = -6$$

Back substitution yields

$$2 \cdot x_2 - 6 \cdot x_3 + 8 \cdot (-6) = 2$$

In this case we use *pseudo solution* $x_3 = \alpha$, $\alpha \in \mathbb{R}$, and get

$$x_2 - 3 \cdot \alpha + 4 \cdot (-6) = 1 \quad \Rightarrow \quad x_2 = 25 + 3\,\alpha$$

$$2 \cdot x_1 + 8 \cdot (25 + 3 \cdot \alpha) + 10 \cdot \alpha + 10 \cdot (-6) = 0$$

$$\Rightarrow \quad x_1 = -70 - 17 \cdot \alpha$$

Example 3

We obtain a solution for each value of α. Using vector notation we obtain

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -70 - 17 \cdot \alpha \\ 25 + 3\,\alpha \\ \alpha \\ -6 \end{pmatrix} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

Thus the solution set of this equation is

$$L = \left\{ \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

Equivalent Representation of Solutions

In Example 3 we also could use $x_2 = \alpha'$ (instead of $x_3 = \alpha$).
Then back substitution yields

$$L' = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -\frac{215}{3} \\ 0 \\ -\frac{25}{3} \\ -6 \end{pmatrix} + \alpha' \begin{pmatrix} -\frac{17}{3} \\ 1 \\ \frac{1}{3} \\ 0 \end{pmatrix} \;\middle|\; \alpha \in \mathbb{R} \right\}$$

However, these two solution sets are equal, $\quad L' = L$!

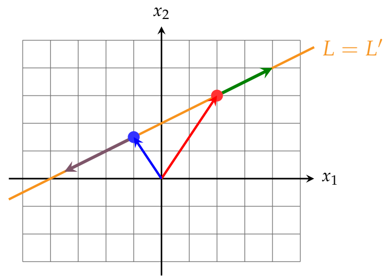
We thus have two different – *but equivalent* – representations of the same set.

The solution set is unique, its representation is not!

Equivalent Representation of Solutions

The set of solution points in Example 3 can be interpreted as a *line* in a (4-dimensional) space.

The representations in L and L' are thus parametric curves in $\mathbb{R}^4$ with the same image.



A Non-Example

Find the solution of equation

$$\begin{aligned} 2\,x_1 + x_2 \phantom{{}-2\,x_3} &= 1 \\ -2\,x_1 + 2\,x_2 - 2\,x_3 &= 4 \\ 4\,x_1 + 9\,x_2 - 3\,x_3 &= 9 \end{aligned}$$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ -2 & 2 & -2 & 4 \\ 4 & 9 & -3 & 9 \end{array}$$

$$R_2 \leftarrow R_2 + R_1, \quad R_3 \leftarrow R_3 - 2 \times R_1$$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ 0 & 7 & -3 & 7 \end{array}$$

A Non-Example

Now one could find $R_3 \leftarrow R_3 - 7 \times R_1$ convenient. However,

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ -14 & 0 & -3 & 0 \end{array}$$

destroys the already created row echelon form in the first column!

Much better: $R_3 \leftarrow 3 \times R_3 - 7 \times R_2$

$$\begin{array}{rrr|r} 2 & 1 & 0 & 1 \\ 0 & 3 & -2 & 5 \\ 0 & 0 & 5 & -14 \end{array}$$

Reduced Row Echelon Form

In **Gauss-Jordan elimination** the augmented matrix is transformed into **reduce row echelon form**, i.e.,

- It is in row echelon form.
- The leading entry in each nonzero row is a 1.
- Each column containing a leading 1 has 0s everywhere else.

$$\begin{pmatrix} 1 & 0 & a_{13} & 0 & 0 \\ 0 & 1 & a_{23} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Back substitution is then simpler.

Gauss-Jordan Elimination / Example 3

Find the solution of equation

$$\begin{aligned} 2\,x_1 + 8\,x_2 + 10\,x_3 + 10\,x_4 &= 0 \\ x_1 + 5\,x_2 + 2\,x_3 + 9\,x_4 &= 1 \\ -3\,x_1 - 10\,x_2 - 21\,x_3 - 6\,x_4 &= -4 \end{aligned}$$

$$\begin{array}{rrrr|r} 2 & 8 & 10 & 10 & 0 \\ 1 & 5 & 2 & 9 & 1 \\ -3 & -10 & -21 & -6 & -4 \end{array}$$

$$R_1 \leftarrow \frac{1}{2} \times R_1, \quad R_2 \leftarrow 2 \times R_2 - R_1, \quad R_3 \leftarrow 2 \times R_3 + 3 \times R_1$$

$$\begin{array}{rrrr|r} 1 & 4 & 5 & 5 & 0 \\ 0 & 2 & -6 & 8 & 2 \\ 0 & 4 & -12 & 18 & -8 \end{array}$$

Gauss-Jordan Elimination / Example 3

$$R_1 \leftarrow R_1 - 2 \times R_2, \quad R_2 \leftarrow \frac{1}{2} \times R_2, \quad R_3 \leftarrow R_3 - 2 \times R_2$$

$$\begin{array}{rrrr|r} 1 & 0 & 17 & -11 & -4 \\ 0 & 1 & -3 & 4 & 1 \\ 0 & 0 & 0 & 2 & -12 \end{array}$$

$$R_1 \leftarrow R_1 + \frac{11}{2} \times R_3, \quad R_2 \leftarrow R_2 - 2 \times R_3, \quad R_3 \leftarrow \frac{1}{2} \times R_3,$$

$$\begin{array}{rrrr|r} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}$$

Gauss-Jordan Elimination / Example 3

$$\begin{array}{rrrr|r} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}$$

The third row immediately implies $x_4 = -6$

Set *pseudo solution* $x_3 = \alpha$, $\alpha \in \mathbb{R}$.

Back substitution yields $x_2 = 25 + 3\,\alpha$

and $x_1 = -70 - 17 \cdot \alpha$

Gauss-Jordan Elimination / Example 3

Thus the solution set of this equation is

$$L = \left\{ \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix} \,\middle|\, \alpha \in \mathbb{R} \right\}$$

Gauss-Jordan Elimination / Example 3

Compare

$$\left(\begin{array}{cccc|c} 1 & 0 & 17 & 0 & -70 \\ 0 & 1 & -3 & 0 & 25 \\ 0 & 0 & 0 & 1 & -6 \end{array}\right) \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} -70 \\ 25 \\ 0 \\ -6 \end{pmatrix} + \alpha \begin{pmatrix} -17 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

The positional vector $(-70, 25, 0, -6)^\mathsf{T}$ follows from the r.h.s. of the reduced row echelon while the direction vector $(-17, 3, 1, 0)^\mathsf{T}$ is given by the column without leading 1.

Inverse of a Matrix

Computation of the inverse $\mathbf{A}^{-1}$ of matrix $\mathbf{A}$ by *Gauss-Jordan elimination*:

(1) Augment matrix $\mathbf{A}$ by the corresponding *identity matrix* to the right.

(2) Transform the augmented matrix such that the identity matrix appears on the left hand side by means of the transformation steps of Gaussian elimination.

(3) Either the procedure is successful. Then we obtain the *inverse matrix* $\mathbf{A}^{-1}$ on the right hand side.

(4) Or the procedure aborts (because we obtain a row of zeros on the l.h.s.). Then the matrix is *singular*.

Example 1

Compute the inverse of

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 3 \\ -3 & -2 & -5 \end{pmatrix}$$

(1) Augment matrix $\mathbf{A}$:

$$\left(\begin{array}{ccc|ccc} 3 & 2 & 6 & 1 & 0 & 0 \\ 1 & 1 & 3 & 0 & 1 & 0 \\ -3 & -2 & -5 & 0 & 0 & 1 \end{array}\right)$$

Example 1

(2) Transform:

$$R_1 \leftarrow \tfrac{1}{3} \times R_1, \quad R_2 \leftarrow 3 \times R_2 - R_1, \quad R_3 \leftarrow R_3 + R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{2}{3} & 2 & \frac{1}{3} & 0 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \tfrac{2}{3} \times R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 3 & -1 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

Example 1

$$R_2 \leftarrow R_2 - 3 \times R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -4 & 3 & -3 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{array}\right)$$

(3) Matrix $\mathbf{A}$ is invertible with inverse

$$\mathbf{A}^{-1} = \begin{pmatrix} 1 & -2 & 0 \\ -4 & 3 & -3 \\ 1 & 0 & 1 \end{pmatrix}$$

Example 2

Compute the inverse of

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 3 \\ 2 & 4 & 1 \\ 5 & 5 & 4 \end{pmatrix}$$

(1) Augment matrix $\mathbf{A}$:

$$\left(\begin{array}{ccc|ccc} 3 & 1 & 3 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ 5 & 5 & 4 & 0 & 0 & 1 \end{array}\right)$$

Example 2

(2) Transform:

$$R_1 \leftarrow \tfrac{1}{3} \times R_1, \quad R_2 \leftarrow 3 \times R_2 - 2 \times R_1, \quad R_3 \leftarrow 3 \times R_3 - 5 \times R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & \frac{1}{3} & 1 & \frac{1}{3} & 0 & 0 \\ 0 & 10 & -3 & -2 & 3 & 0 \\ 0 & 10 & -3 & -5 & 0 & 5 \end{array}\right)$$

$$R_1 \leftarrow R_1 - \tfrac{1}{30} \times R_2, \quad R_2 \leftarrow \tfrac{1}{10} \times R_2, \quad R_3 \leftarrow R_3 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & \frac{11}{10} & \frac{4}{10} & -\frac{1}{10} & 0 \\ 0 & 1 & -\frac{3}{10} & -\frac{2}{10} & \frac{3}{10} & 0 \\ 0 & 0 & 0 & -3 & -3 & 5 \end{array}\right)$$

(4) Matrix $\mathbf{A}$ is *not* invertible.

Summary

- system of linear equations
- Gaussian elimination
- Gauss-Jordan elimination
- computation of inverse matrix