Chapter 2

Matrix Algebra

A Very Simplistic Leontief Model

A community operates the services PUBLIC TRANSPORT, ELECTRICITY and GAS.

Technology matrix and weekly demand (in unit values):

for / expenditure of	transport	electricity	gas	demand
transport	0.0	0.2	0.2	7.0
electricity	0.4	0.2	0.1	12.5
gas	0.0	0.5	0.1	16.5

What is the weekly production that satisfies the demand (but does not create excess)?

A Very Simplistic Leontief Model

We denote the unknown units of production of TRANSPORT, ELECTRICITY and GAS by x_1, x_2,and x_3, resp.
For our production we must have:

$$\begin{aligned} \text{demand} &= \text{ production} - \text{internal expenditur} \\ 7.0 &= x_1 - (0.0\,x_1 + 0.2\,x_2 + 0.2\,x_3) \\ 12.5 &= x_2 - (0.4\,x_1 + 0.2\,x_2 + 0.1\,x_3) \\ 16.5 &= x_3 - (0.0\,x_1 + 0.5\,x_2 + 0.1\,x_3) \end{aligned}$$

Transformation into an equivalent system of equations yields:

$$\begin{aligned} 1.0\,x_1 - 0.2\,x_2 - 0.2\,x_3 &= 7.0 \\ -0.4\,x_1 + 0.8\,x_2 - 0.1\,x_3 &= 12.5 \\ 0.0\,x_1 - 0.5\,x_2 + 0.9\,x_3 &= 16.5 \end{aligned}$$

Which values for x_1, x_2, and x_3 solves these equations simultaneously?

Matrix

An $\boldsymbol{m \times n}$ **matrix** is a rectangular array of mathematical expressions (e.g., numbers) that consists of m *rows* and n *columns*.

$$\boxed{\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = (a_{ij})}$$

Alternative notation: square brackets $[a_{ij}]$.

The terms a_{ij} are called **elements** or **coefficients** of matrix $\mathbf{A}$, the integers i and j are called **row index** and **column index**, resp.

Matrices are denoted by bold upper case Latin letters, its coefficients by the corresponding lower case Latin letters.

Vector

- A (column) **vector** is an $n \times 1$ matrix: $\boxed{\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}}$
- A **row vector** is a $1 \times n$-Matrix: $\boxed{\mathbf{x}^\mathsf{T} = (x_1, \dots, x_n)}$
- The i-th **unit vector** $\mathbf{e}_i$ is a vector where the i-th component is equal to 1 and all other components are 0.

Vectors are denoted by bold *lower case* Latin letters.

We write $\mathbf{A} = (\mathbf{a}_1, \dots, \mathbf{a}_n)$ for a matrix with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$.

Elements of a Matrix

We use the symbol

$$\boxed{[\mathbf{A}]_{ij} = a_{ij}}$$

to denote the coefficient with respective row and column index i and j.

The convenient symbol

$$\boxed{\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}}$$

is called the **Kronecker symbol**.

Example of its usage: $[\mathbf{I}]_{ij} = \delta_{ij}$.

Special Matrices

- An $n \times n$ matrix is called **square matrix**.
- An **upper triangular matrix** is a square matrix where all elements *below* the main diagonal are zero.

$$\mathbf{U} = \begin{pmatrix} -1 & -3 & 1 \\ 0 & 2 & 3 \\ 0 & 0 & -2 \end{pmatrix}$$

Formally:
Matrix $\mathbf{U}$ is an upper triangular matrix if

$$\boxed{[\mathbf{U}]_{ij} = 0 \text{ whenever } i > j.}$$

Special Matrices

- A **lower triangular matrix** is a square matrix where all elements *above* the main diagonal are zero.

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 4 & 0 \end{pmatrix}$$

Formally:
Matrix $\mathbf{L}$ is a lower triangular matrix if

$$\boxed{[\mathbf{L}]_{ij} = 0 \text{ whenever } i < j.}$$

Special Matrices

- A **diagonal matrix** is a square matrix where all elements outside the main diagonal are zero.

$$\mathbf{D} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Formally:
Matrix $\mathbf{D}$ is a diagonal matrix if

$$\boxed{[\mathbf{D}]_{ij} = 0 \text{ whenever } i \neq j.}$$

Special Matrices

- A matrix where all its coefficients are zero is called a **zero matrix** and is denoted by $\mathbf{O}_{n,m}$ or $\mathbf{0}$.
- An **identity matrix** is a diagonal matrix where all its diagonal entries are equal to 1. It is denoted by $\mathbf{I}_n$ or $\mathbf{I}$.
(In German literature also symbol $\mathbf{E}$ is used.)

$$\mathbf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Remark: Both identity matrix $\mathbf{I}_n$ and zero matrix $\mathbf{O}_{n,n}$ are examples of upper and lower triangular matrices and of a diagonal matrix.

Transposed Matrix

We get the **transposed** $\mathbf{A}^\mathsf{T}$ of matrix $\mathbf{A}$ by exchanging rows and columns:

$$\boxed{\left[\mathbf{A}^\mathsf{T}\right]_{ij} = [\mathbf{A}]_{ji}}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$$

Alternative notation: $\mathbf{A}'$

Symmetric Matrix

A matrix $\mathbf{A}$ is called symmetric if

$$\boxed{\mathbf{A}^\mathsf{T} = \mathbf{A}}$$

i.e., if

$$\boxed{[\mathbf{A}]_{ij} = [\mathbf{A}]_{ji} \quad \text{for all } i, j.}$$

Obviously every symmetric matrix is a square matrix.

Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{pmatrix}$ is symmetric.

Scalar Multiplication

A matrix $\mathbf{A}$ can be multiplied by a constant (scalar) $\alpha \in \mathbb{R}$ *component-wise*:

$$\boxed{[\alpha \cdot \mathbf{A}]_{ij} = \alpha\, [\mathbf{A}]_{ij}}$$

$$3 \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 9 & 12 \end{pmatrix}$$

Addition of Matrices

Two $m \times n$ matrices $\mathbf{A}$ and $\mathbf{B}$ are added *component-wise*:

$$\boxed{[\mathbf{A} + \mathbf{B}]_{ij} = [\mathbf{A}]_{ij} + [\mathbf{B}]_{ij}}$$

Addition of two matrices is only possible if their numbers of rows and columns coincide!

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{pmatrix} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}$$

Multiplication of Matrices

The *product* $\mathbf{A} \cdot \mathbf{B}$ of two matrices $\mathbf{A}$ and $\mathbf{B}$ is defined only if the number of columns of the first factor $\mathbf{A}$ coincides with the number of rows of the second factor $\mathbf{B}$.

That is, if $\mathbf{A}$ is an $m \times n$ matrix, then $\mathbf{B}$ must be an $n \times k$ matrix.
The product $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$ then is an $m \times k$ matrix.

Element $[\mathbf{A} \cdot \mathbf{B}]_{ij}$ is then the product of the ith row of $\mathbf{A}$ and the jth column of $\mathbf{B}$ (in the sense of a scalar product):

$$\boxed{[\mathbf{A} \cdot \mathbf{B}]_{ij} = \sum_{s=1}^{n} a_{is} \cdot b_{sj}}$$

Matrix multiplication is **not commutative!**

Falk's Scheme

$$\begin{array}{ccc|cc}
\multicolumn{3}{c|}{\mathbf{A} \cdot \mathbf{B} \rightarrow} & 1 & 2 \\
\multicolumn{3}{c|}{\downarrow} & 3 & 4 \\
 & & & 5 & 6 \\
\hline
1 & 2 & 3 & c_{11} & c_{12} \\
4 & 5 & 6 & c_{21} & c_{22} \\
7 & 8 & 9 & c_{31} & c_{32}
\end{array}$$

$$c_{21} = 1 \cdot 4 + 5 \cdot 3 + 6 \cdot 5 = 49$$

$$\mathbf{A} \cdot \mathbf{B} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \\ 76 & 100 \end{pmatrix}$$

Non-Commutativity

Beware!

Matrix multiplication is **not commutative!**

In general we have

$$\boxed{\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}}$$

Non-Commutativity

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \quad \text{is not defined}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \\ 29 & 40 & 51 \end{pmatrix}$$

while

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 22 & 28 \\ 49 & 64 \end{pmatrix}$$

Non-Commutativity

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 10 & 13 \\ 22 & 29 \end{pmatrix}$$

while

$$\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 11 & 16 \\ 19 & 28 \end{pmatrix}$$

Powers of a Matrix

$$\begin{aligned} \mathbf{A}^2 &= \mathbf{A} \cdot \mathbf{A} \\ \mathbf{A}^3 &= \mathbf{A} \cdot \mathbf{A} \cdot \mathbf{A} \\ &\vdots \\ \mathbf{A}^n &= \underbrace{\mathbf{A} \cdot \ldots \cdot \mathbf{A}}_{n \text{ times}} \end{aligned}$$

Inverse Matrix

Let $\mathbf{A}$ be some square matrix.
If there exists a matrix $\mathbf{A}^{-1}$ with property

$$\boxed{\mathbf{A} \cdot \mathbf{A}^{-1} = \mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I}}$$

then $\mathbf{A}^{-1}$ is called the **inverse matrix** of $\mathbf{A}$.

Matrix $\mathbf{A}$ is called **invertible** if it has an *inverse* matrix.
Otherwise it is called **singular**.

Beware!
Our definition implies that every invertible matrix must be a *square* matrix.

Remark: For any two square matrices $\mathbf{A}$ and $\mathbf{B}$,

$\mathbf{A} \cdot \mathbf{B} = \mathbf{I}$ implies $\mathbf{B} \cdot \mathbf{A} = \mathbf{I}$.

Calculation Rules for Matrices

$$\begin{gathered} \mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \\ (\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \\ \mathbf{A} + \mathbf{0} = \mathbf{A} \\[1em] (\mathbf{A} \cdot \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot (\mathbf{B} \cdot \mathbf{C}) \\ \mathbf{I} \cdot \mathbf{A} = \mathbf{A} \cdot \mathbf{I} = \mathbf{A} \\[1em] (\alpha\, \mathbf{A}) \cdot \mathbf{B} = \alpha (\mathbf{A} \cdot \mathbf{B}) \\ \mathbf{A} \cdot (\alpha\, \mathbf{B}) = \alpha (\mathbf{A} \cdot \mathbf{B}) \\[1em] \mathbf{C} \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{C} \cdot \mathbf{A} + \mathbf{C} \cdot \mathbf{B} \\ (\mathbf{A} + \mathbf{B}) \cdot \mathbf{D} = \mathbf{A} \cdot \mathbf{D} + \mathbf{B} \cdot \mathbf{D} \end{gathered}$$

$\mathbf{A}$ and $\mathbf{B}$ invertible
$\Rightarrow$ $\mathbf{A} \cdot \mathbf{B}$ invertible

$$\begin{gathered} (\mathbf{A} \cdot \mathbf{B})^{-1} = \mathbf{B}^{-1} \cdot \mathbf{A}^{-1} \\ (\mathbf{A}^{-1})^{-1} = \mathbf{A} \\[1em] (\mathbf{A} \cdot \mathbf{B})^{\mathsf{T}} = \mathbf{B}^{\mathsf{T}} \cdot \mathbf{A}^{\mathsf{T}} \\ (\mathbf{A}^{\mathsf{T}})^{\mathsf{T}} = \mathbf{A} \\ (\mathbf{A}^{\mathsf{T}})^{-1} = (\mathbf{A}^{-1})^{\mathsf{T}} \end{gathered}$$

Beware!
In general we have
$\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$

Computations with Matrices

For *appropriate* matrices we have similar calculation rules as for real numbers.
However, we have to keep in mind:

- A *zero matrix* $\mathbf{0}$ is the analog to number 0.
- An *identity matrix* $\mathbf{I}$ corresponds to number 1.
- Matrix multiplication is **not commutative!**
  In general we have $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$.
- There is **no** such thing like **division** by matrices!
  Use multiplication by the *inverse matrix* instead.

Example – Computations with Matrices

$$(\mathbf{A} + \mathbf{B})^2 = (\mathbf{A} + \mathbf{B}) \cdot (\mathbf{A} + \mathbf{B}) = \mathbf{A}^2 + \mathbf{A} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{A} + \mathbf{B}^2$$

$$\begin{aligned} &\mathbf{A}^{-1} \cdot (\mathbf{A} + \mathbf{B}) \cdot \mathbf{B}^{-1}\, \mathbf{x} = \\ &\quad = (\mathbf{A}^{-1} \cdot \mathbf{A} + \mathbf{A}^{-1} \mathbf{B}) \cdot \mathbf{B}^{-1}\, \mathbf{x} \\ &\quad = (\mathbf{I} + \mathbf{A}^{-1} \mathbf{B}) \cdot \mathbf{B}^{-1}\, \mathbf{x} = \\ &\quad = (\mathbf{B}^{-1} + \mathbf{A}^{-1} \cdot \mathbf{B}\, \mathbf{B}^{-1}) \mathbf{x} \\ &\quad = (\mathbf{B}^{-1} + \mathbf{A}^{-1}) \mathbf{x} \\ &\quad = \mathbf{B}^{-1}\, \mathbf{x} + \mathbf{A}^{-1}\, \mathbf{x} \end{aligned}$$

Equations with Matrices

If we multiply an equation with matrices by some matrix $\mathbf{A}$ we have to take care that multiplication is *not commutative*.
That is, $\mathbf{A}$ must be either the first or the second factor of the multiplication on either side of the equality sign!

Beware!
There is **no** such thing like **division** by matrices!

We have to multiply by the inverse matrix instead.

Example – Equations with Matrices

Let $\mathbf{B} + \mathbf{A}\,\mathbf{X} = 2\mathbf{A}$ where $\mathbf{A}$ and $\mathbf{B}$ are known matrices.
Find matrix $\mathbf{X}$?

$$\begin{aligned} \mathbf{B} + \mathbf{A}\,\mathbf{X} &= 2\,\mathbf{A} && \mid\ -\mathbf{B} \\ \mathbf{A}\,\mathbf{X} &= 2\,\mathbf{A} - \mathbf{B} && \mid\ \mathbf{A}^{-1}\cdot \\ \mathbf{A}^{-1}\cdot\mathbf{A}\,\mathbf{X} &= \mathbf{A}^{-1}\cdot(2\,\mathbf{A} - \mathbf{B}) \\ \mathbf{I}\cdot\mathbf{X} &= 2\,\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\cdot\mathbf{B} \\ \mathbf{X} &= 2\,\mathbf{I} - \mathbf{A}^{-1}\cdot\mathbf{B} \end{aligned}$$

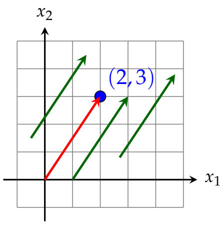
We have to take care that all matrix operations are defined.

Geometric Interpretation I

We have introduced vectors as special cases of matrices.

However, vector $\binom{x_1}{x_2}$ can also be seen as a geometrical object.
It can be interpreted as

- a **point** (x_1, x_2) in the xy-plain.
- an arrow from the origin $(0, 0)$ to point (x_1, x_2) (**position vector**).
- any arrow of the same length, direction and orientation as the position vector. (equivalence class of arrows)

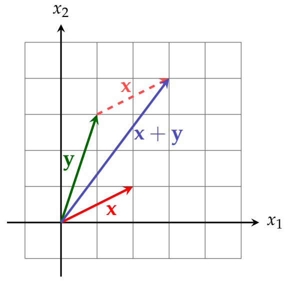


We always choose the representation that fits our needs.

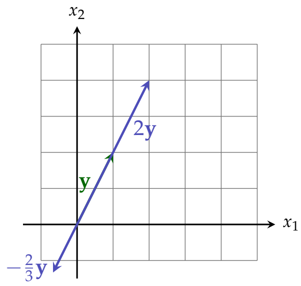
These pictures help us to think about these objects ("thinking crutch").
However, we need formulas to verify our conjectures!

Geometric Interpretation II

Vector addition



Multiplication by a scalar



Scalar Product

The **inner product** (or **scalar product**) of two vectors $\mathbf{x}$ and $\mathbf{y}$:

$$\boxed{\mathbf{x}^\mathsf{T}\,\mathbf{y} = \sum_{i=1}^{n} x_i\, y_i}$$

Two vectors are called **orthogonal** to each other, if $\boxed{\mathbf{x}^\mathsf{T}\,\mathbf{y} = 0}$.

We also say that these vectors are *normal* or *perpendicular* or *in a right angle* to each other.

The inner product of $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix}4\\5\\6\end{pmatrix}$ is given by

$$\mathbf{x}^\mathsf{T}\,\mathbf{y} = 1\cdot 4 + 2\cdot 5 + 3\cdot 6 = 32$$

Norm

The (Euclidean) **norm** $\|\mathbf{x}\|$ of vector $\mathbf{x}$:

$$\boxed{\|\mathbf{x}\| = \sqrt{\mathbf{x}^\mathsf{T}\,\mathbf{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}}$$

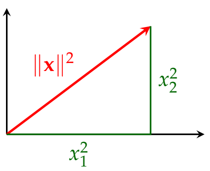
A vector $\mathbf{x}$ is called **normalized**, if $\|\mathbf{x}\| = 1$.

The norm of $\mathbf{x} = \begin{pmatrix}1\\2\\3\end{pmatrix}$ is given by

$$\|\mathbf{x}\| = \sqrt{1^2 + 2^2 + 3^2} = \sqrt{14}$$

Geometric Interpretation

The *norm* of a vector can be interpreted as its *length*:



Pythagorean theorem:

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2$$

The *inner product* measures *angles* between two vectors:

$$\boxed{\cos\sphericalangle(\mathbf{x}, \mathbf{y}) = \frac{\mathbf{x}^\mathsf{T}\,\mathbf{y}}{\|\mathbf{x}\|\cdot\|\mathbf{y}\|}}$$

Properties of the Norm

(i) $\|\mathbf{x}\| \geq 0$.

(ii) $\|\mathbf{x}\| = 0 \quad\Leftrightarrow\quad \mathbf{x} = 0$.

(iii) $\|\alpha\mathbf{x}\| = |\alpha|\cdot\|\mathbf{x}\|$ for all $\alpha\in\mathbb{R}$.

(iv) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. (Triangle inequality)

Inequalities

- **Cauchy-Schwarz inequality**

  $\boxed{|\mathbf{x}^\mathsf{T}\mathbf{y}| \leq \|\mathbf{x}\| \cdot \|\mathbf{y}\|}$

- **Minkowski inequality** (triangle inequality)

  $\boxed{\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|}$

- **Pythagorean theorem**

  For orthogonal vectors $\mathbf{x}$ and $\mathbf{y}$ we have

  $\boxed{\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2}$

Leontief Model

$\mathbf{A}$ ... technology matrix
$\mathbf{x}$ ... production vector
$\mathbf{b}$ ... demand vector
$\mathbf{p}$ ... prices for goods
$\mathbf{w}$ ... wages

Prices must cover production costs:

$$p_j = \sum_{i=1}^n a_{ij} p_i + w_j = a_{1j}p_1 + a_{2j}p_2 + \dots + a_{nj}p_n + w_j$$
$$\mathbf{p} = \mathbf{A}^\mathsf{T}\mathbf{p} + \mathbf{w}$$

So for fixed wages we find:

$$\mathbf{p} = (\mathbf{I} - \mathbf{A}^\mathsf{T})^{-1}\mathbf{w}$$

Moreover, for the input-output model we have:

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b}$$

Leontief Model

Demand is given by the wages for produced goods:

$$\text{demand} = w_1x_1 + w_2x_2 + \dots + w_nx_n = \mathbf{w}^\mathsf{T}\mathbf{x}$$

Supply is given by prices for demanded goods:

$$\text{supply} = p_1b_1 + p_2b_2 + \dots + p_nb_n = \mathbf{p}^\mathsf{T}\mathbf{b}$$

If the following equations hold in a input-output model

$$\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{b} \qquad \text{and} \qquad \mathbf{p} = \mathbf{A}^\mathsf{T}\mathbf{p} + \mathbf{w}$$

then we have market equilibrium, i.e., $\mathbf{w}^\mathsf{T}\mathbf{x} = \mathbf{p}^\mathsf{T}\mathbf{b}$.

Proof:

$$\mathbf{w}^\mathsf{T}\mathbf{x} = (\mathbf{p}^\mathsf{T} - \mathbf{p}^\mathsf{T}\mathbf{A})\mathbf{x} = \mathbf{p}^\mathsf{T}(\mathbf{I} - \mathbf{A})\mathbf{x} = \mathbf{p}^\mathsf{T}(\mathbf{x} - \mathbf{A}\mathbf{x}) = \mathbf{p}^\mathsf{T}\mathbf{b}$$

Summary

- matrix and vector
- triangular and diagonal matrix
- zero matrix and identity matrix
- transposed and symmetric matrix
- inverse matrix
- computations with matrices (matrix algebra)
- equations with matrices
- norm and inner product of vectors