Chapter 19

Control Theory

Economic Growth

Problem: Maximize consumption in period $[0, T]$:

$$\max_{0 \le s(t) \le 1} \int_0^T (1 - s(t)) f(k(t))\, dt$$

$f(k)$ ... production function
$k(t)$ ... capital stock at time t
$s(t)$ ... rate of investment at time t, $s \in [0, 1]$

We can control $s(t)$ at each time freely.
s is called **control function.**

$k(t)$ follows the differential equation

$$k'(t) = s(t) f(k(t)), \quad k(0) = k_0, \quad k(T) \ge k_T .$$

Oil Extraction

$y(t)$ ... amount of oil in reservoir at time t
$u(t)$ ... rate of extraction at time t: $y'(t) = -u(t)$
$p(t)$ ... market price of oil at time t
$C(t, y, u)$ ... extraction costs per unit of time
r ... (constant) discount rate

Problem I: Maximize revenue in fixed time horizon $[0, T]$:

$$\max_{u(t) \ge 0} \int_0^T \left[p(t)u(t) - C(t, y(t), u(t))\right] e^{-rt}\, dt$$

We can control $u(t)$ freely at each time where $u(t) \ge 0$.

$y(t)$ follows the differential equation:

$$y'(t) = -u(t), \quad y(0) = K, \quad y(T) \ge 0 .$$

Oil Extraction

Problem I:
Find an *extraction process* $u(t)$ for a fixed time period $[0, T]$ that optimizes the profit.

Problem II:
Find an *extraction process* $u(t)$ *and time horizon* T that optimizes the profit.

The Standard Problem (T Fixed)

1. Maximize for **objective function** f

$$\max_u \int_0^T f(t, y, u)\, dt, \qquad u \in \mathcal{U} \subseteq \mathbb{R} .$$

   u is the **control function**, $\mathcal{U}$ is the **control region**.

2. *Controlled differential equation* (initial value problem)

$$y' = g(t, y, u), \qquad y(0) = y_0 .$$

3. Terminal value
   (a) $y(T) = y_1$
   (b) $y(T) \ge y_1$ [or: $y(T) \le y_1$]
   (c) $y(T)$ free

(y, u) is called a **feasible pair** if (2) and (3) are satisfied.

Hamiltonian

Analogous to the Lagrange function we define function

$$\mathcal{H}(t, y, u, \lambda) = \lambda_0 f(t, y, u) + \lambda(t) g(t, y, u)$$

which is called the **Hamiltonian** of the standard problem.

Function $\lambda(t)$ is called the **adjoint function**.

Scalar $\lambda_0 \in \{0, 1\}$ can be assumed to be 1.
(However, there exist rare exceptions where $\lambda_0 = 0$.)

In the following we always assume that $\lambda_0 = 1$. Then

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda(t) g(t, y, u)$$

Maximum Principle

Let (y^*, u^*) be an *optimal pair* of the standard problem.
Then there exists a continuous function $\lambda(t)$ such that for all $t \in [0, T]$:

(i) u^* maximizes $\mathcal{H}$ w.r.t. u, i.e.,

$$\mathcal{H}(t, y^*, u^*, \lambda) \ge \mathcal{H}(t, y^*, u, \lambda) \quad \text{for all } u \in \mathcal{U}$$

(ii) λ satisfies the differential equation

$$\lambda' = -\frac{\partial}{\partial y} \mathcal{H}(t, y^*, u^*, \lambda)$$

(iii) Transversality condition
(a) $y(T) = y_1$: $\lambda(T)$ free
(b) $y(T) \ge y_1$: $\lambda(T) \ge 0$ [with $\lambda(T) = 0$ if $y^*(T) > y_1$]
(c) $y(T)$ free: $\lambda(T) = 0$

A Necessary Condition

The maximum principle gives a *necessary* condition for an **optimal pair** of the standard problem, i.e., a feasible pair which solves the optimization problem.

That is, for every optimal pair we can find such a function $\lambda(t)$.

On the other hand if we can find such a function for some feasible pair (y_0, u_0) then (y_0, u_0) need not be optimal.

However, it is a *candidate* for an optimal pair.

(Comparable to the role of stationary points in static constraint optimization problems.)

A Sufficient Condition

Let (y^*, u^*) be a feasible pair of the standard problem and $\lambda(t)$ some function that satisfies the maximum principle.

If $\mathcal{U}$ is convex and $\mathcal{H}(t, y, u, \lambda)$ is concave in (y, u) for all $t \in [0, T]$, then (y^*, u^*) is an optimal pair.

Recipe

1. For every triple (t, y, λ) find a (global) maximum $\hat{u}(t, y, \lambda)$ of $\mathcal{H}(t, y, u, \lambda)$ w.r.t. u.
2. Solve system of differential equations
$$y' = g(t, y, \hat{u}(t, y, \lambda), \lambda)$$
$$\lambda' = -\mathcal{H}_y(t, y, \hat{u}(t, y, \lambda), \lambda)$$
3. Find particular solutions $y^*(t)$ and $\lambda^*(t)$ which satisfy initial condition $y(0) = y_0$ and the transversality condition, resp.
4. We get candidates for an optimal pair by $y^*(t)$ and $u^*(t) = \hat{u}(t, y^*, \lambda^*)$.
5. If $\mathcal{U}$ is convex and $\mathcal{H}(t, y, u, \lambda^*)$ is concave in (y, u), then (y^*, u^*) is an optimal pair.

Example 1

Find optimal control u^* for

$$\max \int_0^2 y(t)\, dt, \quad u \in [0, 1]$$
$$y' = y + u, \quad y(0) = 0, \quad y(2) \text{ free}$$

Heuristically:
Objective function y and thus u should be as large as possible. Therefore we expect that $u^*(t) = 1$ for all t.

Hamiltonian:

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda g(t, y, u) = y + \lambda(y + u)$$

Example 1

$$\mathcal{H}(t, y, u, \lambda) = y + \lambda(y + u)$$

Maximum $\hat{u}$ of $\mathcal{H}$ w.r.t. u:
$$\hat{u} = \begin{cases} 1, & \text{if } \lambda \geq 0, \\ 0, & \text{if } \lambda < 0. \end{cases}$$

Solution of the (inhomogeneous linear) ODE
$$\lambda' = -\mathcal{H}_y = -(1 + \lambda), \quad \lambda(2) = 0$$
$$\Rightarrow \quad \lambda^*(t) = e^{2-t} - 1\,.$$

As $\lambda^*(t) = e^{2-t} - 1 \geq 0$ for all $t \in [0, 2]$ we have $\quad \hat{u}(t) = 1$.

Example 1

Solution of the (inhomogeneous linear) ODE
$$y' = y + \hat{u} = y + 1, \quad y(0) = 0$$
$$\Rightarrow \quad y^*(t) = e^t - 1\,.$$

We thus obtain
$$u^*(t) = \hat{u}(t) = 1\,.$$

Hamiltonian $\mathcal{H}(t, y, u, \lambda) = y + \lambda(y + u)$ is linear and thus concave in (y, u).

$u^*(t) = 1$ is the optimal control we sought for.

Example 2

Find the optimal control u^* for

$$\min \int_0^T \left[y^2(t) + cu^2(t)\right]\, dt, \quad u \in \mathbb{R}, \quad c > 0$$
$$y' = u, \quad y(0) = y_0, \quad y(T) \text{ free}$$

We have to solve the maximization problem

$$\max \int_0^T -\left[y^2(t) + cu^2(t)\right]\, dt$$

Hamiltonian:

$$\mathcal{H}(t, y, u, \lambda) = f(t, y, u) + \lambda g(t, y, u) = -y^2 - cu^2 + \lambda u$$

Example 2

Maximum $\hat{u}$ of $\mathcal{H}$ w.r.t. u:
$$0 = \mathcal{H}_u = -2c\hat{u} + \lambda \qquad \Rightarrow \qquad \hat{u} = \tfrac{\lambda}{2c}$$

Solution of the (system of) differential equations
$$y' = \hat{u} = \frac{\lambda}{2c}$$
$$\lambda' = -\mathcal{H}_y = 2y$$

By differentiating the second ODE we get
$$\lambda'' = 2y' = \frac{\lambda}{c} \qquad \Rightarrow \qquad \lambda'' - \frac{1}{c}\lambda = 0$$

Solution of the (homogeneous linear) ODE of second order
$$\lambda^*(t) = C_1 e^{rt} + C_2 e^{-rt}, \qquad \text{with } r = \tfrac{1}{\sqrt{c}}$$

($\pm\frac{1}{\sqrt{c}}$ are the two roots of the characteristic polynomial.)

Example 2

Initial condition $y(0) = y_0$ and transversality condition, resp., yield
$$\lambda^{*\prime}(0) = 2y(0) = 2y_0$$
$$\lambda^*(T) = 0$$

and thus
$$r(C_1 - C_2) = 2y_0$$
$$C_1 e^{rT} + C_2 e^{-rT} = 0$$

with solutions
$$C_1 = \tfrac{2y_0 e^{-rT}}{r(e^{rT} + e^{-rT})}, \qquad C_2 = -\tfrac{2y_0 e^{rT}}{r(e^{rT} + e^{-rT})}\,.$$

Example 2

Consequently we obtain

$$\lambda^*(t) = \frac{2y_0}{r(e^{rT}+e^{-rT})}\left(e^{-r(T-t)} - e^{r(T-t)}\right)$$

$$y^*(t) = \tfrac{1}{2}\lambda^*(t) = y_0\frac{e^{-r(T-t)}-e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$$

$$u^*(t) = \hat{u}(t,y^*,\lambda^*) = \tfrac{1}{2c}\lambda^*(t) = \frac{y_0}{c}\frac{e^{-r(T-t)}-e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$$

It is easy to verify that Hamiltonian $\mathcal{H}(t,y,u,\lambda) = -y^2 - cu^2 + \lambda u$ is concave in y and u.

$u^*(t) = \frac{y_0}{c}\frac{e^{-r(T-t)}-e^{r(T-t)}}{r(e^{rT}+e^{-rT})}$ is the optimal control.

Standard Problem (T Variable)

If time horizon $[0,T]$ is not fixed in advanced we have to find an optimal time period $[0,T^*]$ in addition to the optimal control u^*.

For this purpose we have to add the following condition to the maximum principle (in addition to (i)–(iii)).

(iv) $\mathcal{H}(T^*, y^*(T^*), u^*(T^*), \lambda(T^*)) = 0$

The recipe for solving the optimization problem remains essentially the same.

Summary

- standard problem
- Hamiltonian function
- maximum principle
- a sufficient condition