Chapter 18

Difference Equation

First Difference

Suppose a state variable y can only be estimated at **discrete** time points $t_1, t_2, t_3, \dots$. In particular we assume that $t_i \in \mathbb{N}$. Thus we can describe the behavior of such a variable by means of a map

$$\mathbb{N} \to \mathbb{R},\ t \mapsto y(t)$$

i.e., a *sequence*. We write y_t instead of $y(t)$.

For the marginal changes of y we have to replace the differential quotient $\frac{dy}{dt}$ by the **difference quotient** $\frac{\Delta y}{\Delta t}$.

So if $\Delta t = 1$ this reduces to the **first difference**

$$\boxed{\Delta y_t = y_{t+1} - y_t}$$

Rules for Differences

For differences similar rules can be applied as for derivatives:

- $\Delta(c\, y_t) = c\, \Delta y_t$
- $\Delta(y_t + z_t) = \Delta y_t + \Delta z_t$ *Summation rule*
- $\Delta(y_t \cdot z_t) = y_{t+1}\, \Delta z_t + z_t\, \Delta y_t$ *Product rule*
- $\Delta\left(\dfrac{y_t}{z_t}\right) = \dfrac{z_t\, \Delta y_t - y_t\, \Delta z_t}{z_t\, z_{t+1}}$ *Quotient rule*

Differences of Higher Order

The k-th derivative $\frac{d^k y}{dt^k}$ has to be replaced by the **difference of order k**:

$$\boxed{\Delta^k y_t = \Delta(\Delta^{k-1} y_t) = \Delta^{k-1} y_{t+1} - \Delta^{k-1} y_t}$$

For example the **second difference** is then

$$\begin{aligned} \Delta^2 y_t &= \Delta(\Delta y_t) = \Delta y_{t+1} - \Delta y_t \\ &= (y_{t+2} - y_{t+1}) - (y_{t+1} - y_t) \\ &= y_{t+2} - 2\, y_{t+1} + y_t \end{aligned}$$

Difference Equation

A **difference equation** is an equation that contains the differences of a sequence. It is of order n if it contains a difference of order n (but not higher).

$\Delta y_t = 3$	difference equation of first order
$\Delta y_t = \frac{1}{2} y_t$	difference equation of first order
$\Delta^2 y_t + 2\, \Delta y_t = -3$	difference equation of second order

If in addition an initial value y_0 is given
we have a so called **initial value problem**.

Equivalent Representation

Difference equations can equivalently written without Δ-notation.

$$\Delta y_t = 3 \Leftrightarrow y_{t+1} - y_t = 3 \quad \Leftrightarrow \quad y_{t+1} = y_t + 3$$

$$\Delta y_t = \tfrac{1}{2} y_t \Leftrightarrow y_{t+1} - y_t = \tfrac{1}{2} y_t \quad \Leftrightarrow \quad y_{t+1} = \tfrac{3}{2} y_t$$

$$\begin{aligned} &\Delta^2 y_t + 2\, \Delta y_t = -3 \Leftrightarrow \\ &\quad \Leftrightarrow (y_{t+2} - 2\, y_{t+1} + y_t) + 2\, (y_{t+1} - y_t) = -3 \\ &\quad \Leftrightarrow \quad y_{t+2} = y_t - 3 \end{aligned}$$

These can be seen as *recursion formulæ* for sequences.

Problem:
Find a sequence y_t which satisfies the given recursion formula *for all* $t \in \mathbb{N}$.

Initial Value Problem and Iterations

Difference equations of first order can be solved by iteratively computing the elements of the sequence if the initial value y_0 is given.

Compute the solution of $y_{t+1} = y_t + 3$ with initial value y_0.

$$\begin{aligned} &y_1 = y_0 + 3 \\ &y_2 = y_1 + 3 = (y_0 + 3) + 3 = y_0 + 2 \cdot 3 \\ &y_3 = y_2 + 3 = (y_0 + 2 \cdot 3) + 3 = y_0 + 3 \cdot 3 \\ &\dots \\ &y_t = y_0 + 3\, t \end{aligned}$$

For initial value $y_0 = 5$ we obtain $y_t = 5 + 3\, t$.

Example – Iterations

Compute the solution of $y_{t+1} = \frac{3}{2} y_t$ with initial value y_0.

$$\begin{aligned} &y_1 = \tfrac{3}{2} y_0 \\ &y_2 = \tfrac{3}{2} y_1 = \tfrac{3}{2}(\tfrac{3}{2} y_0) = \left(\tfrac{3}{2}\right)^2 y_0 \\ &y_3 = \tfrac{3}{2} y_2 = \tfrac{3}{2}(\tfrac{3}{2}^2 y_0) = \left(\tfrac{3}{2}\right)^3 y_0 \\ &\dots \\ &y_t = \left(\tfrac{3}{2}\right)^t y_0 \end{aligned}$$

For initial value $y_0 = 5$ we obtain $y_t = 5 \cdot \left(\frac{3}{2}\right)^t$.

Homogeneous Linear Difference Equation of First Order

A **homogeneous linear difference equation of first order** is of form

$$\boxed{y_{t+1} + a\,y_t = 0}$$

Ansatz for general solution:

$$y_t = C\,\beta^t, \quad C\,\beta \neq 0, \quad \text{for some fixed } C \in \mathbb{R}.$$

It has to satisfy the difference equation for all t:

$$y_{t+1} + a\,y_t = C\,\beta^{t+1} + a\,C\,\beta^t = 0.$$

Division by $C\,\beta^t$ yields $\beta + a = 0$ and thus $\beta = -a$ and

$$\boxed{y_t = C\,(-a)^t}$$

Example – Homogeneous Equation

Homogeneous linear difference equation

$$y_{t+1} - \frac{3}{2} y_t = 0$$

has general solution

$$y_t = C\,\left(\frac{3}{2}\right)^t .$$

Properties of Solutions

The behavior of solution

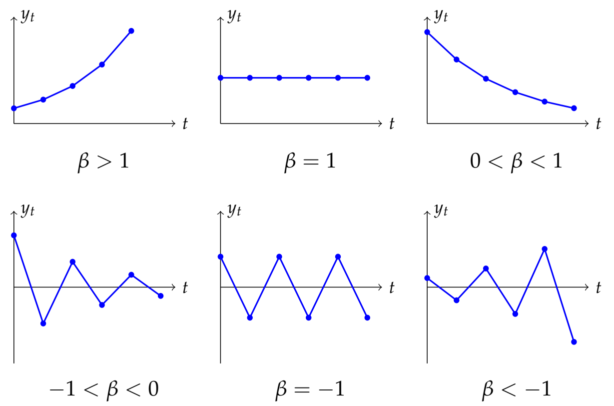
$$y_t = C\,\beta^t = C\,(-a)^t$$

obviously depends on parameter $\beta = -a$ which can be summarized as following:

oscillating	$\Leftrightarrow$	$\beta < 0$
convergent	$\Leftrightarrow$	$\lvert\beta\rvert < 1$

We want to note that β is the root of the *characteristic equation* $\beta + a = 0$.

Properties of Solutions



Inhomogeneous Linear Difference Equation

The general solution of **inhomogeneous linear difference equation**

$$\boxed{y_{t+1} + a\,y_t = s}$$

can be written as

$$\boxed{y_t = y_{h,t} + y_{p,t}}$$

where

- $p_{h,t}$ is the general solution of the corresponding homogeneous equation $y_{t+1} + a\,y_t = 0$, and
- $y_{h,t}$ is some particular solution of the inhomogeneous equation.

How can we find $y_{p,t}$?

Inhomogeneous Linear Difference Equation

As parameters a and s are constant we may set $y_{h,t} = c = \text{const.}$

Then

$$y_{p,t+1} + a\,y_{p,t} = c + a\,c = s$$

which implies

$$y_{p,t} = c = \frac{s}{1+a} \quad \text{if } a \neq -1.$$

If $a = -1$ we set $y_{p,t} = c\,t$. Then

$$c\,(t+1) + (-1)\,c\,t = s$$

which implies $c = s$ and

$$y_{p,t} = s\,t\,.$$

Inhomogeneous Linear Difference Equation

An **inhomogeneous linear difference equation of first order** with *constant coefficients* is of form

$$\boxed{y_{t+1} + a\,y_t = s}$$

The general solution is given by

$$\boxed{y_t = \begin{cases} C\,(-a)^t + \dfrac{s}{1+a} & \text{if } a \neq -1, \\ C + s\,t & \text{if } a = -1. \end{cases}}$$

Observe that $C\,(-a)^t$ is just the solution of the corresponding homogeneous difference equation $y_{t+1} + a\,y_t = 0$.

Asymptotically Stable

Observe that $y_{p,t} = \bar{y} = \frac{s}{1+a}$ is a *fixed point* (or **equilibrium point**) of the inhomogeneous equation $y_{t+1} + a\,y_t = s$.
Obviously solution

$$y_t = C\,(-a)^t + \bar{y} \qquad (C \neq 0)$$

converges to $\bar{y}$ if and only if $\lvert a\rvert < 1$.

In this case $\bar{y}$ is (*locally*) **asymptotically stable**.

Otherwise if $\lvert a\rvert > 1$, y_t diverges and $\bar{y}$ is called **unstable**.

Example – Inhomogeneous Equation

The inhomogeneous linear difference equation

$$y_{t+1} - 2\,y_t = 2$$

has general solution

$$y_t = C\,2^t - 2\,.$$

We get the particular solution of the initial value problem with $y_0 = 1$ by

$$1 = y_0 = C\,2^0 - 2\,.$$

Thus $C = 3$ and consequently

$$y_t = 3 \cdot 2^t - 2\,.$$

Example – Inhomogeneous Equation

The inhomogeneous linear difference equation

$$y_{t+1} - y_t = 3$$

has general solution

$$y_t = C + 3\,t.$$

We get the particular solution of the initial value problem with $y_0 = 4$ by

$$4 = y_0 = C + 3 \cdot 0\,.$$

Thus $C = 4$ and consequently

$$y_t = 4 + 3\,t\,.$$

Model – Dynamic of Market Price

Assume that demand and supply functions are linear:

$$\begin{aligned} q_{d,t} &= \alpha - \beta\,p_t && (\alpha, \beta > 0) \\ q_{s,t} &= -\gamma + \delta\,p_t && (\gamma, \delta > 0) \end{aligned}$$

and the change of price is directly proportional to the difference $(q_d - q_s)$:

$$p_{t+1} - p_t = j\,(q_{d,t} - q_{s,t}) \qquad (j > 0)$$

How does price p_t evolve in time?

$$\begin{aligned} p_{t+1} - p_t = j\,(q_{d,t} - q_{s,t}) &= j\,(\alpha - \beta p_t - (-\gamma + \delta p_t)) \\ &= j\,(\alpha + \gamma) - j\,(\beta + \delta) p_t \end{aligned}$$

i.e., we obtain the inhomogeneous linear difference equation

$$p_{t+1} + (j(\beta + \gamma) - 1)\,p_t = j(\alpha + \gamma)$$

Model – Dynamic of Market Price

The general solution

$$p_{t+1} + (j(\beta + \gamma) - 1)\,p_t = j(\alpha + \gamma)$$

is then

$$p_t = C\,(1 - j(\beta + \delta))^t + \bar{p}$$

where $\bar{p} = \frac{\alpha+\gamma}{\beta+\delta}$ is the price in market equilibrium.

For initial value p_0 we finally obtain the particular solution

$$p_t = (p_0 - \bar{p})(1 - j(\beta + \delta))^t + \bar{p}$$

The difference equation has fixed point $\bar{p}$.
It is asymptotically stable if and only if $j(\beta + \delta) < 2$.

Model – Dynamic of Market Price

Consider the following market model:

$$\begin{aligned} q_{d,t} &= q_{s,t} \\ q_{d,t} &= \alpha - \beta p_t && (\alpha, \beta > 0) \\ q_{s,t} &= -\gamma + \delta p_{t-1} && (\gamma, \delta > 0) \end{aligned}$$

Observe that we have market equilibrium in each period.
The supply depends on the price of the preceding period.
Substituting of the second and third equation onto the first yields the inhomogeneous linear difference equation

$$\beta\,p_t + \delta\,p_{t-1} = \alpha + \gamma \quad \Leftrightarrow \quad p_{t+1} + \frac{\delta}{\beta}\,p_t = \frac{\alpha + \gamma}{\beta}$$

Model – Dynamic of Market Price

Inhomogeneous linear first order difference equation

$$p_{t+1} + \frac{\delta}{\beta}\,p_t = \frac{\alpha + \gamma}{\beta}$$

with initial value p_0 has solution

$$p_t = (p_0 - \bar{p}) \left(-\frac{\delta}{\beta}\right)^t + \bar{p} \qquad \text{where } \bar{p} = \frac{\alpha + \gamma}{\beta + \delta}.$$

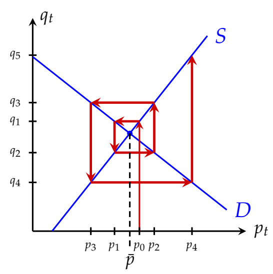
As all constants are positive, root $-\frac{\delta}{\beta} < 0$ and thus all solutions of such a market model oscillate.

The solution converges to the $\bar{p}$ if $\left|\frac{\delta}{\beta}\right| < 1$.

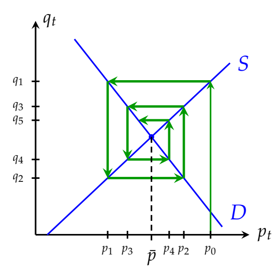
Cobweb Model

We also can analyze this model *graphically*.
Demand and supply are functions of price p:

$$D(p) = \alpha - \beta p, \quad \text{and} \quad S(p) = -\gamma + \delta p$$



$\delta > \beta$



$\delta < \beta$

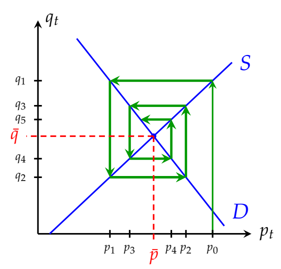
Cobweb Model

↑ We start in period 0 with price p_0 and get supply $q_1 = S(p_0)$ in period 1.

← Market equilibrium implies new price p_1 given implicitly by $D(p_1) = q_1$.

↓ In period 2 price p_1 yields supply $q_2 = S(p_1)$.

→ Market equilibrium implies new price p_2 given implicitly by $D(p_2) = q_2$.



Iterating this procedure spins a **cobweb** around *equilibrium point* $(\bar{p}, \bar{q})$ with $\bar{q} = S(\bar{p}) = D(\bar{p})$.

Cobweb Model – Nonlinear Functions

Cobweb models also work when functions $D(p)$ and $S(p)$ are nonlinear.

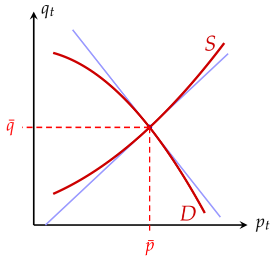
Then there may not exist a solution in closed form.
However, we still have an equilibrium point $\bar{p}$ with $D(\bar{p}) = S(\bar{p})$.

Linearized versions of D and S:

$$\widehat{D}(p) = D(\bar{p}) + D'(\bar{p})(p - \bar{p})$$
$$\widehat{S}(p) = S(\bar{p}) + S'(\bar{p})(p - \bar{p})$$

Equilibrium point $\bar{p}$ is *locally asymptotically stable* if

- $D'(\bar{p}) < 0 < S'(\bar{p})$, and
- $|S'(\bar{p})| < |D'(\bar{p})|$.



Linear Difference Equation of Second Order

A **difference equation** is an equation that contains the differences of second order of a sequence.

We restrict our interest to **linear difference equations of second order with constant coefficients**:

$$\boxed{y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s}$$

Homogeneous Linear Difference Equation

We obtain the general solution of the homogeneous linear ODE

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

by means of the ansatz

$$y_t = C\,\beta^t, \qquad C\,\beta \neq 0$$

which has to satisfies the difference equation :

$$C\,\beta^{t+2} + a_1\, C\,\beta^{t+1} + a_2\, C\,\beta^t = 0\,.$$

Hence β has to satisfy the **characteristic equation**

$$\boxed{\beta^2 + a_1\,\beta + a_2 = 0}$$

Characteristic Equation

The characteristic equation

$$\beta^2 + a_1\,\beta + a_2 = 0$$

has solutions

$$\beta_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}$$

We have three cases:

1. $\frac{a_1^2}{4} - a_2 > 0$: two distinct real solutions
2. $\frac{a_1^2}{4} - a_2 = 0$: exactly one real solution
3. $\frac{a_1^2}{4} - a_2 < 0$: two complex (non-real) solutions

Case: $\frac{a_1^2}{4} - a_2 > 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

is given by

$$\boxed{y(t) = C_1\,\beta_1^t + C_2\,\beta_2^t\,, \quad \text{with} \quad \beta_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}}$$

where C_1 and C_2 are arbitrary real numbers.

Example: $\frac{a_1^2}{4} - a_2 > 0$

Compute the general solution of difference equation

$$y_{t+2} - 3\,y_{t+1} + 2\,y_t = 0\,.$$

Characteristic equation

$$\beta^2 - 3\,\beta + 2 = 0$$

has distinct real solutions

$$\beta_1 = 1 \quad \text{and} \quad \beta_2 = 2\,.$$

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1\,1^t + C_2\,2^t = C_1 + C_2\,2^t\,.$$

Case: $\frac{a_1^2}{4} - a_2 = 0$

The general solution of the homogeneous difference equation

$$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$$

is given by

$$\boxed{y_t = C_1\,\beta^t + C_2\,t\,\beta^t\,, \quad \text{with} \quad \beta = -\frac{a_1}{2}}$$

We can verify the validity of solution $t\,\beta^t$ by a simple (but tedious) straight-forward computation.

Example: $\frac{a_1^2}{4} - a_2 = 0$

Compute the general solution of difference equation

$$y_{t+2} - 4\,y_{t+1} + 4\,y_t = 0\,.$$

Characteristic equation

$$\beta^2 - 4\,\beta + 4 = 0$$

has the unique solution

$$\beta = 2\,.$$

Thus the general solution of the homogeneous equation is given by

$$y_t = C_1\,2^t + C_2\,t\,2^t\,.$$

Case: $\frac{a_1^2}{4} - a_2 < 0$

In this case root $\sqrt{\frac{a_1^2}{4} - a_2}$ is a non-real (imaginary) number:

$\beta_{1,2} = a \pm b\, i$

where

- $a = -\frac{a_1}{2}$ is called the **real part**, and
- $b = \sqrt{\left| a_2 - \frac{a_1^2}{4} \right|}$ the **imaginary part** of root β.

Alternatively β can be represent by so called polar coordinates

$\beta_{1,2} = r(\cos\theta \pm i\, \sin\theta)$

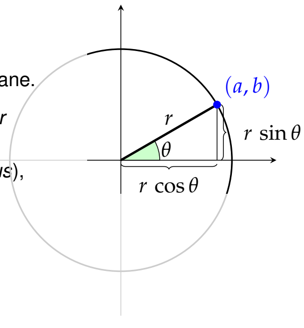
where

- $r = |\beta| = \sqrt{a^2+b^2} = \sqrt{\frac{a_1^2}{4} + a_2 - \frac{a_1^2}{4}} = \sqrt{a_2}$ is called the **modulus** (or *absolute value*) of β, and
- $\theta = \arg(\beta)$ the *argument* of β.

Modulus and Argument

A complex number $z = a + b\, i$ can be interpreted as point (a, b) in the (real) plane.

This point can also can be given by *polar coordinates* with radius $r = |z|$ (*absolute value* or *modulus*), and angle θ (called the *argument* of z).



We then have

$r = |z| = \sqrt{a^2+b^2}$

and

$\tan\theta = \frac{b}{a}$ because $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$.

Case: $\frac{a_1^2}{4} - a_2 < 0$

From the rules for complex numbers one can derive purely real solutions of the homogeneous difference equation

$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = 0$

given by

$$\boxed{\begin{array}{l} y_t = r^t\left[C_1\cos(\theta t) + C_2\sin(\theta t)\right] \\ \qquad \text{with} \quad r = |\beta| = \sqrt{a_2} \quad \text{and} \quad \theta = \arg(\beta) \end{array}}$$

Argument $\arg(\beta)$ is given by

$\cos\theta = \frac{a}{r} = -\frac{a_1}{2\sqrt{a_2}}$

$\sin\theta = \frac{b}{r} = \sqrt{1 - \frac{a_1^2}{4a_2}}$

Example: $\frac{a_1^2}{4} - a_2 < 0$

Compute the general solution of difference equation

$$y_{t+2} + 2\, y_{t+1} + 4\, y_t = 0\,.$$

Characteristic equation

$$\beta^2 + 2\,\beta + 4 = 0$$

has the complex solutions

$$\beta_{1,2} = -1 \pm \sqrt{3}\, i$$

i.e., $a = -1$ and $b = \sqrt{3}$.

Example: $\frac{a_1^2}{4} - a_2 < 0$

Complex root $\beta = a + b\, i$ with $a = -1$ and $b = \sqrt{3}$ has polar coordinates:

- $r = \sqrt{1^2+3} = \sqrt{4} = 2$, and
- $\theta = \frac{2\pi}{3}$, as $\sin\theta = \frac{a}{r} = -\frac{1}{2}$ and $\cos\theta = \frac{b}{r} = \frac{\sqrt{3}}{2}$.

Thus the general solution of the homogeneous equation is given by

$$y_t = 2^t\left[C_1\cos\left(\frac{2\pi}{3}t\right) + C_2\sin\left(\frac{2\pi}{3}t\right)\right]\,.$$

Argument θ can be computed by means of the *arcus tangens* function $\arctan(b/a)$.
A more convenient way is to use function `atan2` which is available in programs like **R**.

Inhomogeneous Linear Difference Equation

The general solution of **inhomogeneous linear difference equation**

$y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s$

can be written as

$$\boxed{y_t = y_{h,t} + y_{p,t}}$$

where

- $y_{h,t}$ is the general solution of the corresponding homogeneous equation $y_{t+2} + a_1\, y_{t+1} + a_2\, y_t = s$, and
- $y_{h,t}$ is some particular solution of the inhomogeneous equation.

How can we find $y_{p,t}$?

Inhomogeneous Linear Difference Equation

By assumption all coefficients a_1, a_2, and s. So we may assume that $y_{p,t} = c = \text{const}$:

$$c + a_1\, c + a_2\, c = s$$

which implies

$$\boxed{y_{p,t} = c = \frac{s}{1 + a_1 + a_2} \qquad \text{if } a_1 + a_2 \neq -1.}$$

If $a_1 + a_2 \neq -1$ we may use $y_{p,t} = ct$ and get

$$\boxed{y_{p,t} = \frac{s}{a_1 + 2}\, t \qquad \text{if } a_1 + a_2 = -1 \text{ and } a_1 \neq -2.}$$

Example – Inhomogeneous Equation

Compute the general solution of difference equation

$$y_{t+2} + 2\, y_{t+1} + 4\, y_t = 14\,.$$

General solution of homogeneous equation $y_{t+2} + 2\, y_{t+1} + 4\, y_t = 0$:

$$y_{h,t} = 2^t\left[C_1\cos\left(\frac{2\pi}{3}t\right) + C_2\sin\left(\frac{2\pi}{3}t\right)\right]\,.$$

As $a_1 + a_2 = 2 + 4 \neq -1$ we use $y_{p,t} = \frac{14}{1+2+4} = 2$ and obtain the general solution of the inhomogeneous equation as

$$y_t = y_{h,t} + y_{p,t} = 2^t\left[C_1\cos\left(\frac{2\pi}{3}t\right) + C_2\sin\left(\frac{2\pi}{3}t\right)\right] + 2\,.$$

Example – Inhomogeneous Equation

Compute the general solution of difference equation

$$y_{t+2} - 3\,y_{t+1} + 2\,y_t = 2\,.$$

General solution of homogeneous equation $y_{t+2} - 3\,y_{t+1} + 2\,y_t = 0$:

$$y_{h,t} = C_1 + C_2 2^t\,.$$

As $a_1 + a_2 = -3 + 2 = -1$ and $a_1 \neq -2$ we use $y_{p,t} = \frac{2}{-3+2} t = -2t$ and obtain the general solution of the inhomogeneous equation as

$$y_t = y_{h,t} + y_{p,t} = C_1 + C_2 2^t - 2t\,.$$

Fixed Point of a Difference Equation

The inhomogeneous linear difference equation

$$y_{t+2} + a_1\,y_{t+1} + a_2\,y_t = s$$

has the special constant solution (for $a_1 + a_2 \neq -1$)

$$y_{p,t} = \bar{y} = \frac{s}{1 + a_1 + a_2} \quad \text{(= constant)}$$

Point $\bar{y}$ is called **fixed point**, or **equilibrium point** of the difference equation.

Stable and Unstable Fixed Points

When we review general solutions of linear difference equations (with constant coefficients) we observe that these solutions converge to a fixed point $\bar{y}$ for all choices of constants C if the absolute values of the roots β of the characteristic equation are less than one:

$$\boxed{y_t \to \bar{y} \text{ for } t \to \infty \quad \text{if} \quad |\beta| < 1.}$$

In this case $\bar{y}$ is called an **asymptotically stable** fixed point.

Summary

- differences of sequences
- difference equation
- homogeneous and inhomogeneous linear difference equation of first order with constant coefficients
- cobweb model
- homogeneous and inhomogeneous linear difference equation of second order with constant coefficients
- stable and unstable fixed points