Chapter 17

Differential Equation

A Simple Growth Model (Domar)

In Domar's growth model we have the following assumptions:

(1) An increase of the rate of investments $I(t)$ increases income $Y(t)$:

$$\frac{dY}{dt} = \frac{1}{s} \cdot \frac{dI}{dt} \qquad (s = \text{constant})$$

(2) Ratio of capital stock $K(t)$ and production capacity $\kappa(t)$ is constant:

$$\frac{\kappa(t)}{K(t)} = \varrho \quad (= \text{constant})$$

(E) In equilibrium we have:

$$Y = \kappa$$

Problem: Which flow of investment causes our model to remain in equilibrium for all times $t \geq 0$?

A simple Growth Model (Domar)

We search for a function $I(t)$ which satisfies model assumptions and equilibrium condition for all times $t \geq 0$.

$Y(t) = \kappa(t)$ for all t implies $Y'(t) = \kappa'(t)$.

We thus find

$$\frac{1}{s} \cdot \frac{dI}{dt} \overset{(1)}{=} \frac{dY}{dt} \overset{(E)}{=} \frac{d\kappa}{dt} \overset{(2)}{=} \varrho \frac{dK}{dt} = \varrho\, I(t)$$

or in short

$$\frac{1}{s} \cdot \frac{dI}{dt} = \varrho\, I(t)$$

This equation contains a **function** and its **derivative**.
It must hold *for all* $t \geq 0$.
The unknown in this equation is a **function**.

Differential Equation of First Order

An **ordinary differential equation (ODE) of first order** is an equation where the unknown is a univariate function and which contains the first (but not any higher) derivative of that function.

$$\boxed{y' = F(t, y)}$$

Examples:

$$y' = a\,y$$
$$y' + a\,y = b$$
$$y' + a\,y = b\,y^2$$

are ODEs of first order which describe exponential, exponentially bounded, and logistic growth, resp.

Remarks

- When time t is the independent variable of a function $y(t)$, then often Newton's notation is used for its derivatives:

$$\dot{y}(t) = \frac{dy}{dt} \qquad \text{and} \qquad \ddot{y}(t) = \frac{d^2y}{dt^2}$$

- The independent variable is often not given explicitly:

  $y' = a\,y$ is short for $y'(t) = a\,y(t)$.

Solution of Domar's Model

Transformation of the differential equation yields

$$\frac{1}{I(t)} I'(t) = \varrho\, s$$

This equation must hold for all t:

$$\ln(I) = \int \frac{1}{I} dI = \int \frac{1}{I(t)} I'(t)\, dt = \int \varrho\, s\, dt = \varrho\, s\, t + c$$

Substitution: $I = I(t) \quad \Rightarrow \quad dI = I'(t)\, dt$

Thus we get

$$I(t) = e^{\varrho s t} \cdot e^{c} = C\, e^{\varrho s t} \qquad (C > 0)$$

General Solution

All solutions of ODE $\quad I' = \varrho s I \quad$ can be written as

$$I(t) = C\, e^{\varrho s t} \qquad (C > 0)$$

This representation is called the **general solution** of the ODE.

We obtain *infinitely many* solutions!

We can easily verify the correctness of these solutions:

$$\frac{dI}{dt} = \varrho s \cdot C\, e^{\varrho s t} = \varrho s \cdot I(t)$$

Initial Value Problem

In our model investment rate $I(t)$ is known at time $t = 0$ (i.e., "now").
So we have *two* equations:

$$\begin{cases} I'(t) = \varrho s \cdot I \\ I(0) = I_0 \end{cases}$$

We have to find some function $I(t)$ which satisfies both the ODE and the *initial value*.

We have to solve the so called **initial value problem**.

We obtain the so called **particular solution** of the initial value problem by substituting the initial values into the general solution of the ODE.

Solution of Domar's Model

We obtain the particular solution of initial value problem

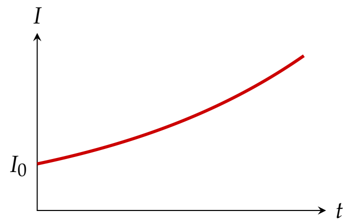
$$\begin{cases} I'(t) = \varrho s \cdot I \\ I(0) = I_0 \end{cases}$$

by substituting into the general solution:

$$I_0 = I(0) = C\, e^{\rho s 0} = C$$

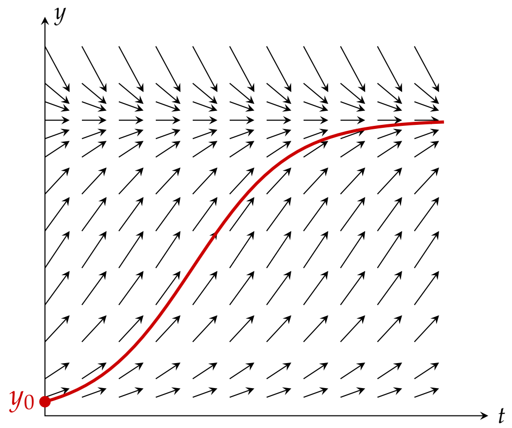
and thus

$$I(t) = I_0\, e^{\varrho s t}$$



Graphical Interpretation

Equation $y' = F(t, y)$ assigns the slope of a tangent to each point (t, y). We get a so called **vector field**.



Separation of Variables

Differential equations of the form

$$\boxed{y' = f(t) \cdot g(y)}$$

can be solved by means of **separation of variables**:

$$\frac{dy}{dt} = f(t) \cdot g(y) \iff \frac{1}{g(y)} dy = f(t)\, dt$$

Integration of either side yields:

$$\int \frac{1}{g(y)} dy = \int f(t)\, dt + c$$

We thus obtain the solution of the ODE as *implicit* function.

We have solved the ODE of Domar's model by separation of variables.

Example – Separation of Variables

Find the solutions of ODE

$$y' + t\, y^2 = 0$$

Separation of variables:

$$\frac{dy}{dt} = -t\, y^2 \quad \Rightarrow \quad -\frac{dy}{y^2} = t\, dt$$

Integration yields

$$-\int \frac{dy}{y^2} = \int t\, dt + c \quad \Rightarrow \quad \frac{1}{y} = \frac{1}{2} t^2 + c$$

and thus we obtain the general solution as

$$y(t) = \frac{2}{t^2 + 2c}$$

Example – Initial Value Problem

Compute the solution of the initial value problem

$$y' + t\,y^2 = 0, \quad y(0) = 1$$

Particular solution by substitution:

$$1 = y(0) = \frac{2}{0^2 + 2c} \quad \Rightarrow \quad c = 1$$

and thus

$$y(t) = \frac{2}{t^2 + 2}$$

Linear ODE of First Order

A **linear differential equation of first order** is of form

$$\boxed{y'(t) + a(t)\,y(t) = s(t)}$$

It is called

- **homogeneous** ODE, if $s = 0$, and
- **inhomogeneous** ODE, if $s \neq 0$.

Homogeneous linear ODE of first order can be solved by separation of variables.

Example – Homogeneous Linear ODE

Find the general solution of the homogeneous linear ODE

$$y' + 3\,t^2 y = 0$$

Separation of variables:

$$\frac{dy}{dt} = -3\,t^2\,y \quad \Rightarrow \quad \frac{1}{y}dy = -3t^2\,dt \quad \Rightarrow \quad \ln y = -t^3 + c$$

General solution thus is

$$y(t) = C\,e^{-t^3}$$

Inhomogeneous Linear ODE of First Order

The general solution of inhomogeneous linear ODE

$$y'(t) + a\,y(t) = s$$

can be written as

$$\boxed{y(t) = y_h(t) + y_p(t)}$$

where

- $y_h(t)$ is the general solution of the corresponding homogeneous equation $y'(t) + a\,y(t) = 0$, and
- $y_p(t)$ is some particular solution of the inhomogeneous equation.

If coefficients a and b are *constants* we set $y_p(t) = \text{const}$. Then $y_p' = 0$ and $y_p(t) = \frac{s}{a}$.

Inhomogeneous Linear ODE of First Order

For the case where all coefficients a and b are *constants* and *non-zero* the general solution of

$$y'(t) + a\,y(t) = s$$

is given as

$$\boxed{y(t) = C\,e^{-at} + \frac{s}{a}}$$

Observe that $C\,e^{-at}$ is just the solution of the corresponding homogeneous ODE $y'(t) + a\,y(t) = 0$.

Inhomogeneous Linear ODE of First Order

For the initial value problem

$$y'(t) + a\,y(t) = s, \quad y(0) = y_0$$

we obtain the particular solution

$$\boxed{y(t) = (y_0 - \bar{y})\;e^{-at} + \bar{y} \qquad \text{with } \bar{y} = \frac{s}{a}}$$

We find this solution by substituting the initial value into the particular solution.

Example – Inhomogeneous Linear ODE

Find the solution of the initial value problem

$$y' - 3y = 6, \quad y(0) = 1$$

We find

$$\bar{y} = \frac{s}{a} = \frac{6}{-3} = -2$$
$$y(t) = (y_0 - \bar{y})\, e^{-at} + \bar{y} = (1 - (-2))\, e^{3t} - 2 = 3e^{3t} - 2$$

The particular solution thus is

$$y(t) = 3e^{3t} - 2$$

Model – Dynamic of Market Price

Assume that demand and supply functions are linear:

$$q_d(t) = \alpha - \beta\, p(t) \qquad (\alpha, \beta > 0)$$
$$q_s(t) = -\gamma + \delta\, p(t) \qquad (\gamma, \delta > 0)$$

The rate of price change is directly proportional to the difference $(q_d - q_s)$:

$$\frac{dp}{dt} = j\,(q_d(t) - q_s(t)) \qquad (j > 0)$$

How does price $p(t)$ evolve in time?

$$\frac{dp}{dt} = j\,(q_d - q_s) = j\,(\alpha - \beta p - (-\gamma + \delta p))$$
$$= j\,(\alpha + \gamma) - j\,(\beta + \delta) p$$

i.e., we obtain the inhomogeneous linear ODE of first order

$$p'(t) + j\,(\beta + \delta)\, p(t) = j\,(\alpha + \gamma)$$

Model – Dynamic of Market Price

The solution of initial value problem

$$p'(t) + j\,(\beta + \delta)\, p(t) = j\,(\alpha + \gamma), \qquad p(0) = p_0$$

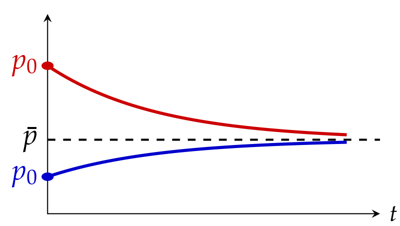
is

$$p(t) = (p_0 - \bar{p})\, e^{-j(\beta+\delta)t} + \bar{p}$$

with

$$\bar{p} = \frac{s}{a} = \frac{j(\alpha + \gamma)}{j(\beta + \delta)} = \frac{\alpha + \gamma}{\beta + \delta}$$

Observe that $\bar{p}$ is just the price in market equilibrium.



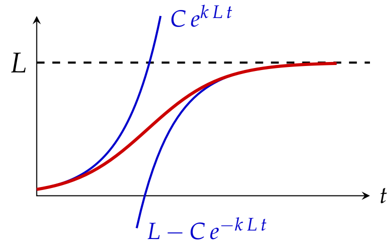
Logistic Differential Equation

A **logistic differential equation** is of form

$$\boxed{y'(t) - k\,y(t)\,(L - y(t)) = 0}$$

where $k, L > 0$ and $0 \le y(t) \le L$.

- $y \approx 0$: $\quad y'(t) - k\,L\,y(t) \approx 0 \quad \Rightarrow \quad y(t) \approx C\,e^{k\,L\,t}$
- $y \approx L$: $\quad y'(t) + k\,L\,y(t) \approx k\,L^2 \quad \Rightarrow \quad y(t) \approx L - C\,e^{-k\,L\,t}$

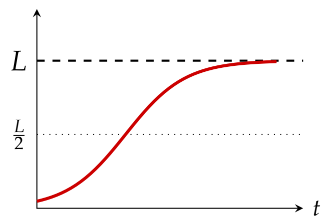


Logistic Differential Equation

We can find general solution by separation of variables:

$$\boxed{y(t) = \frac{L}{1 + C\,e^{-L\,k\,t}}}$$

All solutions have an inflection point in $y = \frac{L}{2}$.



Example – Logistic Differential Equation

A flu epidemic happens in a city with 8100 inhabitants. When the epidemic has been detected 100 persons have been infected. Twenty days later 1000 persons have been infected. It is expected that all inhabitants eventually will be infected.
Give a model for the number of infected persons.

We use a logistic ODE with $L = 8100$.
Let $q(t)$ denote the number of infected persons,
where $q(0) = 100$ and $q(20) = 1000$.

The general solution of this ODE is

$$q(t) = \frac{8100}{1 + C\,e^{-8100kt}}$$

We have to estimate parameters k and C.

Example – Logistic Differential Equation

$$\begin{array}{llll} q(0) = 100 & \Rightarrow \dfrac{8100}{1+C} = 100 & \Rightarrow & C = 80 \\ q(20) = 1000 & \Rightarrow \dfrac{8100}{1+80\,e^{-8100\cdot 20\,k}} = 1000 & \Rightarrow & k = 0.00001495 \end{array}$$

The number of infected persons can be described by means of function

$$q(t) = \frac{8100}{1+80\,e^{-0.121\,t}}\,.$$

Differential Equation of Second Order

An **ordinary differential equation (ODE) of second order** is an equation where the unknown is a univariate function and which contains the second (but not any higher) derivative of that function.

$$\boxed{y'' = F(t, y, y')}$$

We restrict our interest to **linear differential equations of second order with constant coefficients**:

$$\boxed{y''(t) + a_1\,y'(t) + a_2\,y(t) = s}$$

Homogeneous Linear ODE of Second Order

We obtain the general solution of the homogeneous linear ODE

$$y''(t) + a_1\,y'(t) + a_2\,y(t) = 0$$

by means of the ansatz

$$y(t) = C\,e^{\lambda t}$$

where λ satisfies the **characteristic equation**

$$\boxed{\lambda^2 + a_1\lambda + a_2 = 0}$$

This condition immediately follows from

$$\begin{aligned} y''(t) + a_1\,y'(t) + a_2\,y(t) &= \lambda^2\,C\,e^{\lambda t} + a_1\,\lambda\,C\,e^{\lambda t} + a_2\,C\,e^{\lambda t} \\ &= C\,e^{\lambda t}(\lambda^2 + a_1\,\lambda + a_2) = 0 \end{aligned}$$

Characteristic Equation

The characteristic equation

$$\lambda^2 + a_1\lambda + a_2 = 0$$

has solutions

$$\lambda_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}$$

We have three cases:

1. $\frac{a_1^2}{4} - a_2 > 0$: two distinct real solutions
2. $\frac{a_1^2}{4} - a_2 = 0$: exactly one real solution
3. $\frac{a_1^2}{4} - a_2 < 0$: two complex (non-real) solutions

Case: $\frac{a_1^2}{4} - a_2 > 0$

The general solution of the homogeneous ODE is given by

$$\boxed{y(t) = C_1\, e^{\lambda_1 t} + C_2\, e^{\lambda_2 t}\,, \qquad \text{with} \quad \lambda_{1,2} = -\frac{a_1}{2} \pm \sqrt{\frac{a_1^2}{4} - a_2}}$$

where C_1 and C_2 are arbitrary real numbers.

Example: $\frac{a_1^2}{4} - a_2 > 0$

Compute the general solution of ODE

$$y'' - y' - 2y = 0\,.$$

Characteristic equation

$$\lambda^2 - \lambda - 2 = 0$$

has distinct real solutions

$$\lambda_1 = -1 \quad \text{and} \quad \lambda_2 = 2\,.$$

Thus the general solution of the homogeneous ODE is given by

$$y(t) = C_1 e^{-t} + C_2 e^{2t}\,.$$

Case: $\frac{a_1^2}{4} - a_2 = 0$

The general solution of the homogeneous ODE is given by

$$\boxed{y(t) = (C_1 + C_2\, t)\, e^{\lambda t}\,, \qquad \text{with} \quad \lambda = -\frac{a_1}{2}}$$

We can verify the validity of solution $t\, e^{\lambda t}$ by a simple (but tedious) straight-forward computation.

Example: $\frac{a_1^2}{4} - a_2 = 0$

Compute the general solution of ODE

$$y'' + 4y' + 4y = 0\,.$$

Characteristic equation

$$\lambda^2 + 4\lambda + 4 = 0$$

has the unique solution

$$\lambda = -2\,.$$

The general solution of the homogeneous ODE is thus given by

$$y(t) = (C_1 + C_2 t)\, e^{-2t}\,.$$

Case: $\frac{a_1^2}{4} - a_2 < 0$

In this case root $\sqrt{\frac{a_1^2}{4} - a_2}$ is a non-real (imaginary) number.

From the rules for complex numbers one can derive purely real solutions:

$$\boxed{\begin{aligned} & y(t) = e^{at}\left[C_1 \cos(bt) + C_2 \sin(bt)\right] \\ & \qquad \text{with} \quad a = -\tfrac{a_1}{2} \quad \text{and} \quad b = \sqrt{\left|\tfrac{a_1^2}{4} - a_2\right|} \end{aligned}}$$

Notice that a is the real part of the solution of the characteristic equation and b the imaginary part.
Computations with complex numbers however are beyond the scope of this course.

Example: $\frac{a_1^2}{4} - a_2 < 0$

Compute the general solution of ODE

$$y'' + y' + y = 0\,.$$

Characteristic equation

$$\lambda^2 + \lambda + 1 = 0$$

does not have real solutions as $\frac{a_1^2}{4} - a_2 = \frac{1}{4} - 1 = -\frac{3}{4} < 0$.

$$a = -\tfrac{a_1}{2} = -\tfrac{1}{2} \quad \text{and} \quad b = \sqrt{\left|\tfrac{a_1^2}{4} - a_2\right|} = \sqrt{\tfrac{3}{4}} = \tfrac{\sqrt{3}}{2}$$

The general solution of the homogeneous ODE is thus given by

$$y(t) = e^{-\frac{1}{2}t}\left[C_1 \cos\left(\tfrac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]\,.$$

Inhomogeneous Linear ODE of Second Order

We obtain the general solution of the inhomogeneous ODE

$$y''(t) + a_1\, y'(t) + a_2\, y(t) = s$$

by mean so (provide that $a_2 \neq 0$)

$$\boxed{y(t) = y_h(t) + \frac{s}{a_2}}$$

where $y_h(t)$ is the general solution of the corresponding homogeneous ODE

$$y_h''(t) + a_1\, y_h'(t) + a_2\, y_h(t) = 0\,.$$

Example – Inhomogeneous Linear ODE of Second Order

Compute the general solution of ODE

$$y''(t) + y'(t) - 2y(t) = -10$$

Characteristic equation of the homogeneous ODE

$$\lambda^2 + \lambda - 2 = 0$$

has real solutions

$$\lambda_1 = 1 \quad \text{and} \quad \lambda_2 = -2\,.$$

The general solution of the inhomogeneous ODE is thus given by

$$y(t) = C_1\, e^{\lambda_1 t} + C_2\, e^{\lambda_2 t} + \frac{s}{a_2} = C_1\, e^{t} + C_2\, e^{-2t} + \frac{-10}{-2}\,.$$

Initial Value Problem

All general solutions of linear ODEs of second order contain two independent integration constants C_1 and C_2.

Consequently we need two initial values for the particular solution of the **initial value problem**

$$\begin{cases} y''(t) + a_1\, y'(t) + a_2\, y(t) = s \\ y(t_0) = y_0 \\ y'(t_0) = y'_0 \end{cases}$$

Example – Initial Value Problem

Find the particular solution of initial value problem

$$y''(t) + y'(t) - 2y(t) = -10, \qquad y(0) = 12, \quad y'(0) = -2\,.$$

Its general solution is given by

$$\begin{aligned} y(t) &= C_1\, e^t + C_2\, e^{-2t} + 5 \\ y'(t) &= C_1\, e^t - 2C_2\, e^{-2t} \end{aligned}$$

Substitution of the initial values yields equations

$$\begin{aligned} 12 &= y(0) = C_1 + C_2 \\ -2 &= y'(0) = C_1 - 2C_2 \end{aligned}$$

with solutions $C_1 = 4$ and $C_2 = 3$.
Thus the particular solution of the initial value problem is given by

$$y(t) = 4e^t + 3e^{-2t} + 5\,.$$

Fixed Point of an ODE

The inhomogeneous linear ODE

$$y''(t) + a_1\, y'(t) + a_2\, y(t) = s$$

has the special constant solution

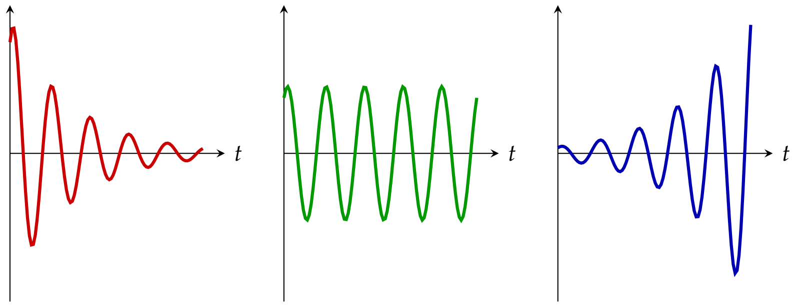
$$y(t) = \bar{y} = \frac{s}{a_2} \quad \text{(= constant)}$$

Point $\bar{y}$ is called **fixed point**, **stationary point**, or **equilibrium point** of the ODE.

Stable and Unstable Fixed Points

The value of a determines the qualitative behavior of solution curve

$$y(t) = e^{at}\left[C_1\cos(bt) + C_2\sin(bt)\right] + \bar{y}\,.$$



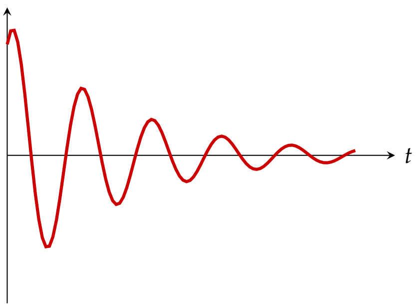
$a < 0$	$a = 0$	$a > 0$
stable fixed point		unstable fixed point

Asymptotically Stable Fixed Point

If $a < 0$, then every solution

$$y(t) = e^{at}\left[C_1\cos(bt) + C_2\sin(bt)\right] + \bar{y}$$

converges to $\bar{y}$. The fixed point $\bar{y}$ is then **asymptotically stable**.

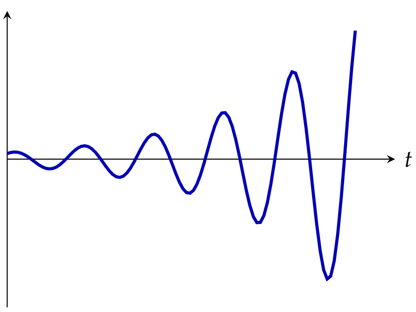


Unstable Fixed Point

If $a > 0$, then every solution

$$y(t) = e^{at}\left[C_1\cos(bt) + C_2\sin(bt)\right] + \bar{y}$$

with initial value $y(0) = y_0 \neq \bar{y}$ diverges.
Such a fixed point $\bar{y}$ is called **unstable**.



Example – Asymptotically Stable Fixed Point

The general solution of

$$y'' + y' + y = 2$$

is given

$$y(t) = 2 + e^{-\frac{1}{2}t}\left[C_1 \cos\left(\tfrac{\sqrt{3}}{2}t\right) + C_2 \sin\left(\tfrac{\sqrt{3}}{2}t\right)\right]$$

Fixed point $\bar{y} = 2$ is asymptotically stable as $a = -\frac{1}{2} < 0$.

Summary

- differential equation of first order
- ODE
- vector field
- separation of variables
- homogeneous and inhomogeneous linear ODE of first order
- logistic ODE
- homogeneous and inhomogeneous linear ODE of second order
- stable and unstable equilibrium points