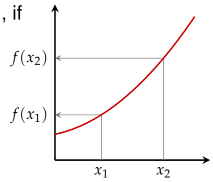
Chapter 13

Convex and Concave

Monotone Functions*

Function f is called **monotonically increasing**, if

$$\boxed{x_1 \leq x_2 \;\Rightarrow\; f(x_1) \leq f(x_2)}$$

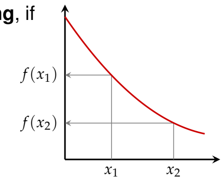


It is called *strictly monotonically increasing*, if

$$\boxed{x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)}$$

Function f is called **monotonically decreasing**, if

$$\boxed{x_1 \leq x_2 \Rightarrow f(x_1) \geq f(x_2)}$$



It is called *strictly monotonically decreasing*, if

$$\boxed{x_1 < x_2 \;\Leftrightarrow\; f(x_1) > f(x_2)}$$

Monotone Functions*

For differentiable functions we have

f monotonically increasing	$\Leftrightarrow$	$f'(x) \geq 0$	for all $x \in D_f$
f monotonically decreasing	$\Leftrightarrow$	$f'(x) \leq 0$	for all $x \in D_f$

f strictly monotonically increasing	$\Leftarrow$	$f'(x) > 0$	for all $x \in D_f$
f strictly monotonically decreasing	$\Leftarrow$	$f'(x) < 0$	for all $x \in D_f$

Function $f\colon (0,\infty),\ x \mapsto \ln(x)$ is strictly monotonically increasing, as

$$f'(x) = (\ln(x))' = \frac{1}{x} > 0 \quad \text{for all } x > 0$$

Locally Monotone Functions*

A function f can be monotonically increasing in some interval and decreasing in some other interval.

For *continuously* differentiable functions (i.e., when $f'(x)$ is continuous) we can use the following procedure:

1. Compute first derivative $f'(x)$.
2. Determine all roots of $f'(x)$.
3. We thus obtain intervals where $f'(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f'(x_i)$.

Example – Locally Monotone Functions*

In which region is function $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ monotonically increasing?

We have to solve inequality $f'(x) \geq 0$:

1. $f'(x) = 6\,x^2 - 24\,x + 18$
2. Roots: $x^2 - 4\,x + 3 = 0 \quad \Rightarrow \quad x_1 = 1,\ x_2 = 3$
3. Obtain 3 intervals: $(-\infty, 1]$, $[1, 3]$, and $[3, \infty)$
4. Sign of $f'(x)$ at appropriate points in each interval: $f'(0) = 3 > 0$, $f'(2) = -1 < 0$, and $f'(4) = 3 > 0$.
5. $f'(x)$ cannot change sign in each interval: $f'(x) \geq 0$ in $(-\infty, 1]$ and $[3, \infty)$.

Function $f(x)$ is monotonically increasing in $(-\infty, 1]$ and in $[3, \infty)$.

Monotone and Inverse Function

If f is *strictly monotonically increasing*, then

$$x_1 < x_2 \;\Leftrightarrow\; f(x_1) < f(x_2)$$

immediately implies

$$x_1 \neq x_2 \;\Leftrightarrow\; f(x_1) \neq f(x_2)$$

That is, f is *one-to-one*.

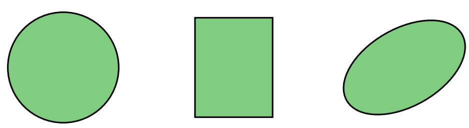
So if f is onto and strictly monotonically increasing (or decreasing), then f is **invertible**.

Convex Set

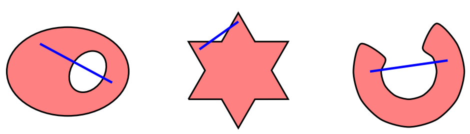
A set $D \subseteq \mathbb{R}^n$ is called **convex**, if for any two points $\mathbf{x}, \mathbf{y} \in D$ the straight line segment between these points also belongs to D, i.e.,

$$(1-h)\,\mathbf{x} + h\,\mathbf{y} \in D \qquad \text{for all } h \in [0,1], \text{ and } \mathbf{x}, \mathbf{y} \in D\,.$$

convex:

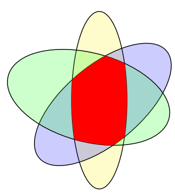


not convex:



Intersection of Convex Sets

Let $S_1, \ldots, S_k$ be convex subsets of $\mathbb{R}^n$. Then their *intersection* $S_1 \cap \ldots \cap S_k$ is also convex.



The union of convex sets need not be convex.

Example – Half-Space

Let $\mathbf{p} \in \mathbb{R}^n$ and $m \in \mathbb{R}$ be fixed, $\mathbf{p} \neq 0$. Then

$$H = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{p}^\mathsf{T} \cdot \mathbf{x} = m\}$$

is a so called **hyper-plane** which partitions the $\mathbb{R}^n$ into two **half-spaces**

$$H_+ = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \geq m\} \,,$$
$$H_- = \{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m\} \,.$$

Sets H, H_+ and H_- are convex.

Let $\mathbf{x}$ be a vector of goods, $\mathbf{p}$ the vector of prices and m the budget.
Then the budget set is convex.

$$\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m, \mathbf{x} \geq 0\}$$
$$= \{\mathbf{x} \colon \mathbf{p}^\mathsf{T} \cdot \mathbf{x} \leq m\} \cap \{\mathbf{x} \colon x_1 \geq 0\} \cap \ldots \cap \{\mathbf{x} \colon x_n \geq 0\}$$

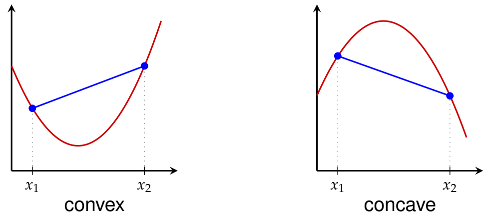
Convex and Concave Functions

Function f is called **convex** in domain $D \subseteq \mathbb{R}^n$, if D is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \leq (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

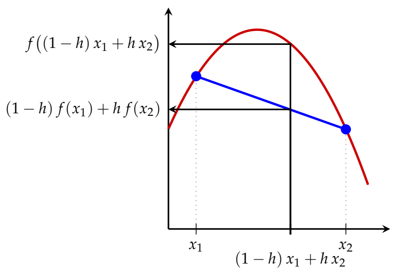
for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ and all $h \in [0,1]$. It is called **concave**, if

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \geq (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$



Concave Function*

$$f\big((1-h)\,x_1 + h\,x_2\big) \;\geq\; (1-h)\,f(x_1) + h\,f(x_2)$$



Secant is below the graph of function f.

Strictly Convex and Concave Functions

Function f is **strictly convex** in domain $D \subseteq \mathbb{R}^n$, if D is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) < (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ and all $h \in (0,1)$.

Function f is **strictly concave** in domain $D \subseteq \mathbb{R}^n$, if D is *convex* and

$$\boxed{f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) > (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2)}$$

for all $\mathbf{x}_1, \mathbf{x}_2 \in D$ with $\mathbf{x}_1 \neq \mathbf{x}_2$ and all $h \in (0,1)$.

Example – Linear Function

Let $\mathbf{a} \in \mathbb{R}^n$ be fixed.
Then $f(\mathbf{x}) = \mathbf{a}^\mathsf{T} \cdot \mathbf{x}$ is a linear map and we find:

$$\begin{aligned} f((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) &= \mathbf{a}^\mathsf{T} \cdot ((1-h)\,\mathbf{x}_1 + h\,\mathbf{x}_2) \\ &= (1-h)\,\mathbf{a}^\mathsf{T} \cdot \mathbf{x}_1 + h\,\mathbf{a}^\mathsf{T} \cdot \mathbf{x}_2 \\ &= (1-h)\,f(\mathbf{x}_1) + h\,f(\mathbf{x}_2) \end{aligned}$$

That is, every *linear function* is both *concave and convex*.

However, a linear function is neither strictly concave nor strictly convex, as the inequality is never strict.

Example – Quadratic Univariate Function

Function $f(x) = x^2$ is *strictly convex*:

$$\begin{aligned} & f((1-h)\,x + h\,y) - \big[(1-h)\,f(x) + h\,f(y)\big] \\ &\quad = ((1-h)\,x + h\,y)^2 - \big[(1-h)\,x^2 + h\,y^2\big] \\ &\quad = (1-h)^2\,x^2 + 2(1-h)h\,xy + h^2\,y^2 - (1-h)\,x^2 - h\,y^2 \\ &\quad = -h(1-h)\,x^2 + 2(1-h)h\,xy - h(1-h)\,y^2 \\ &\quad = -h(1-h)\,(x-y)^2 \\ &\quad < 0 \quad \text{for } x \neq y \text{ and } 0 < h < 1. \end{aligned}$$

Thus
$$f((1-h)\,x + h\,y) < (1-h)\,f(x) + h\,f(y)$$

for all $x \neq y$ and $0 < h < 1$,
i.e., $f(x) = x^2$ is strictly convex, as claimed.

Properties

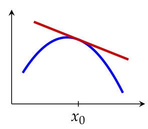
- If $f(\mathbf{x})$ is (strictly) *convex*, then $-f(\mathbf{x})$ is (strictly) *concave* (and vice versa).
- If $f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})$ are *convex* (concave) functions and $\alpha_1, \ldots, \alpha_k > 0$, then

  $$g(\mathbf{x}) = \alpha_1 f_1(\mathbf{x}) + \cdots + \alpha_k f_k(\mathbf{x})$$

  is also *convex* (concave).
- If (at least) one of the functions $f_i(x)$ is *strictly convex* (strictly concave), then $g(x)$ is strictly convex (strictly concave).

Properties

For a differentiable functions the following holds:



- Function f is **concave** if and only if

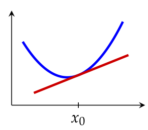
  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) \leq \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}$$

  i.e., the function graph is always below the tangent.
- Function f is **strictly concave** if and only if

  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) < \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \qquad \text{for all } \mathbf{x} \neq \mathbf{x}_0}$$
- Function f is **convex** if and only if

  $$\boxed{f(\mathbf{x}) - f(\mathbf{x}_0) \geq \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}$$

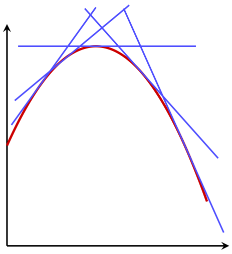
  (Analogous for strictly convex functions.)



Univariate Functions*

For two times differentiable functions we have

f convex	$\Leftrightarrow$	$f''(x) \geq 0$	for all $x \in D_f$
f concave	$\Leftrightarrow$	$f''(x) \leq 0$	for all $x \in D_f$



Derivative $f'(x)$ is monotonically decreasing,

thus $f''(x) \leq 0$.

Univariate Functions*

For two times differentiable functions we have

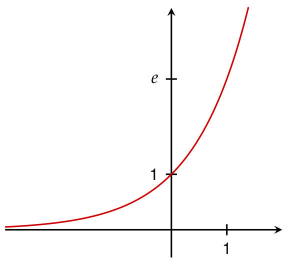
f strictly convex	$\Leftarrow$	$f''(x) > 0$	for all $x \in D_f$
f strictly concave	$\Leftarrow$	$f''(x) < 0$	for all $x \in D_f$

Example – Convex Function*

Exponential function:

$f(x) = e^x$
$f'(x) = e^x$
$f''(x) = e^x > 0$ for all $x \in \mathbb{R}$

$\exp(x)$ is (strictly) convex.

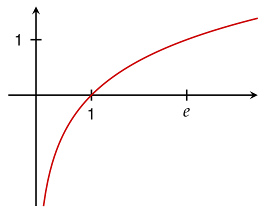


Example – Concave Function*

Logarithm function: $(x > 0)$

$f(x) = \ln(x)$
$f'(x) = \frac{1}{x}$
$f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$

$\ln(x)$ is (strictly) concave.



Locally Convex Functions*

A function f can be convex in some interval and concave in some other interval.

For two times *continuously* differentiable functions (i.e., when $f''(x)$ is continuous) we can use the following procedure:

1. Compute second derivative $f''(x)$.
2. Determine all roots of $f''(x)$.
3. We thus obtain intervals where $f''(x)$ does not change sign.
4. Select appropriate points x_i in each interval and determine the sign of $f''(x_i)$.

Locally Concave Function*

In which region is $f(x) = 2\,x^3 - 12\,x^2 + 18\,x - 1$ concave?

We have to solve inequality $f''(x) \leq 0$.

1. $f''(x) = 12\,x - 24$
2. Roots: $12\,x - 24 = 0 \quad \Rightarrow \quad x = 2$
3. Obtain 2 intervals: $(-\infty, 2]$ and $[2, \infty)$
4. Sign of $f''(x)$ at appropriate points in each interval:
   $f''(0) = -24 < 0$ and $f''(4) = 24 > 0$.
5. $f''(x)$ cannot change sign in each interval: $f''(x) \leq 0$ in $(-\infty, 2]$

Function $f(x)$ is concave in $(-\infty, 2]$.

Univariate Restrictions

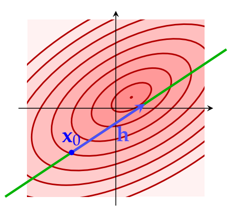
Notice, that by definition a (multivariate) function is convex if and only if every restriction of its domain to a straight line results in a convex univariate function. That is:

Function $f \colon D \subset \mathbb{R}^n \to \mathbb{R}$ is convex

if and only if

$g(t) = f(\mathbf{x}_0 + t \cdot \mathbf{h})$ is convex

for all $\mathbf{x}_0 \in D$ and
all non-zero $\mathbf{h} \in \mathbb{R}^n$.



Quadratic Form

Let $\mathbf{A}$ be a symmetric matrix
and $q_{\mathbf{A}}(\mathbf{x}) = \mathbf{x}^\mathsf{T}\mathbf{A}\mathbf{x}$ be the corresponding quadratic form.

Matrix $\mathbf{A}$ can be diagonalized, i.e., if we use an orthonormal basis of its eigenvectors, then $\mathbf{A}$ becomes a diagonal matrix with the eigenvalues of $\mathbf{A}$ as its elements:

$$q_{\mathbf{A}}(\mathbf{x}) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \cdots + \lambda_n x_n^2 \,.$$

- It is convex if all eigenvalues $\lambda_i \geq 0$
  as it is the sum of convex functions.
- It is concave if all $\lambda_i \leq 0$
  as it is the negative of a convex function.
- It is neither convex nor concave if we have eigenvalues with $\lambda_i > 0$ and $\lambda_i < 0$.

Quadratic Form

We find for a quadratic form $q_{\mathbf{A}}$:

- *strictly convex* $\Leftrightarrow$ *positive definite*
- *convex* $\Leftrightarrow$ *positive semidefinite*
- *strictly concave* $\Leftrightarrow$ *negative definite*
- *concave* $\Leftrightarrow$ *negative semidefinite*
- neither $\Leftrightarrow$ *indefinite*

We can determine the definiteness of $\mathbf{A}$ by means of

- the eigenvalues of $\mathbf{A}$, or
- the (leading) principle minors of $\mathbf{A}$.

Example – Quadratic Form

Let $\mathbf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. Leading principle minors:

$$A_1 = 2 \quad > 0$$

$$A_2 = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5 \quad > 0$$

$$A_3 = |\mathbf{A}| = \begin{vmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{vmatrix} = 8 \quad > 0$$

$\mathbf{A}$ is thus positive definite.
Quadratic form $q_{\mathbf{A}}$ is *strictly convex*.

Example – Quadratic Form

Let $\mathbf{A} = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -2 \end{pmatrix}$. Principle Minors:

$$A_1 = -1 \qquad A_2 = -4 \qquad A_3 = -2$$

$$A_{1,2} = \begin{vmatrix} -1 & 0 \\ 0 & -4 \end{vmatrix} = 4 \quad A_{1,3} = \begin{vmatrix} -1 & 1 \\ 1 & -2 \end{vmatrix} = 1 \quad A_{2,3} = \begin{vmatrix} -4 & 2 \\ 2 & -2 \end{vmatrix} = 4$$

$$A_{1,2,3} = \begin{vmatrix} -1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & -2 \end{vmatrix} = 0$$

$$A_i \quad \leq 0$$
$$A_{i,j} \quad \geq 0$$
$$A_{1,2,3} \quad \leq 0$$

$\mathbf{A}$ is thus negative semidefinite.
Quadratic form $q_{\mathbf{A}}$ is *concave* (but not strictly concave).

Concavity of Differentiable Functions

Le $f\colon D \subseteq \mathbb{R}^n \to \mathbb{R}$ with Taylor series expansion

$$f(\mathbf{x}_0 + \mathbf{h}) = f(\mathbf{x}_0) + \nabla f(\mathbf{x}_0) \cdot \mathbf{h} + \tfrac{1}{2}\mathbf{h}^\mathsf{T} \cdot \mathbf{H}_f(\mathbf{x}_0) \cdot \mathbf{h} + \mathcal{O}(\|\mathbf{h}\|^3)$$

Hessian matrix $\mathbf{H}_f(\mathbf{x}_0)$ determines the concavity or convexity of f around expansion point $\mathbf{x}_0$.

- $\mathbf{H}_f(\mathbf{x}_0)$ *positive definite* $\Rightarrow$ f *strictly convex* around $\mathbf{x}_0$
- $\mathbf{H}_f(\mathbf{x}_0)$ *negative definite* $\Rightarrow$ f *strictly concave* around $\mathbf{x}_0$

- $\mathbf{H}_f(\mathbf{x})$ *positive semidefinite* for all $\mathbf{x} \in D$ $\Leftrightarrow$ f *convex* in D
- $\mathbf{H}_f(\mathbf{x})$ *negative semidefinite* for all $\mathbf{x} \in D$ $\Leftrightarrow$ f *concave* in D

Recipe – Strictly Convex

1. Compute Hessian matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \cdots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \cdots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

2. Compute all *leading principle minors* H_i.
3.
   - f *strictly convex* $\Leftrightarrow$ all $H_k > 0$ for (almost) **all** $\mathbf{x} \in D$
   - f *strictly concave* $\Leftrightarrow$ all $(-1)^k H_k > 0$ for (almost) **all** $\mathbf{x} \in D$

   [$(-1)^k H_k > 0$ implies: $H_1, H_3, \ldots < 0$ and $H_2, H_4, \ldots > 0$]
4. Otherwise f is *neither* **strictly** convex *nor* strictly concave.

Recipe – Convex

1. Compute Hessian matrix

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} f_{x_1x_1}(\mathbf{x}) & f_{x_1x_2}(\mathbf{x}) & \cdots & f_{x_1x_n}(\mathbf{x}) \\ f_{x_2x_1}(\mathbf{x}) & f_{x_2x_2}(\mathbf{x}) & \cdots & f_{x_2x_n}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ f_{x_nx_1}(\mathbf{x}) & f_{x_nx_2}(\mathbf{x}) & \cdots & f_{x_nx_n}(\mathbf{x}) \end{pmatrix}$$

2. Compute all *principle minors* $H_{i_1,\ldots,i_k}$.
   (Only required if $\det(\mathbf{H}_f) = 0$, see below)
3.
   - f *convex* $\Leftrightarrow$ all $H_{i_1,\ldots,i_k} \geq 0$ for **all** $\mathbf{x} \in D$.
   - f *concave* $\Leftrightarrow$ all $(-1)^k H_{i_1,\ldots,i_k} \geq 0$ for **all** $\mathbf{x} \in D$.
4. Otherwise f is *neither* convex *nor* concave.

Recipe – Convex II

Computation of *all* principle minors can be avoided if $\det(\mathbf{H}_f) \neq 0$. Then a function is either strictly convex/concave (and thus convex/concave) or neither convex nor concave.

In particular we have the following recipe:

1. Compute Hessian matrix $\mathbf{H}_f(\mathbf{x})$.
2. Compute all *leading principle minors* H_i.
3. Check if $\det(\mathbf{H}_f) \neq 0$.
4. Check for strict convexity or concavity.
5. If $\det(\mathbf{H}_f) \neq 0$ and f is neither strictly convex nor concave, then f is neither convex nor concave, either.

Example – Strict Convexity

Is function f (strictly) concave or convex?

$$f(x, y) = x^4 + x^2 - 2\,x\,y + y^2$$

1. Hessian matrix: $\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} 12\,x^2 + 2 & -2 \\ -2 & 2 \end{pmatrix}$
2. Leading principle minors:
   $H_1 = 12\,x^2 + 2 \quad > 0$
   $H_2 = |\mathbf{H}_f(\mathbf{x})| = 24\,x^2 \quad > 0$ for all $x \neq 0$.
3. All leading principle minors > 0 for almost all $\mathbf{x}$
   $\Rightarrow$ f is *strictly convex*. (and thus convex, too)

Example – Cobb-Douglas Function

Let $f(x,y) = x^\alpha y^\beta$ with $\alpha, \beta \geq 0$ and $\alpha + \beta \leq 1$,
and $D = \{(x,y)\colon x, y \geq 0\}$.

Hessian matrix at $\mathbf{x}$:

$$\mathbf{H}_f(\mathbf{x}) = \begin{pmatrix} \alpha(\alpha-1)\, x^{\alpha-2} y^\beta & \alpha\beta\, x^{\alpha-1} y^{\beta-1} \\ \alpha\beta\, x^{\alpha-1} y^{\beta-1} & \beta(\beta-1)\, x^\alpha y^{\beta-2} \end{pmatrix}$$

Principle Minors:

$$H_1 = \underbrace{\alpha}_{\geq 0} \underbrace{(\alpha-1)}_{\leq 0} \underbrace{x^{\alpha-2} y^\beta}_{\geq 0} \quad \boxed{\leq 0}$$

$$H_2 = \underbrace{\beta}_{\geq 0} \underbrace{(\beta-1)}_{\leq 0} \underbrace{x^\alpha y^{\beta-2}}_{\geq 0} \quad \boxed{\leq 0}$$

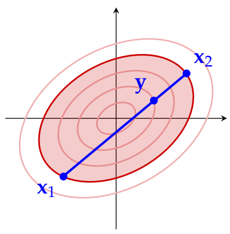
Example – Cobb-Douglas Function

$$\begin{aligned} H_{1,2} &= |\mathbf{H}_f(\mathbf{x})| \\ &= \alpha(\alpha-1)\, x^{\alpha-2} y^\beta \cdot \beta(\beta-1)\, x^\alpha y^{\beta-2} - (\alpha\beta\, x^{\alpha-1} y^{\beta-1})^2 \\ &= \alpha(\alpha-1)\,\beta(\beta-1)\, x^{2\alpha-2} y^{2\beta-2} - \alpha^2\beta^2\, x^{2\alpha-2} y^{2\beta-2} \\ &= \alpha\beta[(\alpha-1)(\beta-1) - \alpha\beta] x^{2\alpha-2} y^{2\beta-2} \\ &= \underbrace{\alpha\beta}_{\geq 0} \underbrace{(1-\alpha-\beta)}_{\geq 0} \underbrace{x^{2\alpha-2} y^{2\beta-2}}_{\geq 0} \quad \boxed{\geq 0} \end{aligned}$$

$H_1 \leq 0$ and $H_2 \leq 0$, and $H_{1,2} \geq 0$ **for all** $(x,y) \in D$.
$f(x,y)$ thus is *concave* in D.

For $0 < \alpha, \beta < 1$ and $\alpha + \beta < 1$ we even find:
$H_1 = H_1 < 0$ and $H_2 = |\mathbf{H}_f(\mathbf{x})| > 0$ **for almost all** $(x,y) \in D$.
$f(x,y)$ is then *strictly concave*.

Lower Level Sets of Convex Functions



Assume that f is *convex*.
Then the **lower level sets** of f

$$\{\mathbf{x} \in D_f \colon f(\mathbf{x}) \leq c\}$$

are *convex*.

Let $\mathbf{x}_1, \mathbf{x}_2 \in \{\mathbf{x} \in D_f \colon f(\mathbf{x}) \leq c\}$,
i.e., $f(\mathbf{x}_1), f(\mathbf{x}_2) \leq c$.

Then for $\mathbf{y} = (1-h)\mathbf{x}_1 + h\mathbf{x}_2$
where $h \in [0,1]$ we find

$$\begin{aligned} f(\mathbf{y}) &= f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \\ &\leq (1-h)\, f(\mathbf{x}_1) + h\, f(\mathbf{x}_2) \\ &\leq (1-h)c + hc = c \end{aligned}$$

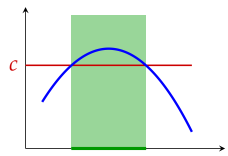
That is, $\mathbf{y} \in \{\mathbf{x} \in D_f \colon f(\mathbf{x}) \leq c\}$, too.

Upper Level Sets of Concave Functions

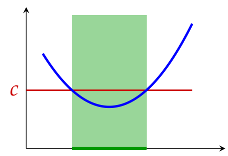
Assume that f is *concave*.
Then the **upper level sets** of f

$$\{\mathbf{x} \in D_f \colon f(\mathbf{x}) \geq c\}$$

are *convex*.



upper level set



lower level set

Extremum and Monotone Transformation

Let $T\colon \mathbb{R} \to \mathbb{R}$ be a *strictly monotonically increasing* function.

If $\mathbf{x}^*$ is a *maximum* of f, then $\mathbf{x}^*$ is also a maximum of $T \circ f$.

As $\mathbf{x}^*$ is a *maximum* of f, we have

$f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x}$.

As T is strictly monotonically increasing,we have

$T(x_1) > T(x_2)$ falls $x_1 > x_2$.

Thus we find

$(T \circ f)(\mathbf{x}^*) = T(f(\mathbf{x}^*)) > T(f(\mathbf{x})) = (T \circ f)(\mathbf{x})$ for all $\mathbf{x}$,

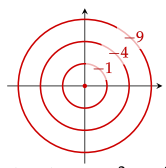
i.e., $\mathbf{x}^*$ is a maximum of $T \circ f$.

As T is one-to-one we also get the converse statement:
If $\mathbf{x}^*$ is a *maximum* of $T \circ f$, then it also is a maximum of f.

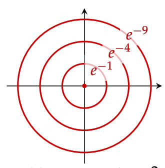
Extremum and Monotone Transformation

A strictly monotonically increasing Transformation T preserves the extrema of f.

Transformation T also preserves the level sets of f:



$f(x,y) = -x^2 - y^2$



$T(f(x,y)) = \exp(-x^2 - y^2)$

Quasi-Convex and Quasi-Concave

Function f is called **quasi-convex** in $D \subseteq \mathbb{R}^n$, if D is *convex* and every *lower level set* $\{\mathbf{x} \in D_f \colon f(\mathbf{x}) \leq c\}$ is *convex*.

Function f is called **quasi-concave** in $D \subseteq \mathbb{R}^n$, if D is *convex* and every *upper level set* $\{\mathbf{x} \in D_f \colon f(\mathbf{x}) \geq c\}$ is *convex*.

Convex and Quasi-Convex

Every *concave* (convex) function also is *quasi-concave* (and quasi-convex, resp.).

However, a quasi-concave function need not be concave.

Let T be a strictly monotonically increasing function.
If function $f(\mathbf{x})$ is *concave* (convex), then $T \circ f$ is *quasi-concave* (and quasi-convex, resp.).

Function $g(x,y) = e^{-x^2-y^2}$ is quasi-concave, as $f(x,y) = -x^2 - y^2$ is concave and $T(x) = e^x$ is strictly monotonically increasing.
However, $g = T \circ f$ is not concave.

A Weaker Condition

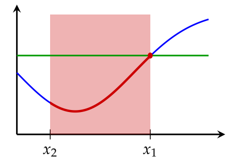
The notion of *quasi-convex* is **weaker** than that of *convex* in the sense that every convex function also is quasi-convex but not vice versa. There are much more quasi-convex functions than convex ones.

The importance of such a weaker notions is based on the observation that a couple of propositions still hold if “convex” is replaced by “quasi-convex”.

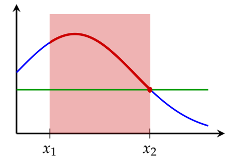
In this way we get a generalization of a theorem, where a *stronger* condition is replaced by a *weaker* one.

Quasi-Convex and Quasi-Concave II

- Function f is *quasi-convex* if and only if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \leq \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$ and $h \in [0, 1]$.
- Function f is *quasi-concave* if and only if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) \geq \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$ and $h \in [0, 1]$.



quasi-convex



quasi-concave

Strictly Quasi-Convex and Quasi-Concave

- Function f is called **strictly quasi-convex** if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) < \max\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$, with $\mathbf{x}_1 \neq \mathbf{x}_2$, and $h \in (0, 1)$.
- Function f is called **strictly quasi-concave** if
$$\boxed{f((1-h)\mathbf{x}_1 + h\mathbf{x}_2) > \min\{f(\mathbf{x}_1), f(\mathbf{x}_2)\}}$$
for all $\mathbf{x}_1, \mathbf{x}_2$, with $\mathbf{x}_1 \neq \mathbf{x}_2$, and $h \in (0, 1)$.

Quasi-convex and Quasi-Concave III

For a differentiable function f we find:

- Function f is *quasi-convex* if and only if
$$\boxed{f(\mathbf{x}) \leq f(\mathbf{x}_0) \quad \Rightarrow \quad \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \leq 0}$$
- Function f is *quasi-concave* if and only if
$$\boxed{f(\mathbf{x}) \geq f(\mathbf{x}_0) \quad \Rightarrow \quad \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) \geq 0}$$

Summary

- monotone function
- convex set
- convex and concave function
- convexity and definiteness of quadratic form
- minors of Hessian matrix
- quasi-convex and quasi-concave function