

## Chapter 12

# Integration

# Antiderivative

A function  $F(x)$  is called an **antiderivative** (or *primitive*) of function  $f(x)$ , if

$$F'(x) = f(x)$$

**Computation:**

Guess and verify

**Example:** We want the antiderivative of  $f(x) = \ln(x)$ .

Guess:  $F(x) = x(\ln(x) - 1)$

Verify:  $F'(x) = (x(\ln(x) - 1))' =$   
 $= 1 \cdot (\ln(x) - 1) + x \cdot \frac{1}{x} = \ln(x)$

But also:  $F(x) = x(\ln(x) - 1) + 5$

# Antiderivative

The antiderivative is denoted by symbol

$$\int f(x) dx + c$$

and is also called the **indefinite integral** of function  $f$ . Number  $c$  is called **integration constant**.

Unfortunately, there are no “*recipes*” for computing antiderivatives (but tools one can try and which may help).

There are functions where antiderivatives cannot be expressed by means of elementary functions.

E.g., the antiderivative of  $\exp(-\frac{1}{2}x^2)$ .

# Basic Integrals

Integrals of some elementary functions:

$f(x)$	$\int f(x) dx$
0	$c$
$x^a$	$\frac{1}{a+1} \cdot x^{a+1} + c$
$e^x$	$e^x + c$
$\frac{1}{x}$	$\ln  x  + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$

(The table is created by swapping the columns in the list of derivatives.)

# Integration Rules

► **Summation rule**

$$\int \alpha f(x) + \beta g(x) dx = \alpha \int f(x) dx + \beta \int g(x) dx$$

► **Integration by parts**

$$\int f \cdot g' dx = f \cdot g - \int f' \cdot g dx$$

► **Integration by substitution**

$$\int f(g(x)) \cdot g'(x) dx = \int f(z) dz$$

with  $z = g(x)$  and  $dz = g'(x) dx$

## Example – Summation Rule

Antiderivative of  $f(x) = 4x^3 - x^2 + 3x - 5$ .

$$\begin{aligned}\int f(x) dx &= \int 4x^3 - x^2 + 3x - 5 dx \\ &= 4 \int x^3 dx - \int x^2 dx + 3 \int x dx - 5 \int dx \\ &= 4 \frac{1}{4} x^4 - \frac{1}{3} x^3 + 3 \frac{1}{2} x^2 - 5x + c \\ &= x^4 - \frac{1}{3} x^3 + \frac{3}{2} x^2 - 5x + c\end{aligned}$$

# Example – Integration by Parts

Antiderivative of  $f(x) = x \cdot e^x$ .

$$\int \underbrace{x}_f \cdot \underbrace{e^x}_{g'} dx = \underbrace{x}_f \cdot \underbrace{e^x}_g - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_g dx = x \cdot e^x - e^x + c$$

$$f = x \quad \Rightarrow \quad f' = 1$$

$$g' = e^x \quad \Rightarrow \quad g = e^x$$

# Example – Integration by Parts

Antiderivative of  $f(x) = x^2 \cos(x)$ .

$$\int \underbrace{x^2}_f \cdot \underbrace{\cos(x)}_{g'} dx = \underbrace{x^2}_f \cdot \underbrace{\sin(x)}_g - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_g dx$$

Integration by parts of the second terms yields:

$$\begin{aligned} \int \underbrace{2x}_f \cdot \underbrace{\sin(x)}_{g'} dx &= \underbrace{2x}_f \cdot \underbrace{(-\cos(x))}_g - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_g dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c \end{aligned}$$

Thus the antiderivative of  $f$  is given by

$$\int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$



# Example – Integration by Substitution

Antiderivative of  $f(x) = 2x \cdot e^{x^2}$ .

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)} dx = \int \exp(z) dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad \Rightarrow \quad dz = g'(x) dx = 2x dx$$

# Integration Rules – Derivation

Integration by parts follows from the product rule for derivatives:

$$\begin{aligned} f(x) \cdot g(x) &= \int (f(x) \cdot g(x))' dx = \int (f'(x) g(x) + f(x) g'(x)) dx \\ &= \int f'(x) g(x) dx + \int f(x) g'(x) dx \end{aligned}$$

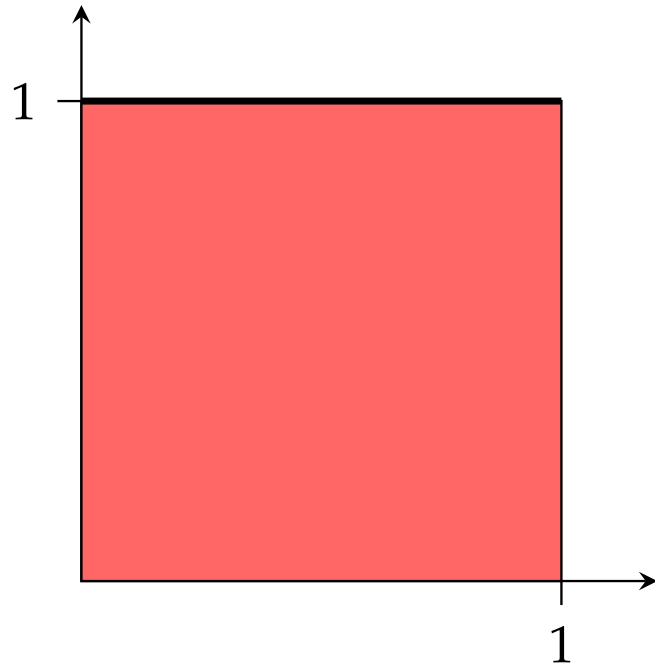
Integration by substitution follows from the chain rule:

Let  $F$  be an antiderivative of  $f$  and let  $z = g(x)$ . Then

$$\begin{aligned} \int f(z) dz &= F(z) = F(g(x)) = \int (F(g(x)))' dx \\ &= \int F'(g(x)) g'(x) dx = \int f(g(x)) g'(x) dx \end{aligned}$$

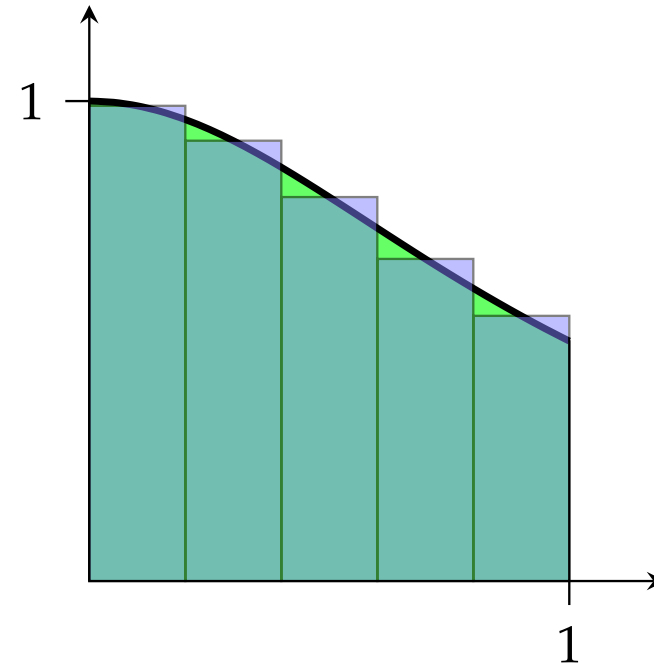
# Area

Compute the areas of the given regions.



$$f(x) = 1$$

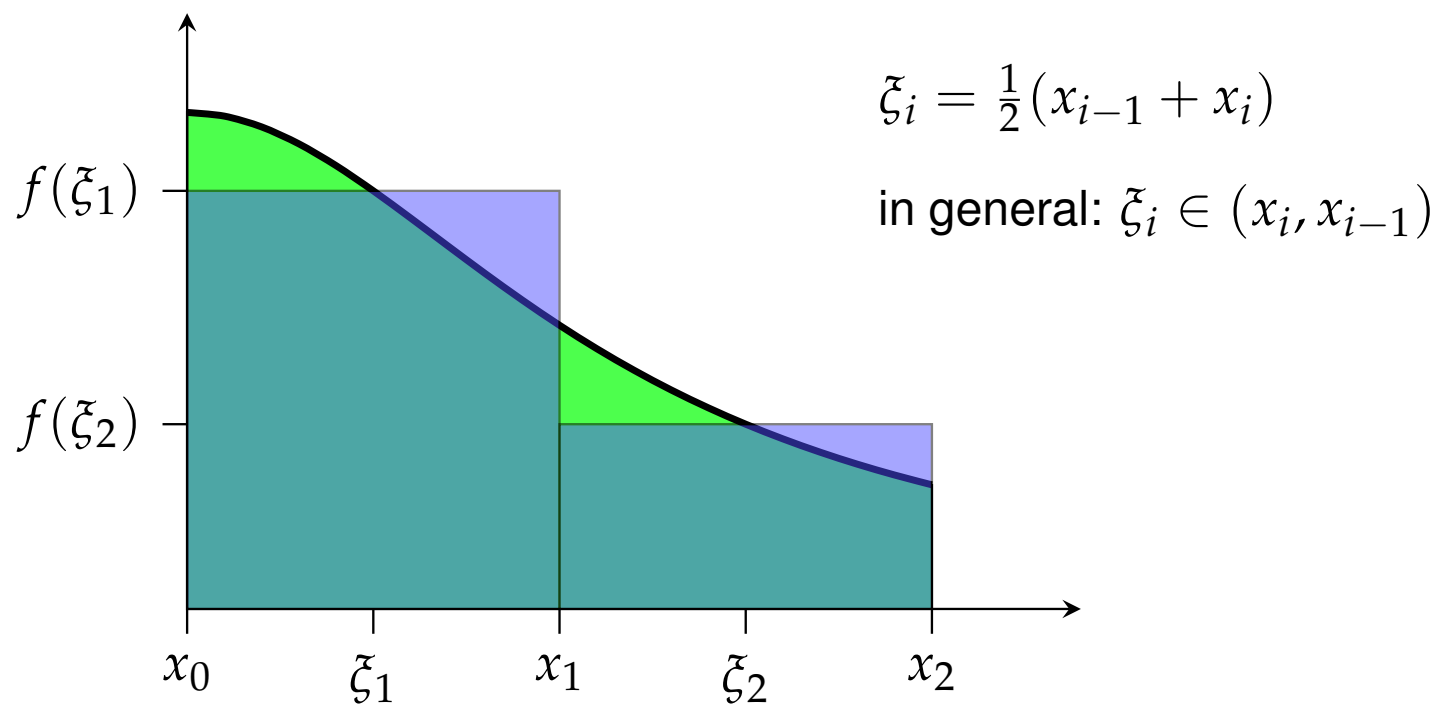
$$\text{Area: } A = 1$$



$$f(x) = \frac{1}{1+x^2}$$

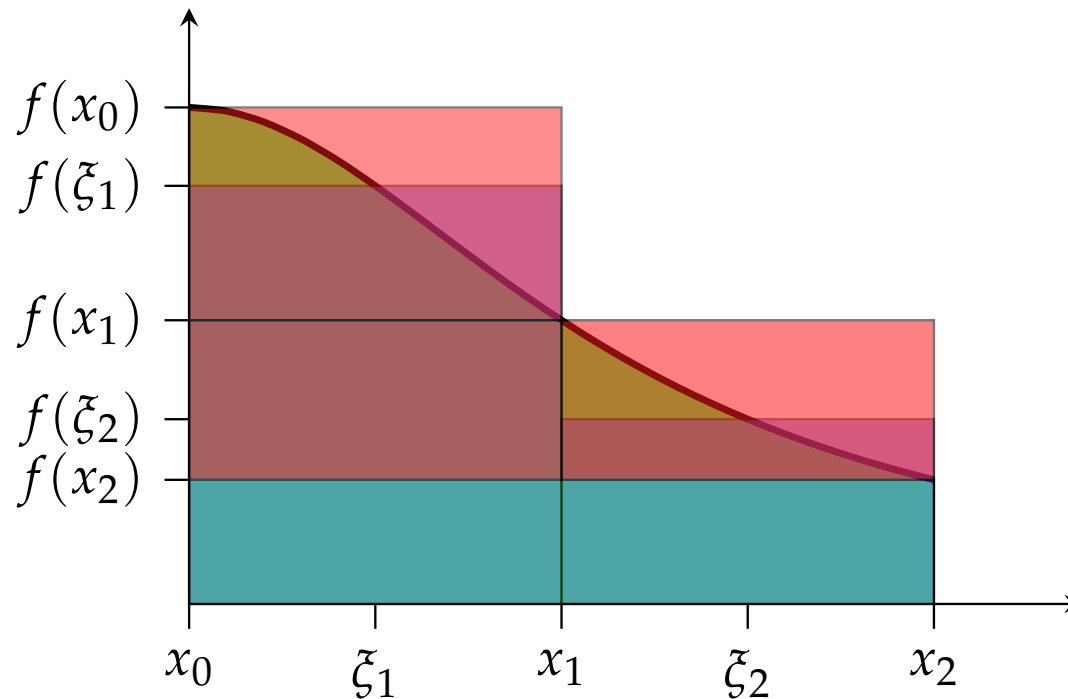
Approximation  
by step function

# Riemann Sum



$$A = \int_a^b f(x) dx \approx \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$$

# Approximation Error



$$\left| \int_a^b f(x) dx - \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1}) \right| \leq (f_{\max} - f_{\min}) (b - a) \frac{1}{n} \rightarrow 0$$

**Assumption:** Function monotone;  $x_0, x_1, \dots, x_n$  equidistant

# Riemann Integral

If all sequences of **Riemann sums**

$$I_n = \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$$

converge, then their (uniquely determined) limit is called the **Riemann integral** of  $f$  and is denoted by  $\int_a^b f(x) dx$ :

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(\xi_i) \cdot (x_i - x_{i-1})$$

Almost all functions in economics have a Riemann integral.

# Riemann Integral – Properties

$$\int_a^b (\alpha f(x) + \beta g(x)) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

$$\int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$\int_a^a f(x) dx = 0$$

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx$$

$$\int_a^b f(x) dx \leq \int_a^b g(x) dx \quad \text{if } f(x) \leq g(x) \text{ for all } x \in [a, b]$$

# Fundamental Theorem of Calculus

Let  $F(x)$  be an antiderivative of a *continuous* function  $f(x)$ , then we find

$$\int_a^b f(x) dx = F(x) \Big|_a^b = F(b) - F(a)$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason  $\int f(x) dx$  is called an *indefinite integral* of  $f$ ; and

$\int_a^b f(x) dx$  is called a **definite integral** of  $f$ .



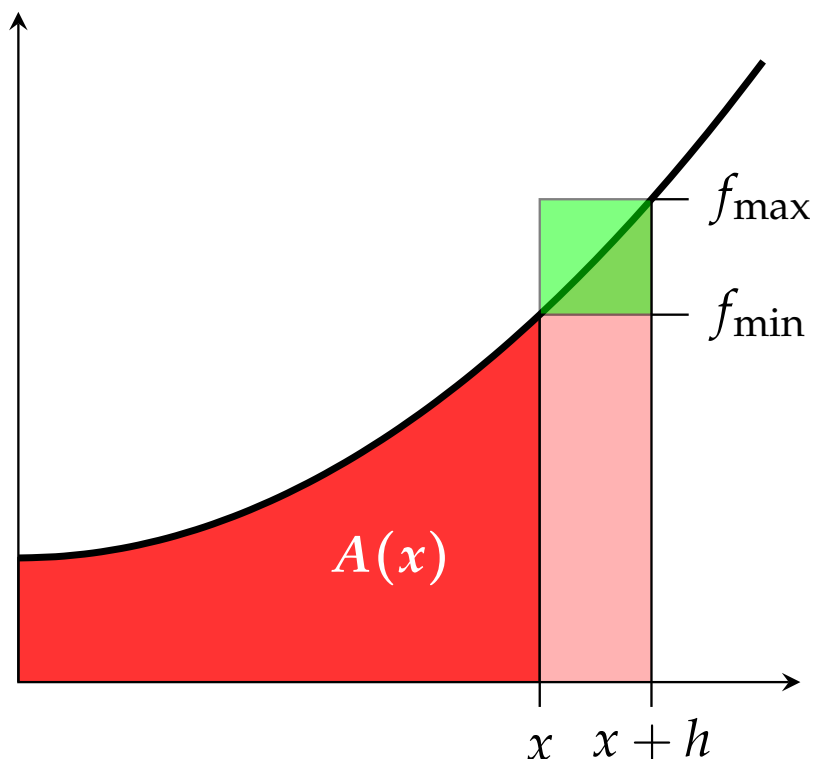
# Example – Fundamental Theorem

Compute the integral of  $f(x) = x^2$  over interval  $[0, 1]$ .

$$\int_0^1 x^2 dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3} \cdot 1^3 - \frac{1}{3} \cdot 0^3 = \frac{1}{3}$$

# Fundamental Theorem / Proof Idea

Let  $A(x)$  be the area between the graph of a continuous function  $f$  and the  $x$ -axis from 0 to  $x$ .



$$f_{\min} \cdot h \leq A(x+h) - A(x) \leq f_{\max} \cdot h$$

$$f_{\min} \leq \frac{A(x+h) - A(x)}{h} \leq f_{\max}$$

Limit for  $h \rightarrow 0$ : ( $\lim_{h \rightarrow 0} f_{\min} = f(x)$ )

$$f(x) \leq \underbrace{\lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h}}_{=A'(x)} \leq f(x)$$

$$A'(x) = f(x)$$

i.e.  $A(x)$  is an antiderivative of  $f(x)$ .

# Integration Rules / (Definite Integrals)

## ► Summation rule

$$\int_a^b \alpha f(x) + \beta g(x) dx = \alpha \int_a^b f(x) dx + \beta \int_a^b g(x) dx$$

## ► Integration by parts

$$\int_a^b f \cdot g' dx = f \cdot g \Big|_a^b - \int_a^b f' \cdot g dx$$

## ► Integration by Substitution

$$\int_a^b f(g(x)) \cdot g'(x) dx = \int_{g(a)}^{g(b)} f(z) dz$$

with  $z = g(x)$  and  $dz = g'(x) dx$

## Example – Integration by Parts

Compute the definite integral  $\int_0^2 x \cdot e^x dx$ .

$$\begin{aligned}\int_0^2 \underbrace{x}_f \cdot \underbrace{e^x}_{g'} dx &= \underbrace{x}_f \cdot \underbrace{e^x}_g \Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_g dx \\ &= x \cdot e^x \Big|_0^2 - e^x \Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0) \\ &= e^2 + 1\end{aligned}$$

Note: we also could use our indefinite integral from above,

$$\int_0^2 x \cdot e^x dx = (x \cdot e^x - e^x) \Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$

# Example – Integration by Substitution

Compute the definite integral  $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x} dx$ .

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x} dx = \int_1^{\ln(10)} \frac{1}{z} dz =$$

$$z = \ln(x) \quad \Rightarrow \quad dz = \frac{1}{x} dx$$

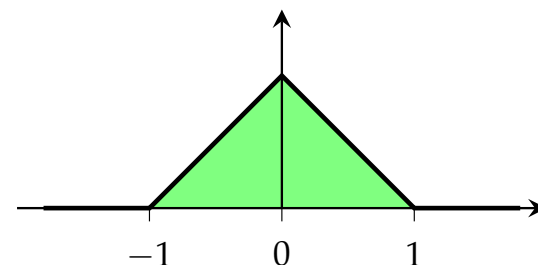
$$= \ln(z) \Big|_1^{\ln(10)} =$$

$$= \ln(\ln(10)) - \ln(1) \approx 0.834$$

# Example – Subdomains

Compute  $\int_{-2}^2 f(x) dx$  for function

$$f(x) = \begin{cases} 1 + x, & \text{for } -1 \leq x < 0, \\ 1 - x, & \text{for } 0 \leq x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \geq 1. \end{cases}$$



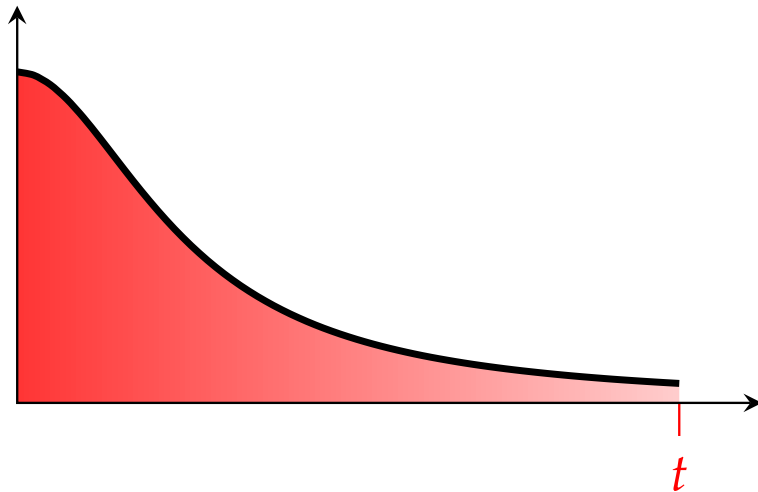
We have

$$\begin{aligned} \int_{-2}^2 f(x) dx &= \int_{-2}^{-1} f(x) dx + \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx + \int_1^2 f(x) dx \\ &= \int_{-2}^{-1} 0 dx + \int_{-1}^0 (1 + x) dx + \int_0^1 (1 - x) dx + \int_1^2 0 dx \\ &= \left(x + \frac{1}{2}x^2\right) \Big|_{-1}^0 + \left(x - \frac{1}{2}x^2\right) \Big|_0^1 \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

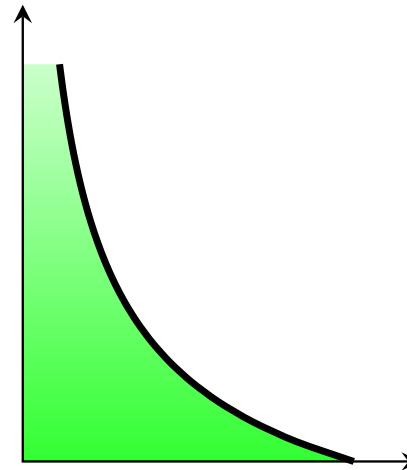
# Improper Integral

An **improper integral** is an integral where

- ▶ the domain of integration is unbounded, or
- ▶ the integrand is unbounded.



$$\int_0^{\infty} f(x) dx = \lim_{t \rightarrow \infty} \int_0^t f(x) dx$$



$$\int_0^1 f(x) dx = \lim_{t \rightarrow 0} \int_t^1 f(x) dx$$

## Example – Improper Integral

$$\int_0^1 \frac{1}{\sqrt{x}} dx = \lim_{t \rightarrow 0} \int_t^1 x^{-\frac{1}{2}} dx = \lim_{t \rightarrow 0} 2\sqrt{x} \Big|_t^1 = \lim_{t \rightarrow 0} (2 - 2\sqrt{t}) = 2$$

$$\int_1^{\infty} \frac{1}{x^2} dx = \lim_{t \rightarrow \infty} \int_1^t x^{-2} dx = \lim_{t \rightarrow \infty} -\frac{1}{x} \Big|_1^t = \lim_{t \rightarrow \infty} -\frac{1}{t} - (-1) = 1$$

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln(x) \Big|_1^t = \lim_{t \rightarrow \infty} \ln(t) - \ln(1) = \infty$$

The improper integral does not exist.



# Two Limits

In probability theory we often have integrals where both boundaries are infinite.

For example, the expectation of random variable  $X$  with density  $f$  is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f(x) dx$$

In such a case we have to separate the domain of integration:

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x \cdot f(x) dx \\ &= \lim_{t \rightarrow -\infty} \int_t^0 x \cdot f(x) dx + \lim_{s \rightarrow \infty} \int_0^s x \cdot f(x) dx \end{aligned}$$

## Beware!

If we yield  $\infty - \infty$ , then the result is **not**  $\infty - \infty = 0$ !

# Leibniz Integration Rule

Let  $f(x, t)$  be *continuously differentiable* function (i.e., all partial derivatives exist and are continuous) and let

$$F(x) = \int_{a(x)}^{b(x)} f(x, t) dt.$$

Then

$$F'(x) = f(x, b(x)) b'(x) - f(x, a(x)) a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt$$

If  $a(x) = a$  and  $b(x) = b$  are constant, then

$$\frac{d}{dx} \left( \int_a^b f(x, t) dt \right) = \int_a^b \frac{\partial}{\partial x} f(x, t) dt$$

## Example – Leibniz Integration Rule

Let  $F(x) = \int_x^{2x} t x^2 dt$  for  $x > 0$ . Compute  $F'(x)$ .

We set  $f(x, t) = t x^2$ ,  $a(x) = x$  and  $b(x) = 2x$  and apply Leibniz's integration rule:

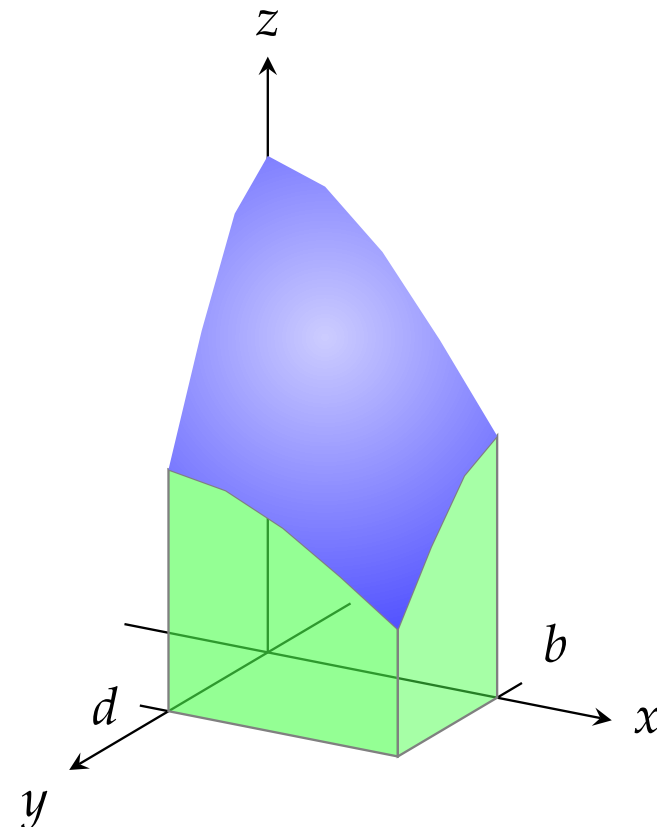
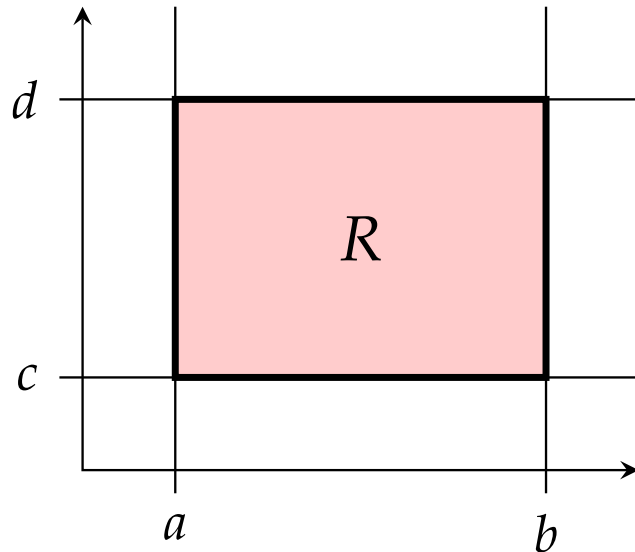
$$\begin{aligned} F'(x) &= f(x, b) \cdot b' - f(x, a) \cdot a' + \int_a^b f_x(x, t) dt \\ &= (2x) x^2 \cdot 2 - (x) x^2 \cdot 1 + \int_x^{2x} 2x t dt \\ &= 4x^3 - x^3 + \left( 2x \frac{1}{2} t^2 \right) \Big|_x^{2x} \\ &= 4x^3 - x^3 + (4x^3 - x^3) \\ &= 6x^3 \end{aligned}$$

# Volume

Let  $f(x, y)$  be a function with domain

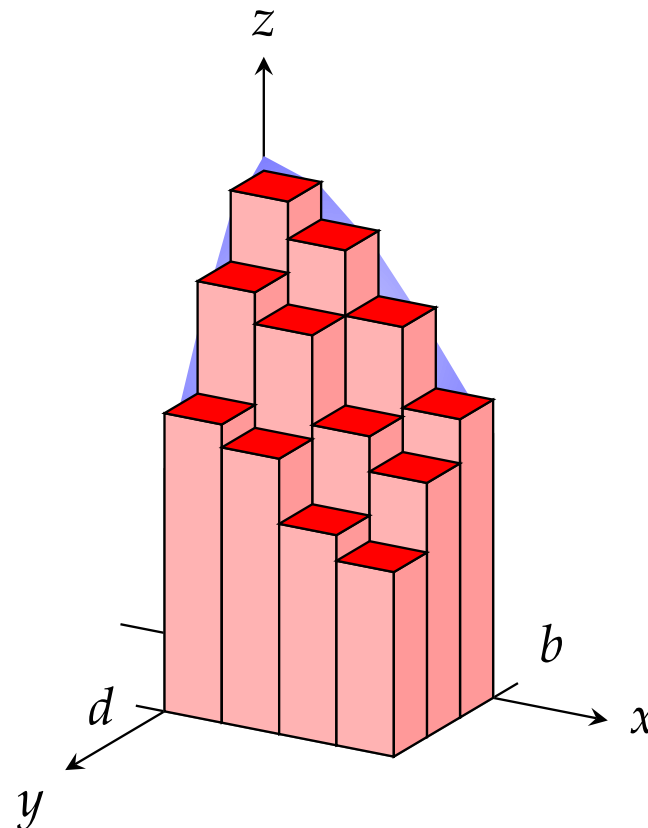
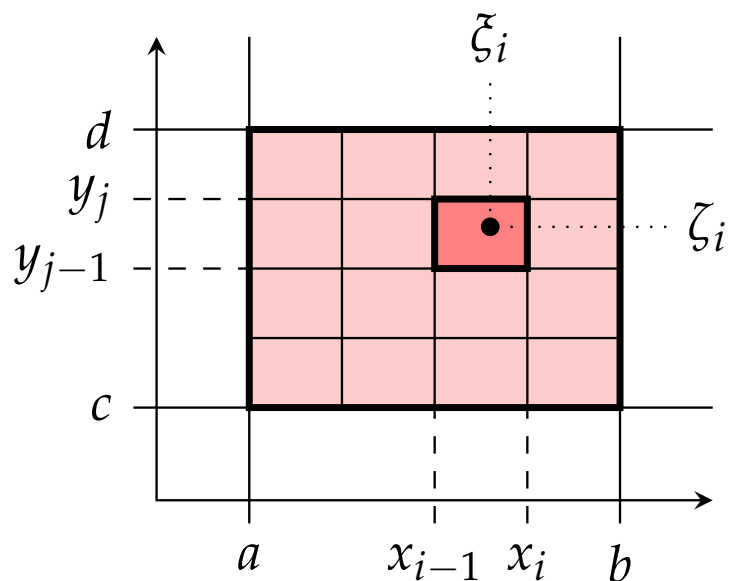
$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d\}$$

What is the Volumen  $V$  below the graph of  $f$ ?



# Riemann Sums

- ▶ Partition  $R$  into smaller rectangles  $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
- ▶ Estimate  $V \approx \sum_{i=1}^n \sum_{j=1}^k f(\zeta_i, \zeta_j) (x_i - x_{i-1}) (y_j - y_{j-1})$



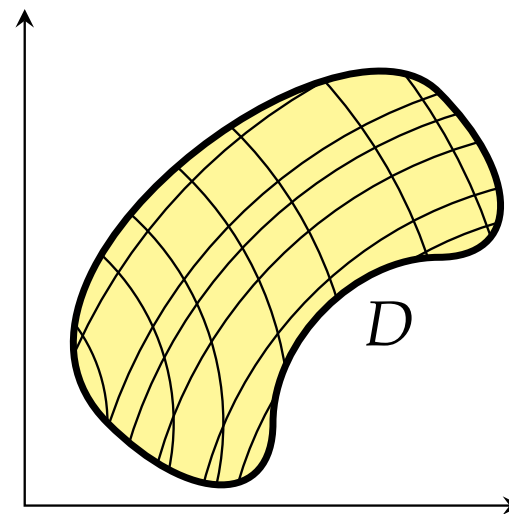
# Riemann Integral

If these **Riemann Sums** converge for partitions of  $R$  with increasing number of rectangles, then the limit is called the **Riemann Integral** of  $f$  over  $R$ :

$$\iint_R f(x, y) dx dy = \lim_{n, k \rightarrow \infty} \sum_{i=1}^n \sum_{j=1}^k f(\xi_i, \zeta_j) (x_i - x_{i-1}) (y_j - y_{j-1})$$

The Riemann integral is defined analogously for arbitrary domains  $D$ .

$$\iint_D f(x, y) dx dy$$



# Fubini's Theorem

Let  $f: R = [a, b] \times [c, d] \subset \mathbb{R}^2 \rightarrow \mathbb{R}$  a *continuous* function. Then

$$\begin{aligned} \iint_R f(x, y) \, dx \, dy &= \int_a^b \left( \int_c^d f(x, y) \, dy \right) dx \\ &= \int_c^d \left( \int_a^b f(x, y) \, dx \right) dy . \end{aligned}$$

Fubini's theorem provides a recipe for computing double integrals stepwise:

1. Treat  $x$  like a constant and compute the inner integral  $\int_c^d f(x, y) \, dy$  w.r.t. variabel  $y$ .
2. Integrate the result from Step 1 w.r.t.  $x$ .

We also may change the order of integration.

## Example – Fubini's Theorem

Compute  $\int_{-1}^1 \int_0^1 (1 - x - y^2 + xy^2) dx dy$ .

We have to integrate twice:

$$\begin{aligned} & \int_{-1}^1 \int_0^1 (1 - x - y^2 + xy^2) dx dy \\ &= \int_{-1}^1 \left( x - \frac{1}{2}x^2 - xy^2 + \frac{1}{2}x^2y^2 \Big|_0^1 \right) dy \\ &= \int_{-1}^1 \left( \frac{1}{2} - \frac{1}{2}y^2 \right) dy = \frac{1}{2}y - \frac{1}{6}y^3 \Big|_{-1}^1 \\ &= \frac{1}{2} - \frac{1}{6} - \left( -\frac{1}{2} + \frac{1}{6} \right) = \frac{2}{3} \end{aligned}$$



# Bounds of Integration

## Beware!

The integration variables and the corresponding integration limits have to be read from **inside** to **outside**.

If we change the order of integration, then we also have to exchange the integration limits:

$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

This may be more obvious if we add (redundant) parenthesis:

$$\int_a^b \left( \int_c^d f(x, y) dy \right) dx = \int_c^d \left( \int_a^b f(x, y) dx \right) dy .$$

# Example – Fubini's Theorem

Integration in reversed order:

$$\begin{aligned} & \int_{-1}^1 \int_0^1 (1 - x - y^2 + xy^2) dx dy \\ &= \int_0^1 \left( \int_{-1}^1 (1 - x - y^2 + xy^2) dy \right) dx \\ &= \int_0^1 \left( y - xy - \frac{1}{3}y^3 + \frac{1}{3}xy^3 \Big|_{-1}^1 \right) dx \\ &= \int_0^1 \left( 1 - x - \frac{1}{3} + \frac{1}{3}x - \left( -1 + x + \frac{1}{3} - \frac{1}{3}x \right) \right) dx \\ &= \int_0^1 \left( \frac{4}{3} - \frac{4}{3}x \right) dx = \frac{4}{3}x - \frac{4}{6}x^2 \Big|_0^1 = \frac{2}{3} \end{aligned}$$

# Fubini's Theorem – Interpretation

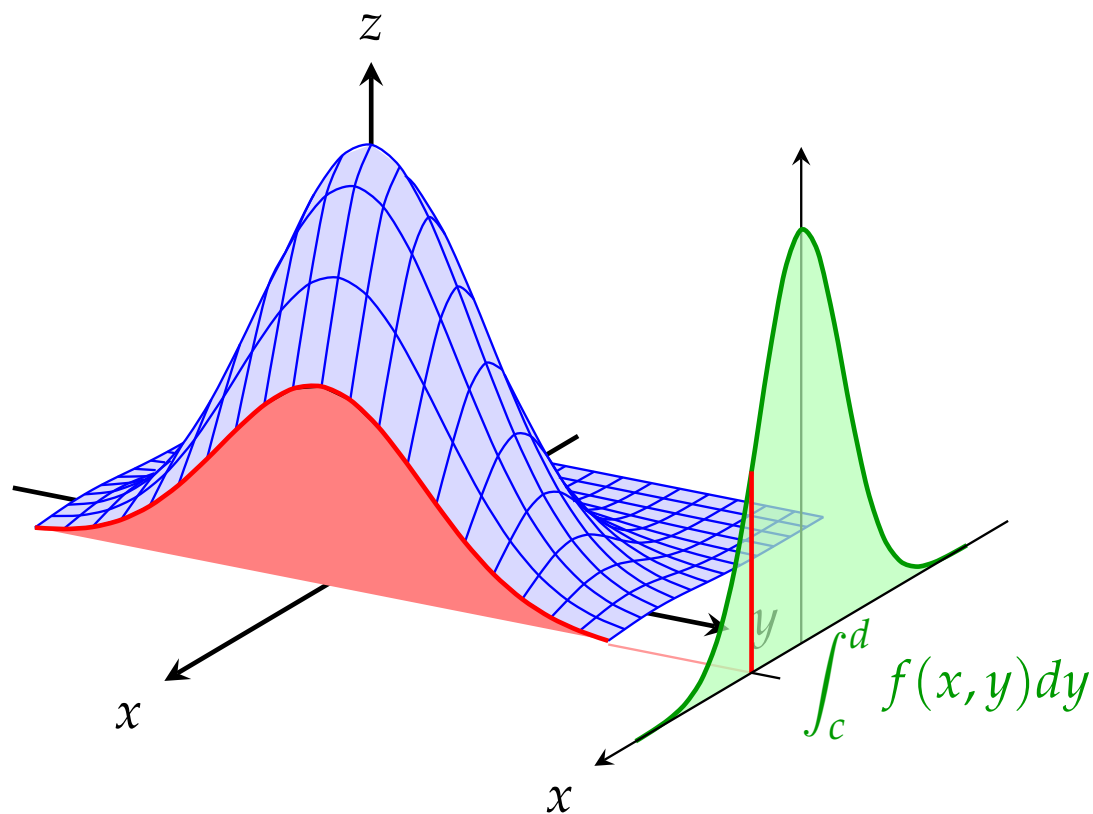
$$\iint_R f(x, y) dx dy = \int_a^b \left( \int_c^d f(x, y) dy \right) dx = \int_a^b A(x) dx$$

If we fix  $x$ , then

$$A(x) = \int_c^d f(x, y) dy$$

is the area below curve

$$g(y) = f(x, y).$$



# Summary

- ▶ antiderivate
- ▶ Riemann sum and Riemann integral
- ▶ indefinite and definite integral
- ▶ Fundamental Theorem of Calculus
- ▶ integration rules
- ▶ Leibniz integration rule
- ▶ improper integral
- ▶ double integral
- ▶ Fubini's theorem