Chapter 12

Integration

Antiderivative

A function $F(x)$ is called an **antiderivative** (or *primitive*) of function $f(x)$, if

$$\boxed{F'(x) = f(x)}$$

Computation:

Guess and verify

Example: We want the antiderivative of $f(x) = \ln(x)$.

Guess: $F(x) = x\,(\ln(x) - 1)$

Verify: $F'(x) = (x\,(\ln(x) - 1)' = 1 \cdot (\ln(x) - 1) + x \cdot \frac{1}{x} = \ln(x)$

But also: $F(x) = x\,(\ln(x) - 1) + 5$

Antiderivative

The antiderivative is denoted by symbol

$$\boxed{\int f(x)\,dx + c}$$

and is also called the **indefinite integral** of function f. Number c is called **integration constant**.

Unfortunately, there are no *“recipes”* for computing antiderivatives (but tools one can try and which may help).

There are functions where antiderivatives cannot be expressed by means of elementary functions.

E.g., the antiderivative of $\exp(-\frac{1}{2}x^2)$.

Basic Integrals

Integrals of some elementary functions:

$f(x)$	$\int f(x)\,dx$
0	c
x^a	$\frac{1}{a+1} \cdot x^{a+1} + c$
e^x	$e^x + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
$\cos(x)$	$\sin(x) + c$
$\sin(x)$	$-\cos(x) + c$

(The table is created by swapping the columns in the list of derivatives.)

Integration Rules

- **Summation rule**

$$\int \alpha f(x) + \beta g(x)\,dx = \alpha \int f(x)\,dx + \beta \int g(x)\,dx$$

- **Integration by parts**

$$\int f \cdot g'\,dx = f \cdot g - \int f' \cdot g\,dx$$

- **Integration by substitution**

$$\int f(g(x)) \cdot g'(x)\,dx = \int f(z)\,dz$$

with $z = g(x)$ and $dz = g'(x)\,dx$

Example – Summation Rule

Antiderivative of $f(x) = 4\,x^3 - x^2 + 3\,x - 5$.

$$\begin{aligned}\int f(x)\,dx &= \int 4\,x^3 - x^2 + 3\,x - 5\,dx\\ &= 4\int x^3\,dx - \int x^2\,dx + 3\int x\,dx - 5\int dx\\ &= 4\frac{1}{4}x^4 - \frac{1}{3}x^3 + 3\frac{1}{2}x^2 - 5x + c\\ &= x^4 - \frac{1}{3}x^3 + \frac{3}{2}x^2 - 5x + c\end{aligned}$$

Example – Integration by Parts

Antiderivative of $f(x) = x \cdot e^x$.

$$\int \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'} \, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g} - \int \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g} \, dx = x \cdot e^x - e^x + c$$

$$\begin{aligned} f &= x & \Rightarrow \quad f' &= 1 \\ g' &= e^x & \Rightarrow \quad g &= e^x \end{aligned}$$

Example – Integration by Parts

Antiderivative of $f(x) = x^2 \cos(x)$.

$$\int \underbrace{x^2}_{f} \cdot \underbrace{\cos(x)}_{g'} \, dx = \underbrace{x^2}_{f} \cdot \underbrace{\sin(x)}_{g} - \int \underbrace{2x}_{f'} \cdot \underbrace{\sin(x)}_{g} \, dx$$

Integration by parts of the second terms yields:

$$\begin{aligned} \int \underbrace{2x}_{f} \cdot \underbrace{\sin(x)}_{g'} \, dx &= \underbrace{2x}_{f} \cdot \underbrace{(-\cos(x))}_{g} - \int \underbrace{2}_{f'} \cdot \underbrace{(-\cos(x))}_{g} \, dx \\ &= -2x \cdot \cos(x) - 2 \cdot (-\sin(x)) + c \end{aligned}$$

Thus the antiderivative of f is given by

$$\int x^2 \cos(x) \, dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + c$$

Example – Integration by Substitution

Antiderivative of $f(x) = 2x \cdot e^{x^2}$.

$$\int \exp(\underbrace{x^2}_{g(x)}) \cdot \underbrace{2x}_{g'(x)} \, dx = \int \exp(z) \, dz = e^z + c = e^{x^2} + c$$

$$z = g(x) = x^2 \quad \Rightarrow \quad dz = g'(x) \, dx = 2x \, dx$$

Integration Rules – Derivation

Integration by parts follows from the product rule for derivatives:

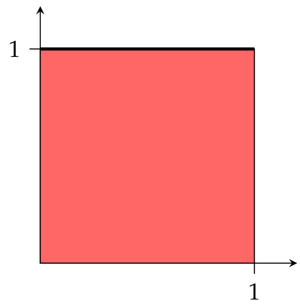
$$f(x) \cdot g(x) = \int (f(x) \cdot g(x))' \, dx = \int \left(f'(x)\, g(x) + f(x)\, g'(x)\right) dx$$
$$= \int f'(x)\, g(x)\, dx + \int f(x)\, g'(x)\, dx$$

Integration by substitution follows from the chain rule:
Let F be an antiderivative of f and let $z = g(x)$. Then

$$\int f(z)\, dz = F(z) = F(g(x)) = \int (F(g(x)))'\, dx$$
$$= \int F'(g(x))\, g'(x)\, dx = \int f(g(x))\, g'(x)\, dx$$

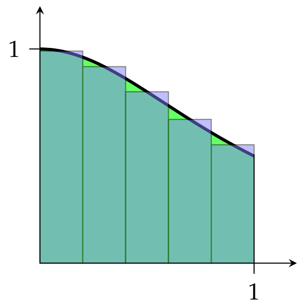
Area

Compute the areas of the given regions.



$f(x) = 1$

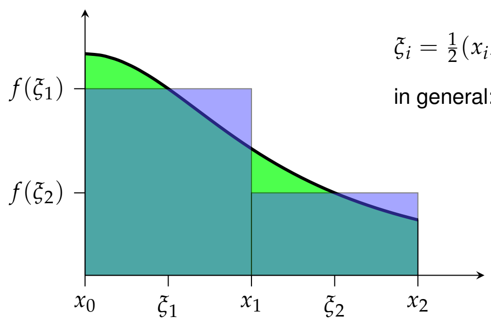
Area: $A = 1$



$f(x) = \frac{1}{1+x^2}$

Approximation
by step function

Riemann Sum

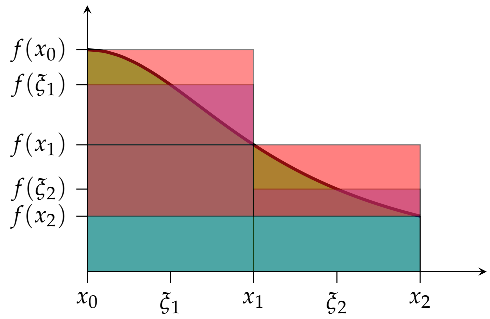


$\xi_i = \frac{1}{2}(x_{i-1} + x_i)$

in general: $\xi_i \in (x_i, x_{i-1})$

$$A = \int_a^b f(x)\, dx \approx \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})$$

Approximation Error



$$\left| \int_a^b f(x)\,dx - \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1}) \right| \le (f_{\max} - f_{\min})\,(b-a)\,\frac{1}{n} \to 0$$

Assumption: Function monotone; $x_0, x_1, \ldots, x_n$ equidistant

Riemann Integral

If all sequences of **Riemann sums**

$$\boxed{I_n = \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})}$$

converge, then their (uniquely determined) limit is called the **Riemann integral** of f and is denoted by $\int_a^b f(x)\,dx$:

$$\boxed{\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i) \cdot (x_i - x_{i-1})}$$

Almost all functions in economics have a Riemann integral.

Riemann Integral – Properties

$$\int_a^b (\alpha f(x) + \beta g(x))\,dx = \alpha \int_a^b f(x)\,dx + \beta \int_a^b g(x)\,dx$$

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

$$\int_a^a f(x)\,dx = 0$$

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx$$

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx \qquad \text{if } f(x) \le g(x) \text{ for all } x \in [a,b]$$

Fundamental Theorem of Calculus

Let $F(x)$ be an antiderivative of a *continuous* function $f(x)$, then we find

$$\boxed{\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(b) - F(a)}$$

By this theorem we can compute Riemann integrals by means of antiderivatives!

For that reason $\int f(x)\,dx$ is called an *indefinite integral* of f; and $\int_a^b f(x)\,dx$ is called a **definite integral** of f.

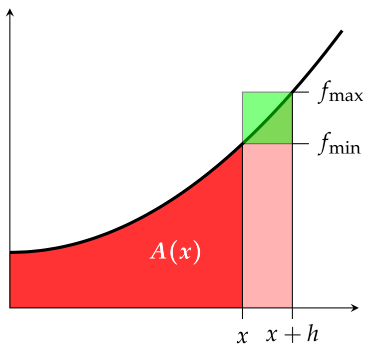
Example – Fundamental Theorem

Compute the integral of $f(x) = x^2$ over interval $[0, 1]$.

$$\int_0^1 x^2\,dx = \tfrac{1}{3}\,x^3\Big|_0^1 = \tfrac{1}{3}\cdot 1^3 - \tfrac{1}{3}\cdot 0^3 = \frac{1}{3}$$

Fundamental Theorem / Proof Idea

Let $A(x)$ be the area between the graph of a continuous function f and the x-axis from 0 to x.



$$f_{\min}\cdot h \le A(x+h) - A(x) \le f_{\max}\cdot h$$

$$f_{\min} \le \frac{A(x+h) - A(x)}{h} \le f_{\max}$$

Limit for $h \to 0$: ($\lim_{h\to 0} f_{\min} = f(x)$)

$$f(x) \le \underbrace{\lim_{h\to 0}\frac{A(x+h) - A(x)}{h}}_{=A'(x)} \le f(x)$$

$$A'(x) = f(x)$$

i.e. $A(x)$ is an antiderivative of $f(x)$.

Integration Rules / (Definite Integrals)

- **Summation rule**

$$\int_a^b \alpha f(x) + \beta g(x)\, dx = \alpha \int_a^b f(x)\, dx + \beta \int_a^b g(x)\, dx$$

- **Integration by parts**

$$\int_a^b f \cdot g'\, dx = f \cdot g\Big|_a^b - \int_a^b f' \cdot g\, dx$$

- **Integration by Substitution**

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(z)\, dz$$

with $z = g(x)$ and $dz = g'(x)\, dx$

Example – Integration by Parts

Compute the definite integral $\int_0^2 x \cdot e^x\, dx$.

$$\int_0^2 \underbrace{x}_{f} \cdot \underbrace{e^x}_{g'}\, dx = \underbrace{x}_{f} \cdot \underbrace{e^x}_{g}\Big|_0^2 - \int_0^2 \underbrace{1}_{f'} \cdot \underbrace{e^x}_{g}\, dx$$

$$= x \cdot e^x\Big|_0^2 - e^x\Big|_0^2 = (2 \cdot e^2 - 0 \cdot e^0) - (e^2 - e^0)$$

$$= e^2 + 1$$

Note: we also could use our indefinite integral from above,

$$\int_0^2 x \cdot e^x\, dx = (x \cdot e^x - e^x)\Big|_0^2 = (2 \cdot e^2 - e^2) - (0 \cdot e^0 - e^0) = e^2 + 1$$

Example – Integration by Substitution

Compute the definite integral $\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\, dx$.

$$\int_e^{10} \frac{1}{\ln(x)} \cdot \frac{1}{x}\, dx \quad = \quad \int_1^{\ln(10)} \frac{1}{z}\, dz =$$

$$z = \ln(x) \quad \Rightarrow \quad dz = \frac{1}{x}\, dx$$

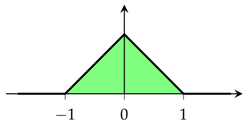
$$= \quad \ln(z)\Big|_1^{\ln(10)} =$$

$$= \quad \ln(\ln(10)) - \ln(1) \approx 0.834$$

Example – Subdomains

Compute $\int_{-2}^{2} f(x)\,dx$ for function

$$f(x) = \begin{cases} 1+x, & \text{for } -1 \le x < 0, \\ 1-x, & \text{for } 0 \le x < 1, \\ 0, & \text{for } x < -1 \text{ and } x \ge 1. \end{cases}$$



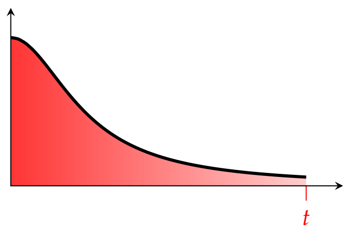
We have

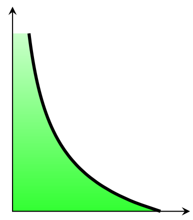
$$\begin{aligned} \int_{-2}^{2} f(x)\,dx &= \int_{-2}^{-1} f(x)\,dx + \int_{-1}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx \\ &= \int_{-2}^{-1} 0\,dx + \int_{-1}^{0} (1+x)\,dx + \int_{0}^{1} (1-x)\,dx + \int_{1}^{2} 0\,dx \\ &= \left(x + \tfrac{1}{2}x^2\right)\Big|_{-1}^{0} + \left(x - \tfrac{1}{2}x^2\right)\Big|_{0}^{1} \\ &= \tfrac{1}{2} + \tfrac{1}{2} = 1 \end{aligned}$$

Improper Integral

An **improper integral** is an integral where

- the domain of integration is unbounded, or
- the integrand is unbounded.





$$\boxed{\int_0^\infty f(x)\,dx = \lim_{t\to\infty} \int_0^t f(x)\,dx}$$

$$\boxed{\int_0^1 f(x)\,dx = \lim_{t\to 0} \int_t^1 f(x)\,dx}$$

Example – Improper Integral

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0} \int_t^1 x^{-\frac{1}{2}}\,dx = \lim_{t\to 0} 2\sqrt{x}\Big|_t^1 = \lim_{t\to 0}(2 - 2\sqrt{t}) = 2$$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t x^{-2}\,dx = \lim_{t\to\infty} -\frac{1}{x}\Big|_1^t = \lim_{t\to\infty} -\frac{1}{t} - (-1) = 1$$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \ln(x)\Big|_1^t = \lim_{t\to\infty} \ln(t) - \ln(1) = \infty$$

The improper integral does not exist.

Two Limits

In probability theory we often have integrals where both boundaries are infinite.
For example, the expectation of random variable X with density f is defined as

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

In such a case we have to separate the domain of integration:

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} x \cdot f(x)\, dx \\ &= \lim_{t\to-\infty} \int_t^0 x \cdot f(x)\, dx + \lim_{s\to\infty} \int_0^s x \cdot f(x)\, dx\end{aligned}$$

Beware!
If we yield $\infty - \infty$, then the result is **not** $\infty - \infty = 0$!

Leibniz Integration Rule

Let $f(x,t)$ be *continously differentiable* function
(i.e., all partial derivatives exist and are continuous) and let
$F(x) = \int_{a(x)}^{b(x)} f(x,t)\, dt$.

Then

$$\boxed{F'(x) = f(x,b(x))\; b'(x) - f(x,a(x))\; a'(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\, dt}$$

If $a(x) = a$ and $b(x) = b$ are constant, then

$$\boxed{\frac{d}{dx}\left(\int_a^b f(x,t)\, dt\right) = \int_a^b \frac{\partial}{\partial x} f(x,t)\, dt}$$

Example – Leibniz Integration Rule

Let $F(x) = \int_x^{2x} t\, x^2\, dt$ for $x > 0$. Compute $F'(x)$.

We set $f(x,t) = t\, x^2$, $a(x) = x$ and $b(x) = 2x$
and apply Leibniz's integration rule:

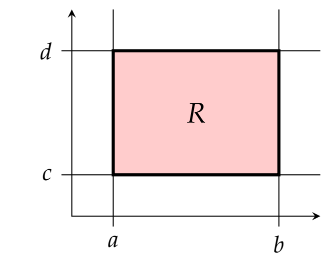
$$\begin{aligned}F'(x) &= f(x,b) \cdot b' - f(x,a) \cdot a' + \int_a^b f_x(x,t)\, dt \\ &= (2x)\, x^2 \cdot 2 - (x)\, x^2 \cdot 1 + \int_x^{2x} 2x\, t\, dt \\ &= 4x^3 - x^3 + \left(2x\, \tfrac{1}{2} t^2\right)\Big|_x^{2x} \\ &= 4x^3 - x^3 + (4x^3 - x^3) \\ &= 6x^3\end{aligned}$$

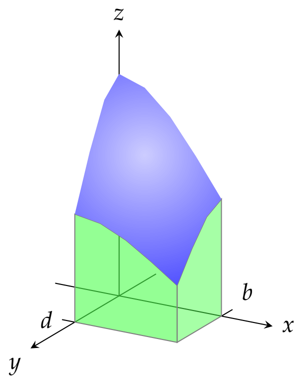
Volume

Let $f(x, y)$ be a function with domain

$$R = [a, b] \times [c, d] = \{(x, y) \in \mathbb{R}^2 \colon a \le x \le b,\ c \le y \le d\}$$

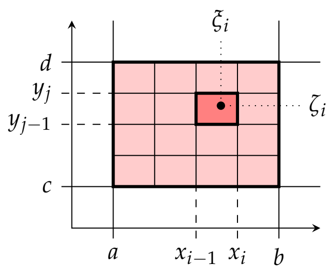
What is the Volumen V below the graph of f?

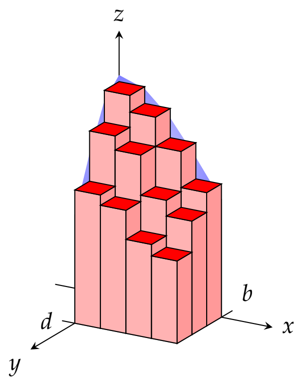




Riemann Sums

- Partition R into smaller rectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$
- Estimate $V \approx \sum_{i=1}^{n} \sum_{j=1}^{k} f(\xi_i, \zeta_j)\,(x_i - x_{i-1})\,(y_j - y_{j-1})$





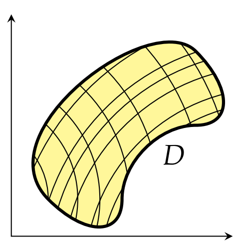
Riemann Integral

If these **Riemann Sums** converge for partitions of R with increasing number of rectangles, then the limit is called the **Riemann Integral** of f over R:

$$\boxed{\iint_R f(x, y)\,dx\,dy = \lim_{n,k\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{k} f(\xi_i, \zeta_j)\,(x_i - x_{i-1})\,(y_j - y_{j-1})}$$

The Riemann integral is defined analogously for arbitrary domains D.

$$\iint_D f(x, y)\,dx\,dy$$



Fubini's Theorem

Let $f\colon R = [a,b] \times [c,d] \subset \mathbb{R}^2 \to \mathbb{R}$ a *continuous* function. Then

$$\boxed{\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx \\ &= \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy\,.\end{aligned}}$$

Fubini's theorem provides a recipe for computating double integrals stepwise:

1. Treat x like a constant and compute the inner integral $\int_c^d f(x,y)\,dy$ w.r.t. variabel y.
2. Integrate the result from Step 1 w.r.t. x.

We also may change the order of integration.

Example – Fubini's Theorem

Compute $\displaystyle\int_{-1}^1 \int_0^1 (1 - x - y^2 + xy^2)\,dx\,dy$.

We have to integrate twice:

$$\begin{aligned}&\int_{-1}^1 \int_0^1 (1 - x - y^2 + xy^2)\,dx\,dy \\ &\qquad = \int_{-1}^1 \left(x - \frac{1}{2}x^2 - xy^2 + \frac{1}{2}x^2y^2 \Big|_0^1 \right) dy \\ &\qquad = \int_{-1}^1 \left(\frac{1}{2} - \frac{1}{2}y^2\right) dy = \frac{1}{2}y - \frac{1}{6}y^3 \Big|_{-1}^1 \\ &\qquad = \frac{1}{2} - \frac{1}{6} - \left(-\frac{1}{2} + \frac{1}{6}\right) = \frac{2}{3}\end{aligned}$$

Bounds of Integration

Beware!
The integration variables and the corresponding integration limits have to be read from **inside** to **outside**.

If we change the order of integration, then we also have to exchange the integration limits:

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

This may be more obvious if we add (redundant) parenthesis:

$$\int_a^b \left(\int_c^d f(x,y)\,dy\right) dx = \int_c^d \left(\int_a^b f(x,y)\,dx\right) dy\,.$$

Example – Fubini's Theorem

Integration in reversed order:

$$\int_{-1}^{1}\int_{0}^{1}(1-x-y^2+xy^2)\,dx\,dy$$

$$= \int_0^1 \left(\int_{-1}^{1}(1-x-y^2+xy^2)\,dy\right)\,dx$$

$$= \int_0^1 \left(y - xy - \frac{1}{3}y^3 + \frac{1}{3}xy^3\Big|_{-1}^{1}\right)\,dx$$

$$= \int_0^1 \left(1 - x - \frac{1}{3} + \frac{1}{3}x - \left(-1 + x + \frac{1}{3} - \frac{1}{3}x\right)\right)\,dx$$

$$= \int_0^1 \left(\frac{4}{3} - \frac{4}{3}x\right)\,dx = \frac{4}{3}x - \frac{4}{6}x^2\Big|_0^1 = \frac{2}{3}$$

Fubini's Theorem – Interpretation

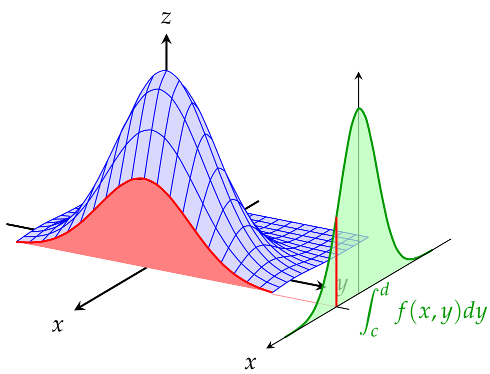
$$\iint_R f(x,y)\,dx\,dy = \int_a^b \left(\int_c^d f(x,y)\,dy\right)\,dx = \int_a^b A(x)dx$$

If we fix x, then

$$A(x) = \int_c^d f(x,y)\,dy$$

is the area below curve

$$g(y) = f(x,y).$$



Summary

- antiderivate
- Riemann sum and Riemann integral
- indefinite and definite integral
- Fundamental Theorem of Calculus
- integration rules
- Leibniz integration rule
- improper integral
- double integral
- Fubini's theorem