Chapter 11

Taylor Series

First-Order Approximation

We want to approximate function f by some *simple* function.

Best possible approximation by a **linear function**:

$$\boxed{f(x) \doteq f(x_0) + f'(x_0)\,(x - x_0)}$$

$\doteq$ means "*first-order approximation*".

If we use this approximation, we calculate the value of the tangent at x instead of f.

Polynomials

We get a better approximation (i.e., with smaller approximation error) if we use a **polynomial** $P_n(x) = \sum_{k=0}^{n} a_k x^k$ of higher degree.

Ansatz:

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + R_n(x)$$

Remainder term $R_n(x)$ gives the approximation error when we replace function f by approximation $P_n(x)$.

Idea:
Choose coefficients a_i such that the derivatives of f and P_n coincide at $x_0 = 0$ up to order n.

Derivatives

$$\begin{aligned}
f(x) &= a_0 + a_1 x + \cdots + a_n x^n && = P_n(x) \\
&\Rightarrow f(0) = a_0 \\[2ex]
f'(x) &= a_1 + 2 \cdot a_2 x + \cdots + n \cdot a_n x^{n-1} && = P_n'(x) \\
&\Rightarrow f'(0) = a_1 \\[2ex]
f''(x) &= 2 \cdot a_2 + 3 \cdot 2 \cdot a_3 x + \cdots + n \cdot (n-1) \cdot a_n x^{n-2} && = P_n''(x) \\
&\Rightarrow f''(0) = 2\, a_2 \\[2ex]
f'''(x) &= 3 \cdot 2 \cdot a_3 + \cdots + n \cdot (n-1) \cdot (n-2) \cdot a_n x^{n-3} && = P_n'''(x) \\
&\Rightarrow f'''(0) = 3!\, a_3 \\
&\vdots \\
f^{(n)}(x) &= n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 1 \cdot a_n && = P_n^{(n)}(x) \\
&\Rightarrow f^{(n)}(0) = n!\, a_n
\end{aligned}$$

MacLaurin Polynomial

Thus we find for the coefficients of the polynomial

$$a_k = \frac{f^{(k)}(0)}{k!}$$

$f^{(k)}(x_0)$ denotes the k-th derivatives of f at x_0, $\quad f^{(0)}(x_0) = f(x_0)$.

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k + R_n(x)}$$

This polynomial is called the **MacLaurin polynomial** of degree n of f:

$$f(0) + f'(0)\, x + \frac{f''(0)}{2!}\, x^2 + \cdots + \frac{f^{(n)}(0)}{n!}\, x^n$$

Taylor Polynomial

This idea can be generalized to arbitrary exansion points x_0.
We then get the **Taylor polynomial** of degree n of f around point x_0:

$$\boxed{f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\,(x - x_0)^k + R_n(x)}$$

Taylor Series

The (infinite) series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$$

is called the **Taylor series** of f around x_0.

If $\lim_{n\to\infty} R_n(x) = 0$, then the Taylor series converges to $f(x)$.

We then say that we **expand** f into a *Taylor series* around **expansion point** x_0.

Example – Exponential Function

Taylor series expansion of $f(x) = e^x$ around $x_0 = 0$:

$$f(x) = f(0) + f'(0)\,x + \tfrac{f''(0)}{2!}\,x^2 + \tfrac{f'''(0)}{3!}\,x^3 + \cdots + \tfrac{f^{(n)}(0)}{n!}\,x^n + R_n(x)$$

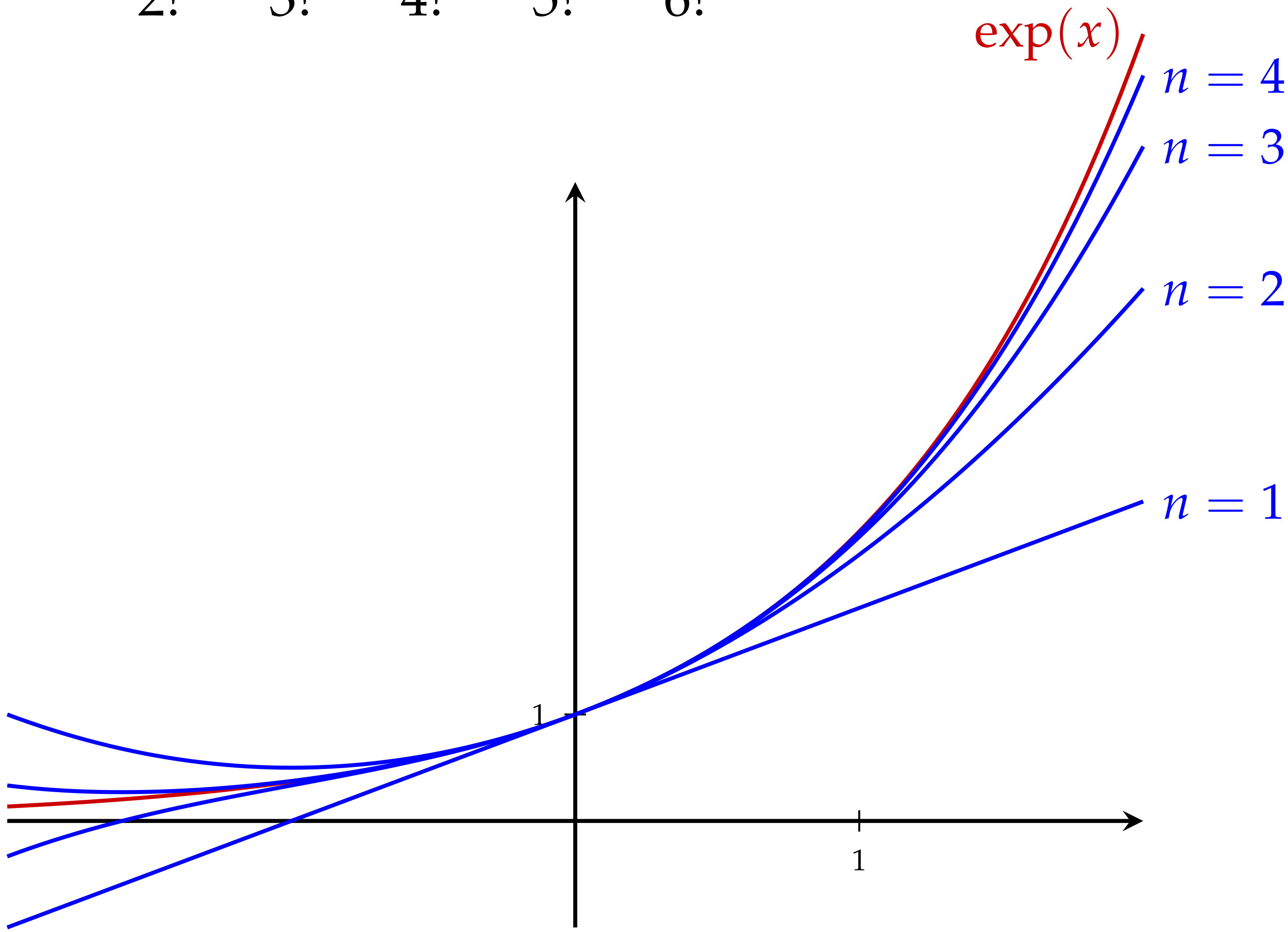
$$\begin{aligned}
f(x) &= e^x & \Rightarrow \quad & f(0) = 1 \\
f'(x) &= e^x & \Rightarrow \quad & f'(0) = 1 \\
f''(x) &= e^x & \Rightarrow \quad & f''(0) = 1 \\
f'''(x) &= e^x & \Rightarrow \quad & f'''(0) = 1 \\
&\vdots \\
f^{(n)}(x) &= e^x & \Rightarrow \quad & f^{(n)}(0) = 1
\end{aligned}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

The Taylor series converges for all $x \in \mathbb{R}$.

Example – Exponential Function

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$



Example – Logarithm

Taylor series expansion of $f(x) = \ln(1+x)$ around $x_0 = 0$:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots + \frac{f^{(n)}(0)}{n!}\,x^n + R_n(x)$$

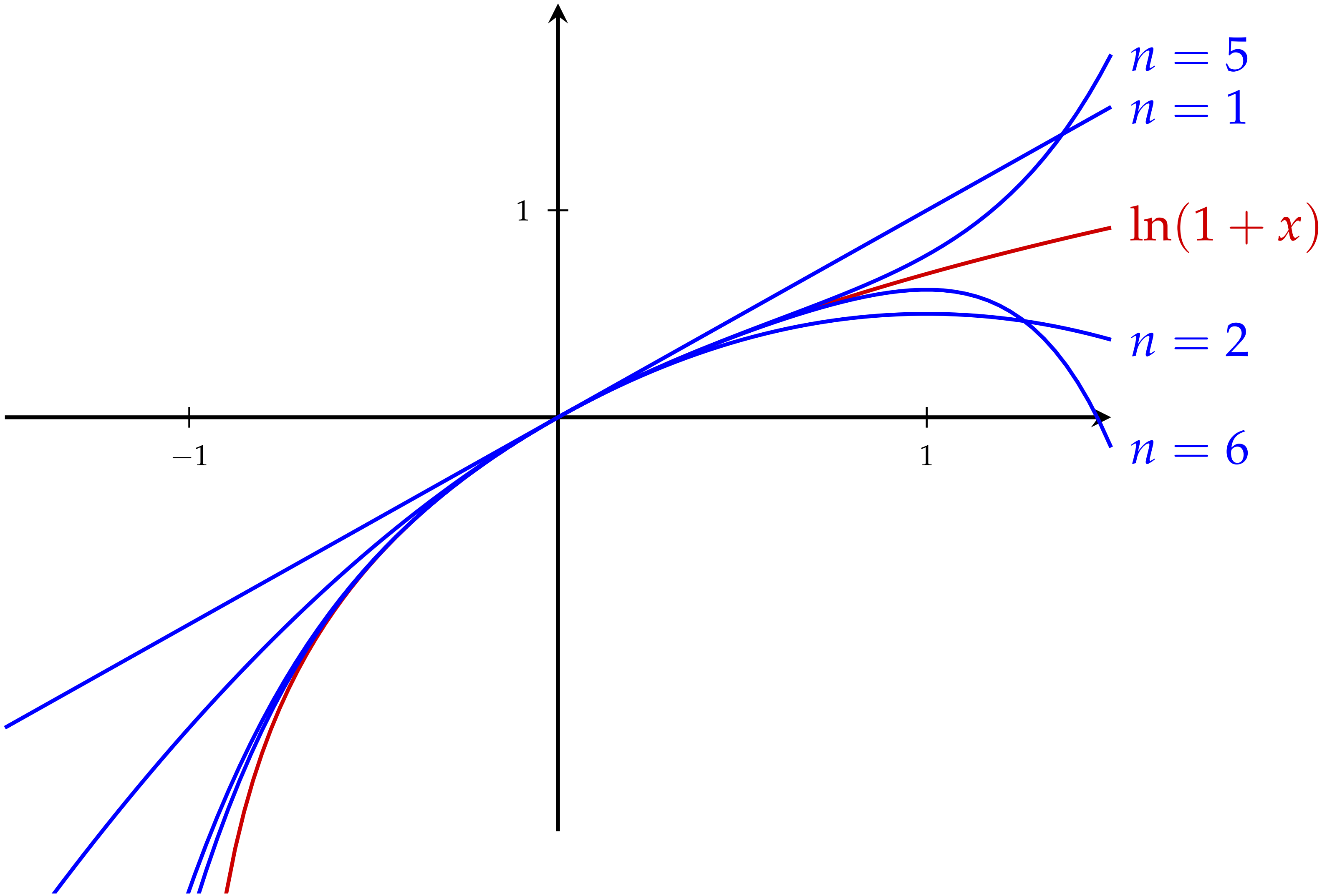
$$\begin{aligned}
f(x) &= \ln(1+x) && \Rightarrow \quad f(0) = 0 \\
f'(x) &= (1+x)^{-1} && \Rightarrow \quad f'(0) = 1 \\
f''(x) &= -1 \cdot (1+x)^{-2} && \Rightarrow \quad f''(0) = -1 \\
f'''(x) &= 2 \cdot 1 \cdot (1+x)^{-3} && \Rightarrow \quad f'''(0) = 2! \\
&\vdots \\
f^{(n)}(x) &= (-1)^{n-1}(n-1)!(1+x)^{-n+1} && \Rightarrow \quad f^{(n)}(0) = (-1)^{n-1}(n-1)!
\end{aligned}$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots$$

The Taylor series converges for all $x \in (-1, 1)$.

Example – Logarithm

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$



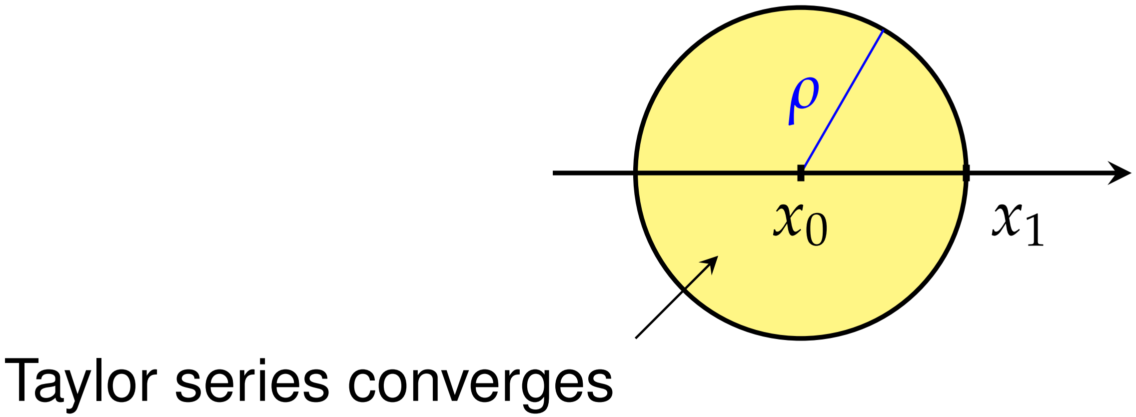
Radius of Convergence

Some Taylor series do not converge for all $x \in \mathbb{R}$.
For example: $\ln(1+x)$

At least the following holds:

If a Taylor series converges for some x_1 with $|x_1 - x_0| = \rho$,
then it also converges for all x with $|x - x_0| < \rho$.

The maximal value for ρ is called the **radius of convergence** of the Taylor series.

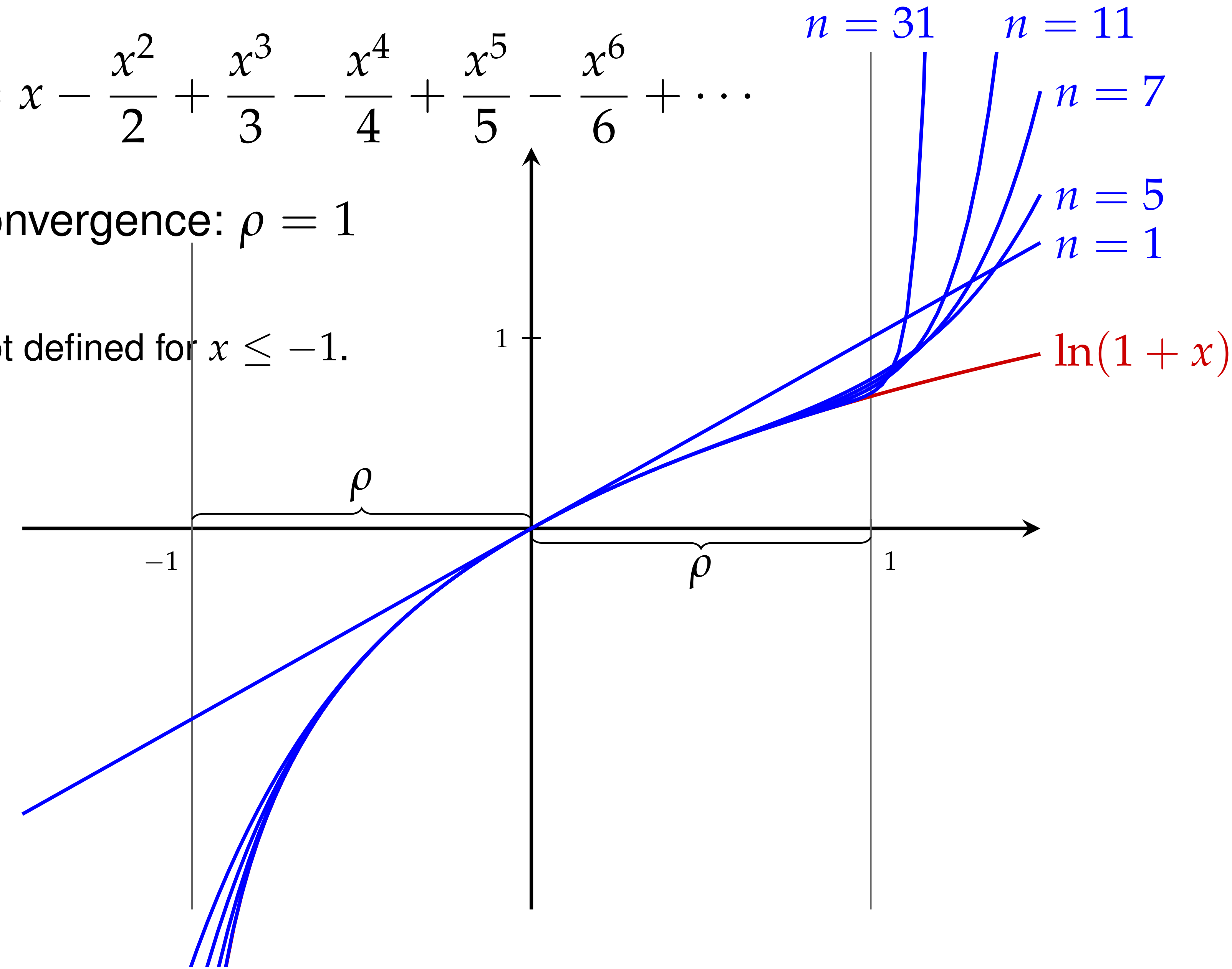


Example – Radius of Convergence

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \cdots$$

Radius of convergence: $\rho = 1$

Indication:
$\ln(1+x)$ is not defined for $x \leq -1$.



Approximation Error

The remainder term indicates the error of the approximation by a Taylor polynomial.
It can be estimated by means of **Lagrange's form of the remainder**:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}$$

for some point $\xi \in (x, x_0)$.

If the Taylor series converges we have

$$\boxed{R_n(x) = \mathcal{O}\left((x - x_0)^{n+1}\right) \quad \text{for} \quad x \to x_0}$$

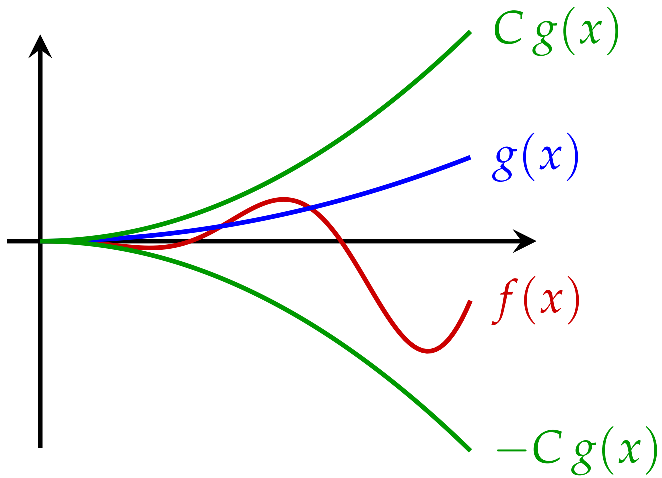
We say: the remainder is of big O of x^{n+1} as x tends to x_0.

Big O Notation

Let $f(x)$ and $g(x)$ be two functions.

We write

$$\boxed{f(x) = \mathcal{O}\big(g(x)\big) \quad \text{as} \quad x \to x_0}$$



if there exist reals numbers C and ε such that

$$|f(x)| < C \cdot |g(x)|$$

for all x with $|x - x_0| < \varepsilon$.

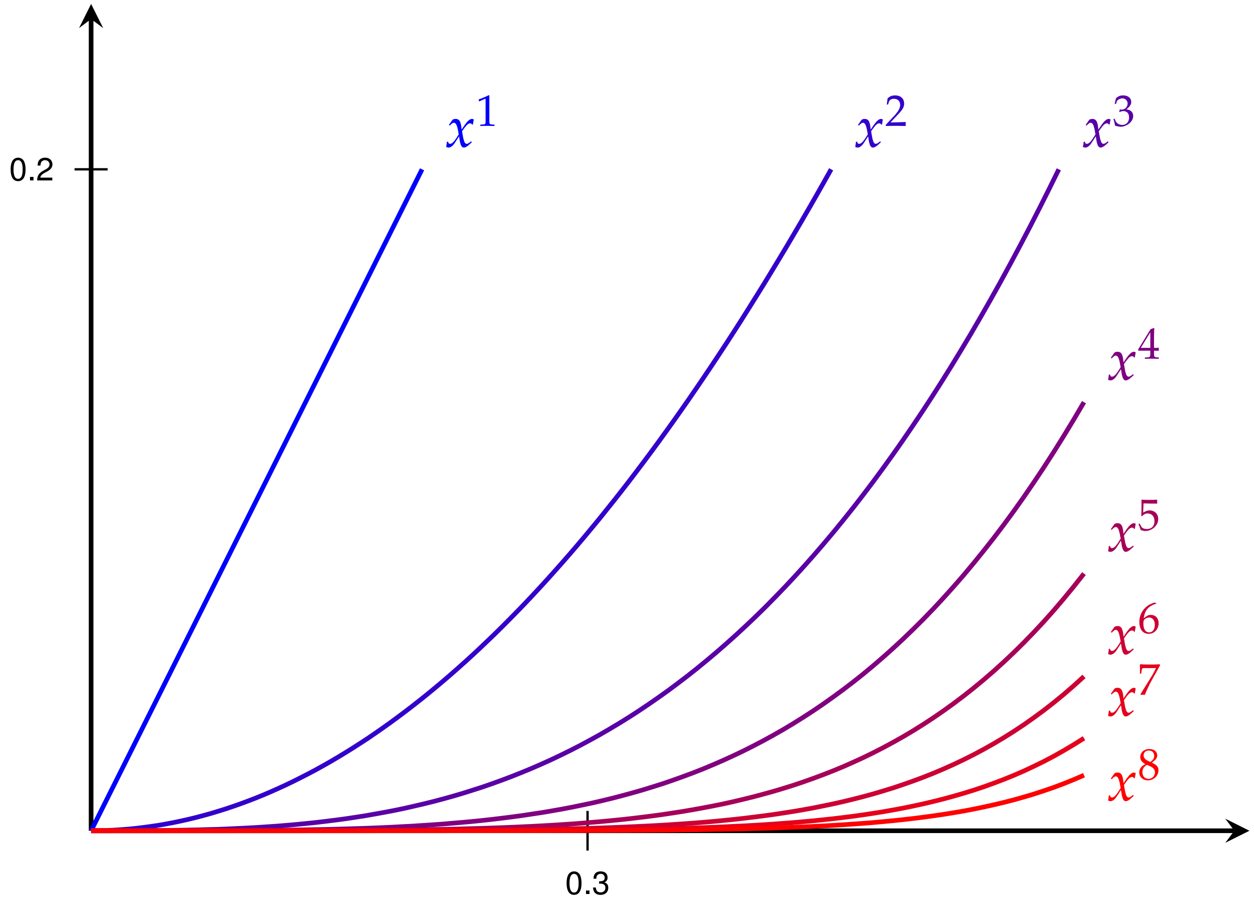
We say that $f(x)$ is of big O of $g(x)$ as x tends to x_0.

Symbol $\mathcal{O}(\cdot)$ belongs to the family of *Bachmann-Landau* notations.

Some books use notation $\quad f(x) \in \mathcal{O}\big(g(x)\big)$ as $x \to x_0$

Impact of Order of Powers

The higher the order of a monomial is,
the smaller is its contribution to the summation.



Important Taylor Series

$f(x)$		MacLaurin Series	ρ
$\exp(x)$	$=$	$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$	∞
$\ln(1+x)$	$=$	$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$	1
$\sin(x)$	$=$	$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$	∞
$\cos(x)$	$=$	$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$	∞
$\frac{1}{1-x}$	$=$	$1 + x + x^2 + x^3 + x^4 + \cdots$	1

Calculations with Taylor Series

Taylor series can be conveniently

- added (term by term)
- differentiated (term by term)
- integrated (term by term)
- multiplied
- divided
- substituted

Therefore Taylor series are also used for the *Definition* of some function.

For example:

$$\exp(x) := 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

Example – Derivative

We get the first derivative of $\exp(x)$ by computing the derivative of its Taylor series:

$$
\begin{aligned}
(\exp(x))' &= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right)' \\
&= 0 + 1 + \frac{2\,x}{2!} + \frac{3\,x^2}{3!} + \frac{4\,x^3}{4!} + \cdots \\
&= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \\
&= \exp(x)
\end{aligned}
$$

Example – Product

We get the MacLaurin series of $f(x) = x^2 \cdot e^x$ by multiplying the MacLaurin series of x^2 with the MacLaurin series of $\exp(x)$:

$$\begin{aligned} x^2 \cdot e^x &= x^2 \cdot \left(1 + x + \tfrac{x^2}{2!} + \tfrac{x^3}{3!} + \tfrac{x^4}{4!} + \cdots\right) \\ &= x^2 + x^3 + \frac{x^4}{2!} + \frac{x^5}{3!} + \frac{x^6}{4!} + \cdots \end{aligned}$$

Example – Substitution

We get the MacLaurin series of $f(x) = \exp(-x^2)$ by substituting of $-x^2$ into the MacLaurin series of $\exp(x)$:

$$\begin{aligned}
e^u &= 1 + u && + \tfrac{u^2}{2!} && + \tfrac{u^3}{3!} && + \tfrac{u^4}{4!} && + \cdots \\
e^{-x^2} &= 1 + (-x^2) && + \tfrac{(-x^2)^2}{2!} && + \tfrac{(-x^2)^3}{3!} && + \tfrac{(-x^2)^4}{4!} && + \cdots \\
&= 1 - x^2 && + \tfrac{x^4}{2!} && - \tfrac{x^6}{3!} && + \tfrac{x^8}{4!} && - \cdots
\end{aligned}$$

Polynomials

The concept of Taylor series can be generalized to multivariate functions.

A polynomial of degree n in *two* variables has the form

$$\begin{aligned} P_n(x_1, x_2) = \quad & a_0 \\ & + a_{10}\, x_1 + a_{11}\, x_2 \\ & + a_{20}\, x_1^2 + a_{21}\, x_1\, x_2 + a_{22}\, x_2^2 \\ & + a_{30}\, x_1^3 + a_{31}\, x_1^2\, x_2 + a_{32}\, x_1\, x_2^2 + a_{33}\, x_2^3 \\ & \quad \vdots \\ & + a_{n0}\, x_1^n + a_{n1}\, x_1^{n-1}\, x_2 + a_{n2}\, x_1^{n-2}\, x_2^2 + \cdots + a_{nn}\, x_2^n \end{aligned}$$

We choose coefficients a_{kj} such that all its partial derivatives in expansion point $\mathbf{x}_0 = 0$ up to order n coincides with the respective derivatives of f.

Taylor Polynomial of Degree 2

We obtain the coefficients as

$$a_{kj} = \frac{1}{k!} \binom{k}{j} \frac{\partial^k f(0)}{(\partial x_1)^{k-j} (\partial x_2)^j} \qquad k \in \mathbb{N},\ j = 0, \cdots, k$$

In particular we find for the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$\begin{aligned} f(\mathbf{x}) = \quad & f(0) \\ & + f_{x_1}(0)\, x_1 + f_{x_2}(0)\, x_2 \\ & + \tfrac{1}{2} f_{x_1 x_1}(0)\, x_1^2 + f_{x_1 x_2}(0)\, x_1\, x_2 + \tfrac{1}{2} f_{x_2 x_2}(0)\, x_2^2 + \cdots \end{aligned}$$

Taylor Polynomial of Degree 2

Observe that the linear term can be written by means of the gradient:

$$f_{x_1}(0)\,x_1 + f_{x_2}(0)\,x_2 = \nabla f(0) \cdot \mathbf{x}$$

The quadratic term can be written by means of the Hessian matrix:

$$f_{x_1 x_1}(0)\,x_1^2 + 2\,f_{x_1 x_2}(0)\,x_1\,x_2 + f_{x_2 x_2}(0)\,x_2^2 = \mathbf{x}^\mathsf{T} \cdot \mathbf{H}_f(0) \cdot \mathbf{x}$$

So we find for the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$\boxed{f(\mathbf{x}) = f(0) + \nabla f(0) \cdot \mathbf{x} + \tfrac{1}{2}\,\mathbf{x}^\mathsf{T} \cdot \mathbf{H}_f(0) \cdot \mathbf{x} + \mathcal{O}(\|\mathbf{x}\|^3)}$$

or in different notation

$$\boxed{f(\mathbf{x}) = f(0) + f'(0)\mathbf{x} + \tfrac{1}{2}\,\mathbf{x}^\mathsf{T} f''(0)\mathbf{x} + \mathcal{O}(\|\mathbf{x}\|^3)}$$

Example – Bivariate Function

Compute the Taylor polynomial of degree 2 around $\mathbf{x}_0 = 0$

$$f(x,y) = e^{x^2-y^2} + x$$

$$\begin{aligned}
f(x,y) &= e^{x^2-y^2} + x & \Rightarrow\quad & f(0,0) = 1\\
f_x(x,y) &= 2x\,e^{x^2-y^2} + 1 & \Rightarrow\quad & f_x(0,0) = 1\\
f_y(x,y) &= -2y\,e^{x^2-y^2} & \Rightarrow\quad & f_y(0,0) = 0\\
f_{xx}(x,y) &= 2\,e^{x^2-y^2} + 4x^2\,e^{x^2-y^2} & \Rightarrow\quad & f_{xx}(0,0) = 2\\
f_{xy}(x,y) &= -4xy\,e^{x^2-y^2} & \Rightarrow\quad & f_{xy}(0,0) = 0\\
f_{yy}(x,y) &= -2\,e^{x^2-y^2} + 4y^2\,e^{x^2-y^2} & \Rightarrow\quad & f_{yy}(0,0) = -2
\end{aligned}$$

gradient:

$$\nabla f(0) = (1,0)$$

Hessian matrix:

$$\mathbf{H}_f(0) = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Example – Bivariate Function

Thus we find for the Taylor polynomial

$$
\begin{aligned}
f(x,y) &\approx f(0) + \nabla f(0) \cdot \mathbf{x} + \tfrac{1}{2}\, \mathbf{x}^{\mathsf{T}} \cdot \mathbf{H}_f(0) \cdot \mathbf{x} \\
&= 1 + (1,0) \cdot \begin{pmatrix} x \\ y \end{pmatrix} + \tfrac{1}{2}\,(x,y) \cdot \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \\
&= 1 + x + x^2 - y^2
\end{aligned}
$$

Summary

- MacLaurin and Taylor polynomial
- Taylor series expansion
- radius of convergence
- calculations with Taylor series
- Taylor series of multivariate functions