Chapter 10

Inverse and Implicit Functions

Inverse Function

Let $\mathbf{f}\colon D_f \subseteq \mathbb{R}^n \to W_f \subseteq \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x})$. A Function

$$\boxed{\mathbf{f}^{-1}\colon W_f \to D_f,\ \mathbf{y} \mapsto \mathbf{x} = \mathbf{f}^{-1}(\mathbf{y})}$$

is called **inverse function** of $\mathbf{f}$, if

$$\boxed{\mathbf{f}^{-1} \circ \mathbf{f} = \mathbf{f} \circ \mathbf{f}^{-1} = \mathbf{id}}$$

where $\mathbf{id}$ denotes the **identity function**, $\mathbf{id}(\mathbf{x}) = \mathbf{x}$:

$$\mathbf{f}^{-1}(\mathbf{f}(\mathbf{x})) = \mathbf{f}^{-1}(\mathbf{y}) = \mathbf{x} \qquad \text{and} \qquad \mathbf{f}(\mathbf{f}^{-1}(\mathbf{y})) = \mathbf{f}(\mathbf{x}) = \mathbf{y}$$

$\mathbf{f}^{-1}$ exists if and only if $\mathbf{f}$ is bijective.

We then obtain $\mathbf{f}^{-1}(\mathbf{y})$ as the *unique* solution $\mathbf{x}$ of equation $\mathbf{y} = \mathbf{f}(\mathbf{x})$.

Linear Function

Let $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = a\,x + b$.

$$y = a\,x + b \quad \Leftrightarrow \quad a\,x = y - b \quad \Leftrightarrow \quad x = \tfrac{1}{a}\,y - \tfrac{b}{a}$$

That is,

$$f^{-1}(y) = a^{-1}\,y - a^{-1}\,b$$

Provided that $\quad a \neq 0 \qquad [\, a = f'(x) \,]$

Observe:

$$(f^{-1})'(y) = a^{-1} = \frac{1}{a} = \frac{1}{f'(x)}$$

Linear Function

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{x} \mapsto \mathbf{y} = \mathbf{f}(\mathbf{x}) = \mathbf{A}\,\mathbf{x} + \mathbf{b}$ for some $m \times n$ matrix $\mathbf{A}$.

$$\mathbf{y} = \mathbf{A}\,\mathbf{x} + \mathbf{b} \quad \Leftrightarrow \quad \mathbf{x} = \mathbf{A}^{-1}\,\mathbf{y} - \mathbf{A}^{-1}\,\mathbf{b}$$

That is,

$$\mathbf{f}^{-1}(\mathbf{y}) = \mathbf{A}^{-1}\,\mathbf{y} - \mathbf{A}^{-1}\,\mathbf{b}\ .$$

Provided that $\mathbf{A}$ is invertible, $\qquad [\ \mathbf{A} = D\mathbf{f}(\mathbf{x})\ ]$
(and thus: $n = m$)

Observe:

$$(\mathbf{f}^{-1})'(\mathbf{y}) = \mathbf{A}^{-1} = (\mathbf{f}'(\mathbf{x}))^{-1}$$

Locally Invertible Function

Function

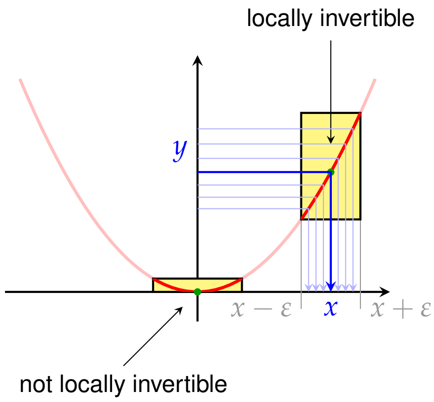
$$f\colon \mathbb{R} \to [0,\infty),\ x \mapsto f(x) = x^2$$

is not bijective. Thus
f^{-1} does *not* exist *globally*.

For some x_0 there exists an *open* interval $(x_0 - \varepsilon, x_0 + \varepsilon)$ where $y = f(x)$ can be solved w.r.t. x.

We say:
f is **locally invertible** around x_0.

For other x_0 such an interval does not exist (even if it is very short).



Existence and Derivative

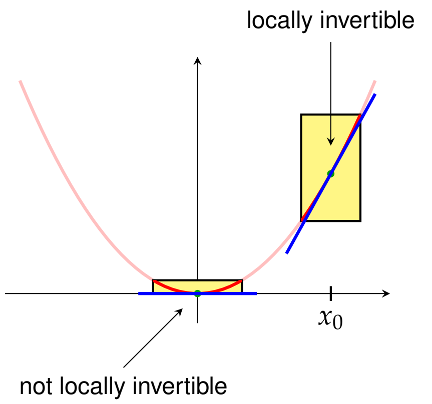
1. For which $\mathbf{x}_0$ is $\mathbf{f}$ locally invertible?
2. What is the derivative of $\mathbf{f}^{-1}$ at $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$.

Idea:
Replace $\mathbf{f}$ by its differential:

$$\mathbf{f}(\mathbf{x}_0 + \mathbf{h}) \approx \mathbf{f}(\mathbf{x}_0) + \mathbf{f}'(\mathbf{x}_0) \cdot \mathbf{h}$$

Hence:

1. $\mathbf{f}'(\mathbf{x}_0)$ must be invertible.
2. $(\mathbf{f}^{-1})'(\mathbf{y}_0) = (\mathbf{f}'(\mathbf{x}_0))^{-1}$



Inverse Function Theorem

Let $f\colon D_f \subseteq \mathbb{R} \to \mathbb{R}$ be a function and x_0 some point with $f'(x_0) \neq 0$.

Then there exist open intervals U around x_0 and V around $y_0 = f(x_0)$ such that $f\colon U \to V$ is one-to-one and onto,
i.e., the inverse function $f^{-1}\colon \mathcal{V} \to \mathcal{U}$ exists.

Moreover, we find for its derivative:

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$

Example – Inverse Function Theorem

Let $f\colon \mathbb{R} \to \mathbb{R}$, $x \mapsto y = f(x) = x^2$ and $x_0 = 3$, $y_0 = f(x_0) = 9$.

As $f'(x_0) = 6 \neq 0$, f is locally invertible around $x_0 = 3$ and

$$(f^{-1})'(9) = \frac{1}{f'(3)} = \frac{1}{6}$$

For $x_0 = 0$ we *cannot apply* this theorem as $f'(0) = 0$.
(The inverse function theorem provides a *sufficient* condition.)

Inverse Function Theorem II

Let $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{x}_0$ and x_0 some point with $|\mathbf{f}'(\mathbf{x}_0)| \neq 0$.

Then there exist open hyper-rectangles U around $\mathbf{x}_0$ and V around $\mathbf{y}_0 = \mathbf{f}(\mathbf{x}_0)$ such that $\mathbf{f}\colon U \to V$ is one-to-one and onto,
i.e., the inverse function $\mathbf{f}^{-1}\colon \mathcal{V} \to \mathcal{U}$ exists.

Moreover, we find for its derivative:

$$\boxed{(\mathbf{f}^{-1})'(\mathbf{y}_0) = (\mathbf{f}'(\mathbf{x}_0))^{-1}}$$

The **Jacobian determinant** $|\mathbf{f}'(\mathbf{x}_0)|$ is also denoted by

$$\boxed{\frac{\partial(f_1, \dots, f_n)}{\partial(x_1, \dots, x_n)} = |\mathbf{f}'(\mathbf{x}_0)|}$$

Example – Inverse Function Theorem

Let $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^2$, $\mathbf{x} \mapsto \mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_1^2 - x_2^2 \\ x_1\, x_2 \end{pmatrix}$ and $\mathbf{x}_0 = (1,1)^\mathsf{T}$.

Then $\quad \mathbf{f}'(\mathbf{x}) = \begin{pmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{pmatrix}$ and

$$\frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \begin{vmatrix} 2x_1 & -2x_2 \\ x_2 & x_1 \end{vmatrix} = 2x_1^2 + 2x_2^2 \neq 0 \qquad \text{for all } \mathbf{x} \neq 0.$$

That is, $\mathbf{f}$ is locally invertible around all $\mathbf{x}_0 \neq 0$.
In particular for $\mathbf{x}_0 = (1,1)^\mathsf{T}$ we find

$$(\mathbf{f}^{-1})'(\mathbf{f}(1,1)) = (\mathbf{f}'(1,1))^{-1} = \begin{pmatrix} 2 & -2 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{4} & \frac{2}{4} \\ -\frac{1}{4} & \frac{2}{4} \end{pmatrix}$$

However, $\mathbf{f}$ is not bijective: $\quad \mathbf{f}(1,1) = \mathbf{f}(-1,-1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Explicit and Implicit Function

The relation between two variables x and y can be described by an

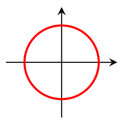
explicit function:

$y = f(x)$

Example:

$y = x^2$

does not exist



implicit function:

$F(x, y) = 0$

Example:

$y - x^2 = 0$

$x^2 + y^2 - 1 = 0$

Questions:

- When can an implicit function be represented (**locally**) by an explicit function?
- What is the derivative of y w.r.t. variable x?

Case: Linear Function

For a linear function

$$F(x, y) = ax + by$$

both questions can be easily answered:

$$ax + by = 0 \qquad \Rightarrow \quad y = -\frac{a}{b}x \qquad \textbf{(if } F_y = b \neq 0\textbf{)}$$

$$\frac{dy}{dx} = -\frac{a}{b} = -\frac{F_x}{F_y}$$

Case: General Function

Let $F(x, y)$ be a function and (x_0, y_0) some point with $F(x_0, y_0) = 0$.

If F is not linear, then we can compute the derivative $\frac{dy}{dx}$ in x_0 by replacing F *locally* by its total differential

$$dF = F_x\,dx + F_y\,dy = d0 = 0$$

and yield[1]

$$\boxed{\frac{dy}{dx} = -\frac{F_x}{F_y}}$$

[1] The given "computation" is not correct but yields the correct result. Note that the differential quotient is not the quotient of differentials.

Example – Implicit Derivative

Compute the implicit derivative $\frac{dy}{dx}$ of

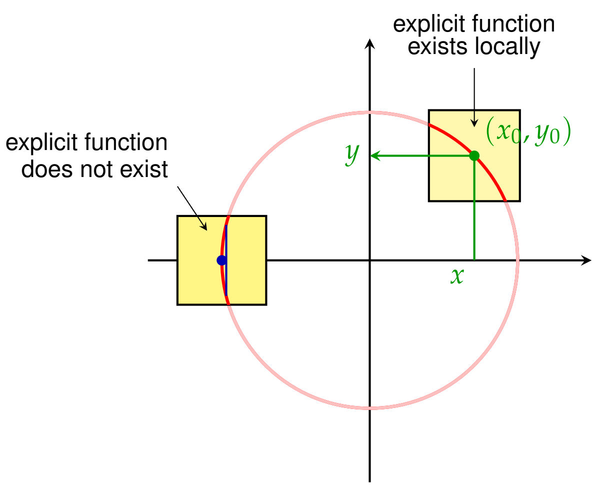
$$F(x, y) = x^2 + y^2 - 1 = 0\,.$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x}{2y} = -\frac{x}{y}$$

We also can compute the derivative of x w.r.t. variable y:

$$\frac{dx}{dy} = -\frac{F_y}{F_x} = -\frac{2y}{2x} = -\frac{y}{x}$$

Local Existence of an Explicit Function



$y = f(x)$ exists locally, if $F_y \neq 0$.

Implicit Function Theorem

Let $F\colon \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be some point with

$$F(x_0, y_0) = 0 \qquad \text{and} \qquad F_y(x_0, y_0) \neq 0\,.$$

Then there exists a rectangle R around (x_0, y_0) such that

- $F(x, y) = 0$ has a unique solution $y = f(x)$ in R, and
- $\boxed{\dfrac{dy}{dx} = -\dfrac{F_x}{F_y}}$

Example – Implicit Function Theorem

Let $F(x, y) = x^2 + y^2 - 8$ and $(x_0, y_0) = (2, 2)$.

As $F(x_0, y_0) = 0$ and $F_y(x_0, y_0) = 2\,y_0 = 4 \neq 0$,
variable y can be represented locally as a function of variable x and

$$\frac{dy}{dx}(x_0) = -\frac{F_x(x_0, y_0)}{F_y(x_0, y_0)} = -\frac{2x_0}{2y_0} = -1\,.$$

Implicit Function Theorem II

Let $F\colon \mathbb{R}^{n+1} \to \mathbb{R}$, $(\mathbf{x}, y) \mapsto F(\mathbf{x}, y) = F(x_1, \dots, x_n, y)$, and
let $(\mathbf{x}_0, y_0)$ be some point with

$$F(\mathbf{x}_0, y_0) = 0 \qquad \text{and} \qquad F_y(\mathbf{x}_0, y_0) \neq 0\,.$$

Then there exists a hyper-rectangle R around $(\mathbf{x}_0, y_0)$ such that

- $F(\mathbf{x}, y) = 0$ has a unique solution $y = f(\mathbf{x})$ in R, where $f\colon \mathbb{R}^n \to \mathbb{R}$, and
- $\boxed{\dfrac{\partial y}{\partial x_i} = -\dfrac{F_{x_i}}{F_y}}$

The independent variable y can be any of the variables of F and need not be in the last position.

Example – Implicit Function Theorem

Compute $\frac{\partial x_2}{\partial x_3}$ of the implicit function

$$F(x_1, x_2, x_3, x_4) = x_1^2 + x_2\,x_3 + x_3^2 - x_3\,x_4 - 1 = 0$$

at point $(x_1, x_2, x_3, x_4) = (1, 0, 1, 1)$.

As $F(1, 0, 1, 1) = 0$ and $F_{x_2}(1, 0, 1, 1) = 1 \neq 0$ we can represent x_2 locally as a function of (x_1, x_3, x_4): $\quad x_2 = f(x_1, x_3, x_4)$.

The partial derivative w.r.t. x_3 is given by

$$\frac{\partial x_2}{\partial x_3} = -\frac{F_{x_3}}{F_{x_2}} = -\frac{x_2 + 2\,x_3 - x_4}{x_3} = -1$$

At $(1, 1, 1, 1)$ and $(1, 1, 0, 1)$ the implicit function theorem cannot be applied for independent variable x_2:
$F(1, 1, 1, 1) \neq 0$ and $F_{x_2}(1, 1, 0, 1) = 0$.

Jacobian Matrix

Let

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) \end{pmatrix} = 0$$

then matrix

$$\frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \dots & \frac{\partial F_1}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \dots & \frac{\partial F_m}{\partial y_m} \end{pmatrix}$$

is called the **Jacobian matrix** of $\mathbf{F}(\mathbf{x}, \mathbf{y})$ w.r.t. $\mathbf{y}$.

Analogous: $\frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{x}}$

Implicit Function Theorem III

Let $\mathbf{F}\colon \mathbb{R}^{n+m} \to \mathbb{R}^m$,

$$(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{F}(\mathbf{x}, \mathbf{y}) = \begin{pmatrix} F_1(x_1, \dots, x_n, y_1, \dots, y_m) \\ \vdots \\ F_m(x_1, \dots, x_n, y_1, \dots, y_m) \end{pmatrix}$$

and let $(\mathbf{x}_0, \mathbf{y}_0)$ be a point with

$$\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = 0 \quad \text{and} \quad \left| \frac{\partial \mathbf{F}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} \right| \neq 0 \quad \text{for } (\mathbf{x}, \mathbf{y}) = (\mathbf{x}_0, \mathbf{y}_0).$$

Then there exists a hyper-rectangle R around $(\mathbf{x}_0, \mathbf{y}_0)$ such that

- $\mathbf{F}(\mathbf{x}, \mathbf{y}) = 0$ has a unique solution $\mathbf{y} = \mathbf{f}(\mathbf{x})$ in R, where $\mathbf{f}\colon \mathbb{R}^n \to \mathbb{R}^m$, and
- $\boxed{\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}\right)^{-1} \cdot \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right)}$

Example – Implicit Function Theorem

Let $\mathbf{F}(\mathbf{x},\mathbf{y}) = \begin{pmatrix} F_1(x_1,x_2,y_1,y_2) \\ F_2(x_1,x_2,y_1,y_2) \end{pmatrix} = \begin{pmatrix} x_1^2 + x_2^2 - y_1^2 - y_2^2 + 3 \\ x_1^3 + x_2^3 + y_1^3 + y_2^3 - 11 \end{pmatrix}$
and $(\mathbf{x}_0,\mathbf{y}_0) = (1,1,\,1,2)$.

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} 2x_1 & 2x_2 \\ 3x_1^2 & 3x_2^2 \end{pmatrix} \qquad \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(1,1,\,1,2) = \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{y}} = \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \frac{\partial F_1}{\partial y_2} \\ \frac{\partial F_2}{\partial y_1} & \frac{\partial F_2}{\partial y_2} \end{pmatrix} = \begin{pmatrix} -2y_1 & -2y_2 \\ 3y_1^2 & 3y_2^2 \end{pmatrix} \quad \frac{\partial \mathbf{F}}{\partial \mathbf{y}}(1,1,\,1,2) = \begin{pmatrix} -2 & -4 \\ 3 & 12 \end{pmatrix}$$

As $\mathbf{F}(1,1,\,1,2) = 0$ and $\left|\frac{\partial \mathbf{F}(\mathbf{x},\mathbf{y})}{\partial \mathbf{y}}\right| = -12 \neq 0$ we can apply the implicit function theorem and we find

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = -\left(\frac{\partial \mathbf{F}}{\partial \mathbf{y}}\right)^{-1} \cdot \left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right) = -\frac{1}{-12} \begin{pmatrix} 12 & 4 \\ -3 & -2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 0 & 0 \end{pmatrix}$$

Summary

- local existence of an inverse function
- derivative of an inverse Function
- inverse function theorem
- explicit and implicit function
- explicit representation of an implicit function
- derivative of an implicit function
- implicit function theorem