Chapter 1

Logic, Sets and Maps

Proposition

We need some elementary knowledge about **logic** for doing mathematics. The central notion is “proposition”.

A **proposition** is a sentence with is either **true** (T) or **false** (F).

- *“Vienna is located at river Danube.”* is a true proposition.
- *“Bill Clinton was president of Austria.”* is a false proposition.
- *“19 is a prime number.”* is a true proposition.
- *“This statement is false.”* is not a proposition.

Logical Connectives

We get compound propositions by connecting (simpler) propositions by using **logical connectives**.

This is done by means of words *"and"*, *"or"*, *"not"*, or *"if ... then"*, known from everyday language.

Connective	Symbol	Name
not P	$\neg P$	negation
P and Q	$P \wedge Q$	conjunction
P or Q	$P \vee Q$	disjunction
if P then Q	$P \Rightarrow Q$	implication
P if and only if Q	$P \Leftrightarrow Q$	equivalence

Truth Table

Truth values of logical connectives.

P	Q	$\neg P$	$P \wedge Q$	$P \vee Q$	$P \Rightarrow Q$	$P \Leftrightarrow Q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

Let $P =$ "x is divisible by 2" and $Q =$ "x is divisible by 3".
Proposition $P \wedge Q$ is true if and only if x is divisible by 2 and 3 (i.e., by 6).

Negation and Disjunction

- *Negation* $\neg P$ is not the “opposite” of proposition P.

  Negation of $P =$ “all cats are black”
  is $\neg P =$ “Not all cats are black”

  (And not “all cats are not black” or even “all cats are white”!)

- *Disjunction* $P \vee Q$ is in a non-exclusive sense:

  $P \vee Q$ is true if and only if
  - P is true, or
  - Q is true, or
  - both P and Q are true.

Implication

The truth value of *implication* $P \Rightarrow Q$ seems a bit mysterious.

Note that $P \Rightarrow Q$ does not make any proposition about the truth value of P or Q!

Which of the following propositions is true?

- "*If* Bill Clinton is Austrian citizen, *then* he can be elected for Austrian president."
- "*If* Karl (born 1970) is Austrian citizen, *then* he can be elected for Austrian president."
- "*If* x is a prime number larger than 2, *then* x is odd."

Implication $P \Rightarrow Q$ is *equivalent* to $\neg P \vee Q$:

$$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$$

A Simple Logical Proof

We can derive the truth value of proposition $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ by means of a truth table:

P	Q	$\neg P$	$(\neg P \vee Q)$	$(P \Rightarrow Q)$	$(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

That is, proposition $(P \Rightarrow Q) \Leftrightarrow (\neg P \vee Q)$ is always true independently from the truth values for P and Q.

It is a so called *tautology*.

Theorems

Mathematics consists of propositions of the form: P implies Q,
but you never ask whether P is true. *(Bertrand Russell)*

A **mathematical statement** (*theorem*, *proposition*, *lemma*, *corollary*) is a proposition of the form $P \Rightarrow Q$.

P is called a **sufficient** condition for Q.

A *sufficient* condition P guarantees that proposition Q is true. However, Q can be true even if P is false.

Q is called a **necessary** condition for P, $\quad Q \Leftarrow P$.

A *necessary* condition Q must be true to allow P to be true. It does not guarantee that P is true.

Necessary conditions often are used to find *candidates* for valid answers to our problems.

Quantors

Mathematical texts often use the expressions *“for all”* and *“there exists”*, resp.

In formal notation the following symbols are used:

Quantor	Symbol
for all	$\forall$
there exists a	$\exists$
there exists exactly one	$\exists!$
there does not exists	$\nexists$

Set*

The notion of *set* is fundamental in modern mathematics.

We use a simple definition from naïve set theory:

> A **set** is a collection of *distinct* objects.

An object a of a set A is called an **element** of the set. We write:

$$\boxed{a \in A}$$

Sets are defined by *enumerating* or a *description* of their elements within *curly brackets* $\{\dots\}$.

$A = \{1, 2, 3, 4, 5, 6\} \qquad B = \{x \mid x \text{ is an integer divisible by 2}\}$

Important Sets*

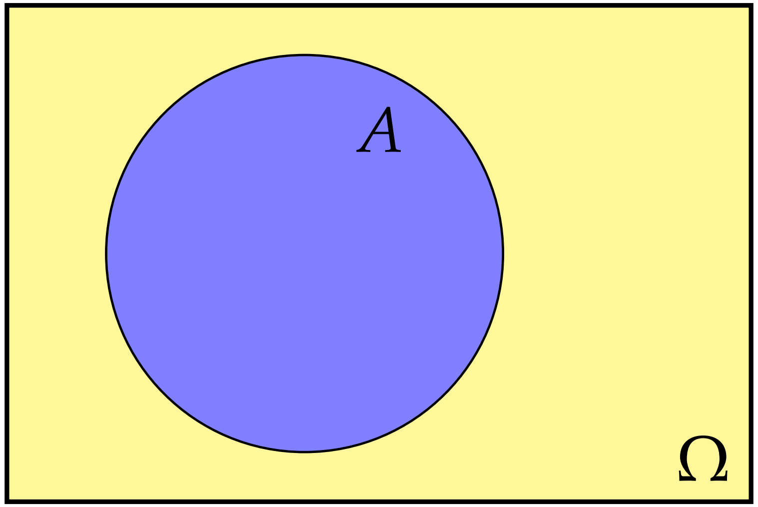
Symbol	Description
$\emptyset$	empty set sometimes: $\{\}$
$\mathbb{N}$	natural numbers $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	integers $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
$\mathbb{Q}$	rational numbers $\{\frac{k}{n} \mid k, n \in \mathbb{Z}, n \neq 0\}$
$\mathbb{R}$	real numbers
$[a, b]$	closed interval $\{x \in \mathbb{R} \mid a \leq x \leq b\}$
(a, b)	open interval[a] $\{x \in \mathbb{R} \mid a < x < b\}$
$[a, b)$	half-open interval $\{x \in \mathbb{R} \mid a \leq x < b\}$
$\mathbb{C}$	complex numbers $\{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$

[a]also: $]a, b[$

Venn Diagram*

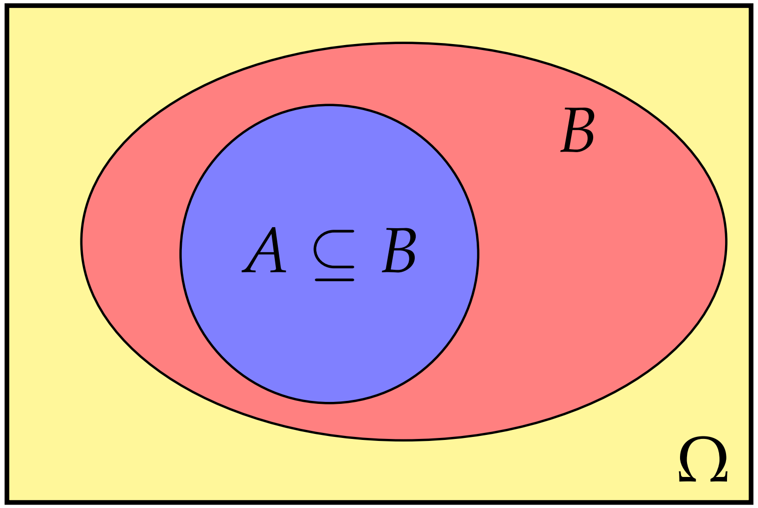
We assume that all *sets* are subsets of some universal **superset** Ω.

Sets can be represented by **Venn diagrams** where Ω is a rectangle and sets are depicted as circles or ovals.



Subset and Superset*

Set A is a **subset** of B, $\boxed{A \subseteq B}$, if all elements of A also belong to B,
$x \in A \Rightarrow x \in B$.



Vice versa, B is then called a **superset** of A, $\boxed{B \supseteq A}$.

Set A is a **proper subset** of B, $\boxed{A \subset B}$ (or: $A \subsetneqq B$),
if $A \subseteq B$ and $A \neq B$.

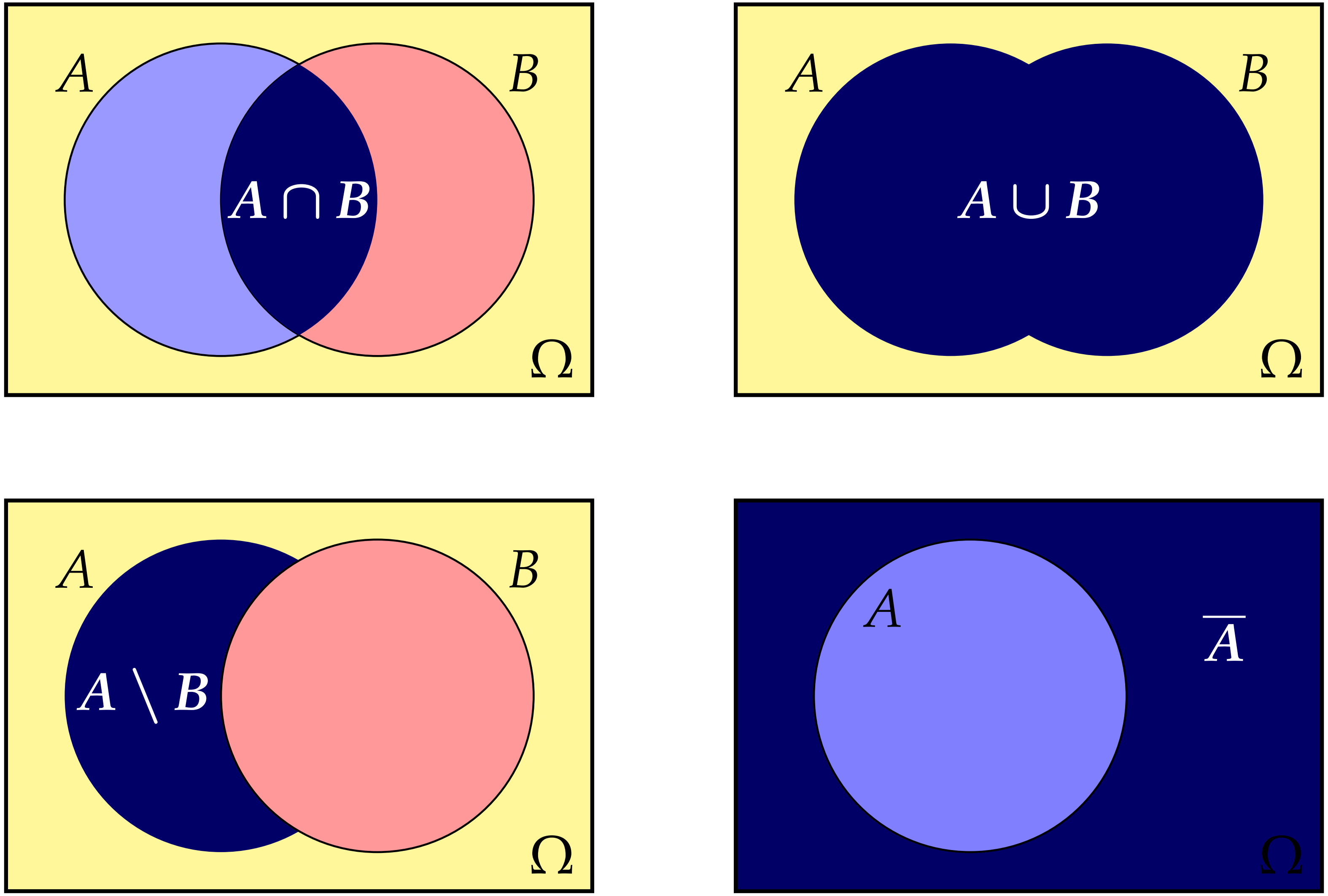
Basic Set Operations*

Symbol	Definition	Name
$A \cap B$	$\{x \mid x \in A$ and $x \in B\}$	**intersection**
$A \cup B$	$\{x \mid x \in A$ or $x \in B\}$	**union**
$A \setminus B$	$\{x \mid x \in A$ and $x \notin B\}$	**set-theoretic difference**[a]
$\overline{A}$	$\Omega \setminus A$	**complement**

[a] also: $A - B$

Two sets A and B are **disjoint** if $A \cap B = \emptyset$.

Basic Set Operations*



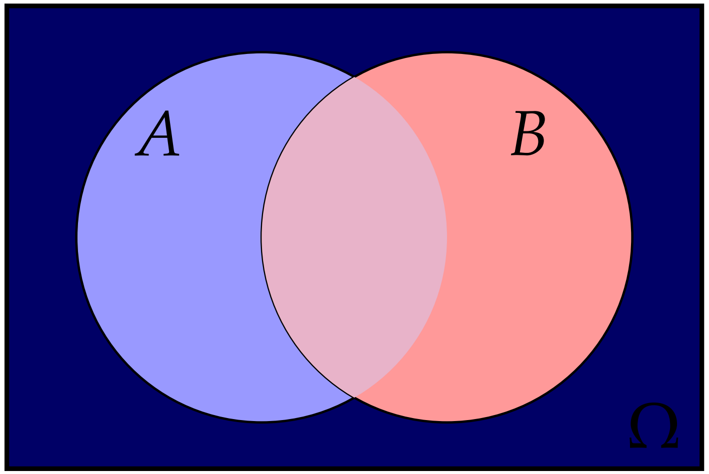
Rules for Basic Operations*

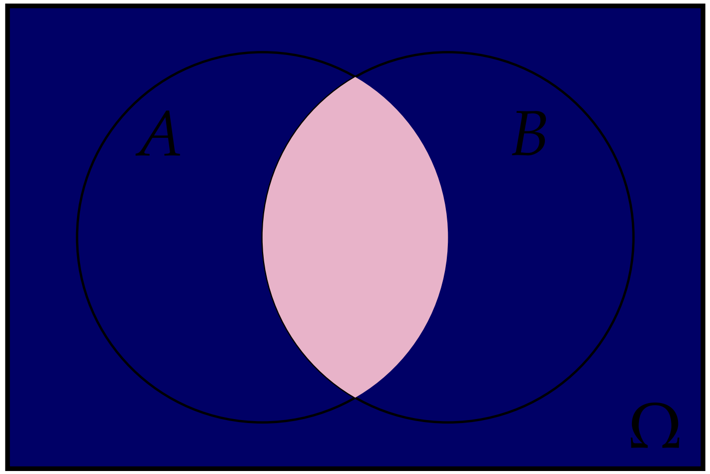
Rule	Name
$A \cup A = A \cap A = A$	*Idempotence*
$A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$	*Identity*
$(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$	*Associativity*
$A \cup B = B \cup A$ and $A \cap B = B \cap A$	*Commutativity*
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	*Distributivity*

$\overline{A} \cup A = \Omega$ and $\overline{A} \cap A = \emptyset$ and $\overline{\overline{A}} = A$

De Morgan's Law*

$$\overline{(A \cup B)} = \overline{A} \cap \overline{B} \quad \text{and} \quad \overline{(A \cap B)} = \overline{A} \cup \overline{B}$$





A union B complemented is the equivalent of A complemented intersected with B complemented.

Cartesian Product*

The set

$$\boxed{A \times B = \{(x,y) | x \in A, \, y \in B\}}$$

is called the **Cartesian product** of *sets* A and B.

Given two sets A and B the Cartesian product $A \times B$ is the set of all unique *ordered pairs* where the first element is from set A and the second element is from set B.

In general we have $\quad A \times B \neq B \times A$.

Cartesian Product*

The Cartesian product of $A = \{0,1\}$ and $B = \{2,3,4\}$ is

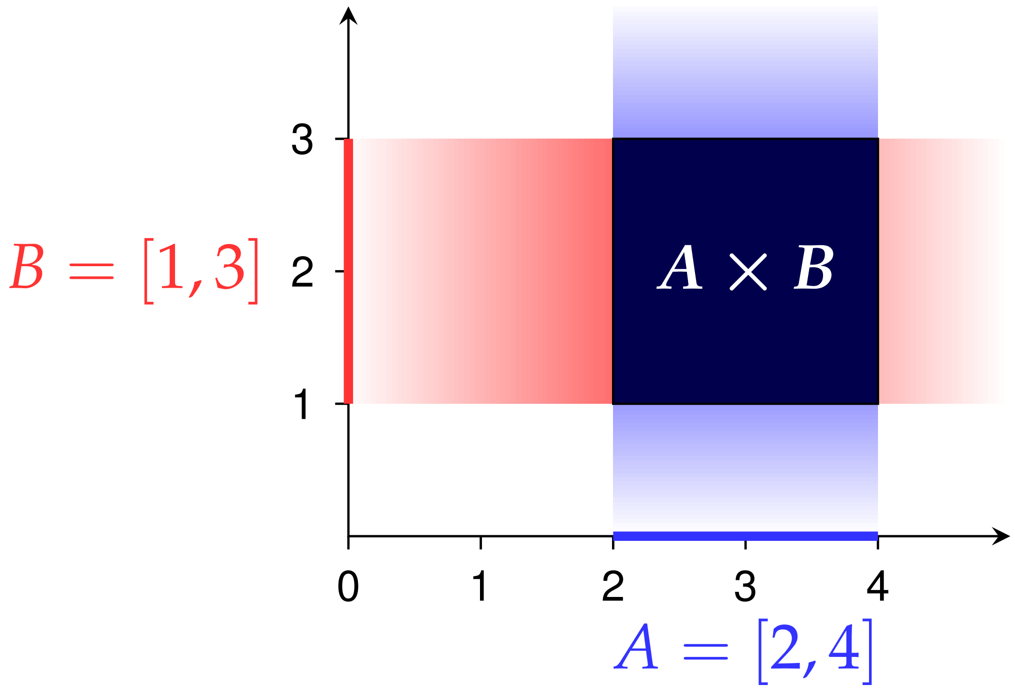
$$A \times B = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}.$$

$A \times B$	2	3	4
0	$(0,2)$	$(0,3)$	$(0,4)$
1	$(1,2)$	$(1,3)$	$(1,4)$

Cartesian Product*

The Cartesian product of $A = [2,4]$ and $B = [1,3]$ is

$$A \times B = \{(x,y) \mid x \in [2,4] \text{ and } y \in [1,3]\}.$$



Map*

A **map** (or **mapping**) f is defined by

(i) a **domain** D_f,

(ii) a **codomain** (**target set**) W_f and

(iii) a **rule**, that maps each element of D to *exactly one* element of W.

$$\boxed{f\colon D \to W, \quad x \mapsto y = f(x)}$$

- x is called the **independent** variable, y the **dependent** variable.
- y is the **image** of x, x is the **preimage** of y.
- $f(x)$ is the **function term**, x is called the **argument** of f.
- $f(D) = \{y \in W : y = f(x) \text{ for some } x \in D\}$
  is the **image** (or **range**) of f.

Other names: *function*, *transformation*

Injective · Surjective · Bijective*

Each argument has exactly one image.
Each $y \in W$, however, may have any number of preimages.
Thus we can characterize maps by their possible number of preimages.

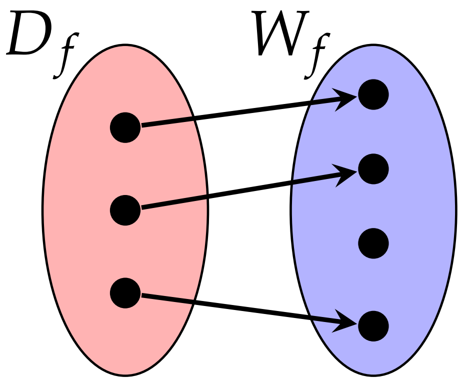
- A map f is called **one-to-one** (or **injective**), if each element in the codomain has *at most one* preimage.
- It is called **onto** (or **surjective**), if each element in the codomain has *at least one* preimage.
- It is called **bijective**, if it is both one-to-one and onto, i.e., if each element in the codomain has *exactly one* preimage.

Injections have the important property

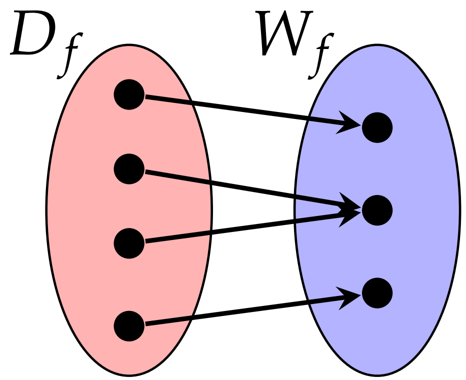
$$f(x) \neq f(y) \quad \Leftrightarrow \quad x \neq y$$

Injective · Surjective · Bijective*

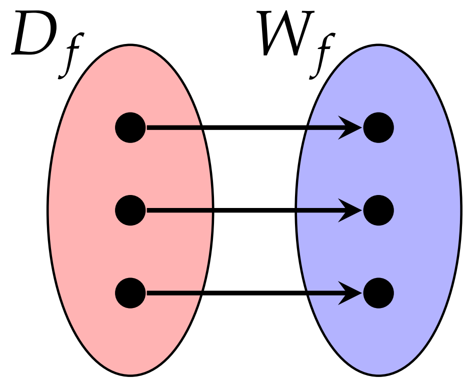
Maps can be visualized by means of arrows.



one-to-one
(not onto)



onto
(not one-to-one)



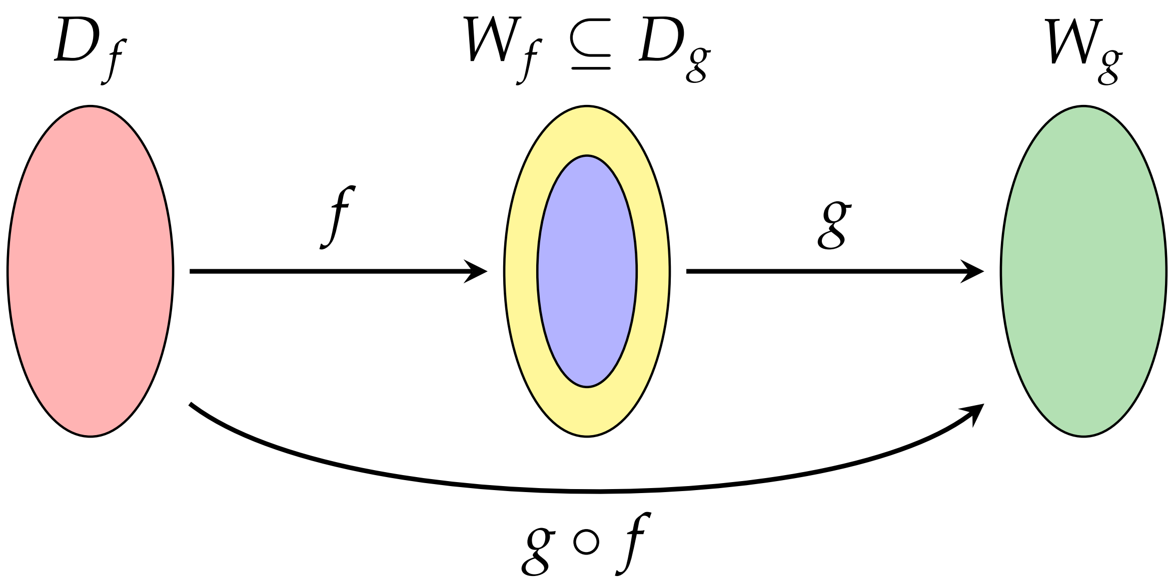
one-to-one and onto
(bijective)

Function Composition*

Let $f\colon D_f \to W_f$ and $g\colon D_g \to W_g$ be functions with $W_f \subseteq D_g$.
Function

$$\boxed{g \circ f\colon D_f \to W_g,\; x \mapsto (g \circ f)(x) = g(f(x))}$$

is called **composite function**.
(read: "g composed with f", "g circle f", or "g after f")



Inverse Map*

If $f\colon D_f \to W_f$ is a **bijection**, then every $y \in W_f$ can be uniquely mapped to its preimage $x \in D_f$.

Thus we get a map

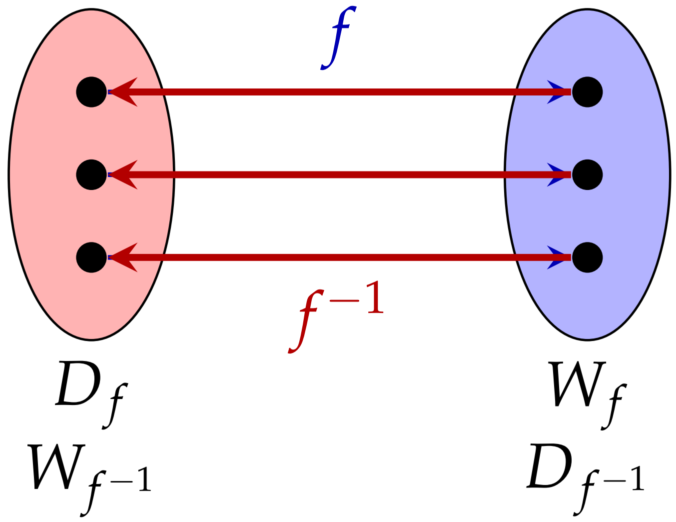
$$\boxed{f^{-1}\colon W_f \to D_f,\ y \mapsto x = f^{-1}(y)}$$

which is called the **inverse map** of f.

We obviously have for all $x \in D_f$ and $y \in W_f$,

$$f^{-1}(f(x)) = f^{-1}(y) = x \quad \text{and} \quad f(f^{-1}(y)) = f(x) = y\,.$$

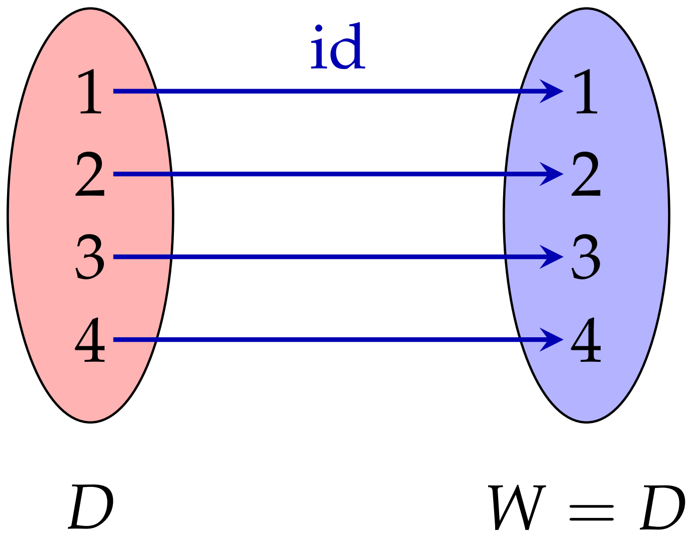
Inverse Map*



Identity*

The most elementary function is the **identity map** id, which maps its argument to itself, i.e.,

$$\boxed{\mathrm{id}\colon D \to W = D,\ x \mapsto x}$$



Identity*

The identity map has a similar role for compositions of functions as 1 has for multiplications of numbers:

$$f \circ \mathrm{id} = f \qquad \text{and} \qquad \mathrm{id} \circ f = f$$

Moreover,

$$\boxed{f^{-1} \circ f = \mathrm{id}\colon D_f \to D_f \qquad \text{and} \qquad f \circ f^{-1} = \mathrm{id}\colon W_f \to W_f}$$

Real-valued Functions*

Maps where domain and codomain are (subsets of) *real* numbers are called **real-valued functions**,

$$\boxed{f : \mathbb{R} \to \mathbb{R},\ x \mapsto f(x)}$$

and are the most important kind of functions.

The term **function** is often exclusively used for *real-valued* maps.

We will discuss such functions in more details later.

Summary

- mathematical logic
- theorem
- necessary and sufficient condition
- sets, subsets and supersets
- Venn diagram
- basic set operations
- de Morgan’s law
- Cartesian product
- maps
- one-to-one and onto
- inverse map and identity