Matrix Algebra

$[\alpha \boldsymbol{A}]_{ij} = \alpha[\boldsymbol{A}]_{ij}$
$[\boldsymbol{A}+\boldsymbol{B}]_{ij} = [\boldsymbol{A}]_{ij} + [\boldsymbol{B}]_{ij}$
$[\boldsymbol{A}\cdot\boldsymbol{B}]_{ij} = \sum_{s=1}^{n}[\boldsymbol{A}]_{is}[\boldsymbol{B}]_{sj}$
$\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x}^t\boldsymbol{x}}$

Determinant

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\boldsymbol{A}^{*t}$$

Eigenvalues

$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{A}$

$$A_k = \begin{vmatrix} a_{11} & \dots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \dots & a_{kk} \end{vmatrix}$$

$$A_{i_1,\dots,i_k} = \begin{vmatrix} a_{i_1,i_1} & \dots & a_{i_1,i_k} \\ \vdots & \ddots & \vdots \\ a_{i_k,i_1} & \dots & a_{i_k,i_k} \end{vmatrix}$$

$\boldsymbol{A}$ positive definite $\Leftrightarrow$ all $\lambda_k > 0 \Leftrightarrow$ all $A_k > 0$

Limits

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$
if $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = 0\ (\pm\infty)$

Derivatives

$(x^\alpha)' = \alpha\cdot x^{\alpha-1}$
$(e^x)' = e^x$
$(\ln(x))' = \frac{1}{x}$
$(\sin(x))' = \cos(x)$
$(\cos(x))' = -\sin(x)$

$(f(x)\cdot g(x))' = f'(x)\cdot g(x) + f(x)\cdot g'(x)$
$(f(g(x)))' = f'(g(x))\cdot g'(x)$

$$\varepsilon_f(x) = x\cdot\frac{f'(x)}{f(x)}$$

Multivariate Analysis

gradient: $\nabla f(\boldsymbol{x}) = (f_{x_1}(\boldsymbol{x}),\dots,f_{x_n}(\boldsymbol{x}))$

directional derivative: $\frac{\partial f}{\partial \boldsymbol{h}} = \nabla f(\boldsymbol{x})\cdot\boldsymbol{h}\quad,\quad ||\boldsymbol{h}|| = 1$

Jacobian Matrix:

$$f'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) & \dots & \frac{\partial f_1}{\partial x_n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1}(\boldsymbol{x}) & \dots & \frac{\partial f_m}{\partial x_n}(\boldsymbol{x}) \end{pmatrix}$$

$(g\circ f)'(\boldsymbol{x}) = g'(f(\boldsymbol{x}))\cdot f'(\boldsymbol{x})$

Hessian matrix:

$$\boldsymbol{H}_f(\boldsymbol{x}) = \begin{pmatrix} f_{x_1x_1}(\boldsymbol{x}) & \dots & f_{x_1x_n}(\boldsymbol{x}) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(\boldsymbol{x}) & \dots & f_{x_nx_n}(\boldsymbol{x}) \end{pmatrix}$$

Inverse and Implicit Functions

$\left(f^{-1}\right)'(\boldsymbol{y}) = \left(f'(\boldsymbol{x})\right)^{-1}\ ,\quad \boldsymbol{y} = f(\boldsymbol{x})$

Derivative of implicit function $\boldsymbol{F}(x_1,\dots,x_n) = 0$:

$$\frac{\partial x_i}{\partial x_k} = -\frac{F_{x_k}}{F_{x_i}}$$

Taylor Series

$$f(x) \approx f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = \sum_{k=0}^{n}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$$

Important Taylor series:

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$
$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

$$f(\boldsymbol{x}) \approx f(\boldsymbol{0}) + \nabla f(\boldsymbol{0})^t\boldsymbol{x} + \frac{1}{2}\boldsymbol{x}^t\boldsymbol{H}_f(\boldsymbol{0})\boldsymbol{x}$$

$$f(\boldsymbol{x}) \approx f(\boldsymbol{x}_0) + \nabla f(\boldsymbol{x}_0)^t(\boldsymbol{x}-\boldsymbol{x}_0) + \tfrac{1}{2}(\boldsymbol{x}-\boldsymbol{x}_0)^t\boldsymbol{H}_f(\boldsymbol{0})(\boldsymbol{x}-\boldsymbol{x}_0)$$

Integration

$\int x^a\,dx = \frac{1}{a+1}\cdot x^{a+1} + c$

$\int e^x\,dx = e^x + c$

$\int \frac{1}{x}\,dx = \ln|x| + c$

$\int \cos(x)\,dx = \sin(x) + c$

$\int \sin(x)\,dx = -\cos(x) + c$

$$\int f(x)\cdot g'(x)\,dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)\,dx$$

$$\int f(g(x))\cdot g'(x)\,dx = \int f(z)\,dz \text{ with } z = g(x)$$

Optimization

Lagrange function:
$\mathcal{L}(x,y;\lambda) = f(x,y) + \lambda\,(c - g(x,y))$

bordered Hessian:

$$\bar{\boldsymbol{H}}(\boldsymbol{x}) = \begin{pmatrix} 0 & g_x & g_y \\ g_x & \mathcal{L}_{xx} & \mathcal{L}_{xy} \\ g_y & \mathcal{L}_{yx} & \mathcal{L}_{yy} \end{pmatrix}$$

$|\bar{\boldsymbol{H}}(\boldsymbol{x})| > 0 \Rightarrow \boldsymbol{x}$ is local maximum

Kuhn-Tucker conditions:
$\mathcal{L}_{x_j} \le 0,\quad x_j \ge 0,\quad x_j\,\mathcal{L}_{x_j} = 0$
$\mathcal{L}_{\lambda_i} \ge 0,\quad \lambda_i \ge 0,\quad \lambda_i\,\mathcal{L}_{\lambda_i} = 0$