Portfolio Credit Risk Models with Interacting Default Intensities: a Markovian Approach

Rüdiger Frey and Jochen Backhaus

Department of Mathematics
University of Leipzig*

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Abstract

We consider reduced-form models for portfolio credit risk with interacting default intensities. In this class of models the impact of default of some firm on the default intensities of surviving firms is exogenously specified and the dependence structure of the default times is endogenously determined. We construct and study the model using Markov process techniques. We analyze in detail a model where the interaction between firms is of the mean-field type. Moreover, we discuss the pricing of portfolio related credit products such as basket default swaps and CDOs in our model.

Keywords: Portfolio credit risk, default correlation, credit derivatives, mean-field interaction, Markov processes

1 Introduction

A major cause of concern in the pricing and management of the credit risk in a given loan or bond portfolio is the occurrence of disproportionately many defaults of different counterparties in the portfolio, a risk which is directly linked to the structure of the dependence between default events. Dependence between defaults stems from at least two non-exclusive sources. First the financial health of a firm varies with randomly fluctuating macroeconomic factors such changes in economic growth. Since different firms are affected by common macroeconomic factors, we have dependence between their defaults. This type of dependence between defaults can and has been modelled in the standard reduced-form credit risk models with conditionally independent defaults; see for instance Duffie & Singleton (2003) or Lando (2004) for an overview.

Moreover, dependence between defaults is caused by direct economic links between firms. These direct links lead to default contagion and counterparty risk. Loosely speaking this means that the conditional default probability of non-defaulted firms given the additional information that some firm has defaulted is higher than the unconditional

*Postal address: Mathematisches Institut, Universität Leipzig, Augustusplatz 10/11 D-04109 Leipzig, Email: frey@math.uni-leipzig.de, backhaus@math.uni-leipzig.de. We are grateful to Stefan Weber for helpful suggestions.

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default probability of these firms. As a consequence the credit spread of bonds issued by non-defaulted firms increases given the news that some other firm has defaulted. In mathematical terms in reduced-form models default contagion and counterparty risk lead to (upward) jumps in the default intensity of non-defaulted firms at the default time of other firms in the portfolio. The impact of the default of some firm on the conditional default probability of other firms can arise via different non-exclusive channels. On the one hand this impact can be caused by direct economic links between firms such as an intense business relation or a strong borrower-lender relationship. For instance the default probability of a corporate bank is likely to increase if one of its major borrowers defaults. This direct channel of default interaction is termed counterparty risk. On the other hand, changes in the conditional default probability of non-defaulted firms can be caused by information effects: investors might revise their estimate of the financial health of non-defaulted firms in light of the news that a particular firm has defaulted. This phenomenon is usually termed (information-based) default contagion.

There is substantial empirical evidence for interaction between default events. A recent example is provided by the downfall of the energy giant Enron in autumn 2001. The news that Enron had used illegal accounting practices led to rising credit spreads for many other corporations, as bond investors lost confidence in the accounting statements of these corporations – a striking example of default contagion. Moreover, the stock price of major lenders to Enron fell in anticipation of large losses on these loans, reflecting counterparty risk. More formal empirical evidence for default contagion and counterparty risk is for instance provided by Lang & Stulz (1992) or by Collin-Dufresne, Goldstein & Helwege (2003b).

The modelling of default contagion and counterparty risk has generated a lot of interest in the recent literature. The existing reduced-form models with these features can be divided into two groups, copula models such as Schönbucher & Schubert (2001) and models with interacting intensities. In the copula models the copula and hence the dependence structure of the default times is exogenously specified; the default intensities and the amount of default contagion (the reaction of default intensities to defaults of other firms in the portfolio) are then endogenously derived from the model primitives. Copula models are quite popular in practice, since they are easy to calibrate to prices of defaultable bonds or Credit Default Swap (CDS) spreads. However, in general copula models the precise form of the default contagion depends on higher order derivatives of the copula, which makes the copula parameterization of default contagion quite unintuitive. This problem is less pronounced in the so-called factor copula models, which use ideas from survival analysis to model information based default contagion; see for instance Laurent & Gregory (2003) or Schönbucher (2004).

As the title suggests, in the present paper we are interested in models with interacting intensities. In this class of models the impact of defaults on the default intensities of surviving firms is exogenously specified; the joint distribution of the default times is then endogenously derived. This leads to a very intuitive parameterization of counterparty risk and dependence between defaults in general. On the downside, the calibration of the model to defaultable term structure data can be more evolved. At least to our knowledge models with interacting intensities were first proposed by Jarrow & Yu (2001) and Davis & Lo (2001). Unfortunately, the construction of default processes in Jarrow & Yu (2001) works only for a very special type of interaction between defaults, the so-called primary secondary framework, which excludes many interesting examples of cyclical default dependence. This and other mathematical aspects of the Jarrow-Yu model are discussed in Kusuoka (1999), Bielecki & Rutkowski (2002), and Collin-Dufresne, Goldstein & Hugonnier (2003). Yu (2004) improves upon the original Jarrow-Yu paper and provides a rigorous construction of the model using the general hazard construction from survival analysis. Moreover, he prices certain simple credit derivatives using simulation. Credit risk models with explicitly specified interaction between default intensities are conceptually and mathematically close to models for interacting particle systems developed in statistical physics. Föllmer (1994) contains an inspiring discussion of the relevance of ideas from the interacting particle systems literature for financial modelling; the link to credit risk is explored by Giesecke & Weber (2002, 2003) and Horst (2004). Finally, Egloff, Leippold & Vanini (2004) study credit contagion in a firm-value model.

In the present paper we propose several extensions to the literature on models with interacting intensities. To begin with, we model the default indicator process of the firms in our portfolio as conditional finite-state Markov chain; the states of this chain are given by the default state of all obligors in the portfolio at a given point in time and the transition rates correspond to the default intensities. This yields a natural and at the same time completely rigorous construction of models with interacting intensities. Moreover, computational tools for Markov chains can be employed fruitfully in the analysis of the model. These results, which are similar in spirit to in Davis & Lo (2001), are presented in Section 2.

In Section 3 we take a closer look at the modelling of the interaction between the default intensities. This is a major challenge, in particular if the portfolio is large: the model should capture essential features of counterparty risk, and should at the same time be parsimonious to ensure ease of calibration. To achieve these goals we split our portfolio in several homogeneous groups and propose a model where the default intensity of a given firm depends only on the distribution of defaulted firms in these groups - in the simplest case of a one-group model just the proportion of companies which have defaulted so far. This type of interaction, which is called mean-field interaction in the literature on interacting particle systems, makes immediate sense in the context of portfolio credit risk. For instance, if a financial institution has incurred unusually many losses in its loan portfolio, it is less likely to extend credit lines, if another obligor experiences financial distress. Obviously, this raises the default probability of the remaining obligors. Moreover, unusually many defaults might have a negative impact on the business climate in general. From a mathematical viewpoint we are automatically led to models based on mean-field interaction, if we assume that our portfolio consists of several homogeneous groups within which default times are exchangeable. We will show that homogeneous-group models with mean-field interaction are relatively easy to treat. Using results on the convergence in distribution of Markov processes we study the asymptotic behavior of the mean-field model as the portfolio size becomes large. In order to quantify the impact of counterparty risk on default correlations and credit loss distribution we carry out a simulation study. It will turn out that default correlations and quantiles of the loss distributions increase substantially, if the amount of interaction in the portfolio is increased.

In Section 4 we finally study the pricing of portfolio credit derivatives such as basket default swaps and CDOs in our Markovian model. This is a prime area of application for dynamic portfolio credit risk models. In particular, we show how computational tools for Markov chains can be fruitfully employed to find semi-analytical solutions for many pricing problems. Again we provide some numerical examples to illustrate our findings.

2 A Markovian Model with Interacting Intensities

Our setup. We consider a portfolio of m firms, indexed by $i \in \{1, \ldots, m\}$. Its default state is described by a default indicator process $\mathbf{Y} = \big(Y_t(1), \ldots, Y_t(m)\big)_{t \geq 0}$ with values in $S := \{0,1\}^m$; here $Y_t(i) = 1$ if firm i has defaulted by time t and $Y_t(i) = 0$ else. The corresponding default times are denoted by $\tau_i = \inf\{t \geq 0 : Y_t(i) = 1\}$. Throughout our analysis we restrict ourselves to models without simultaneous defaults. We may therefore define the *ordered default times* $T_0 < T_1 < \ldots < T_m$ recursively by

$$T_0 = 0 \text{ and } T_n = \min\{\tau_i : 1 \leq i \leq m, \, \tau_i > T_{n-1}\}, \; 1 \leq n \leq m. \tag{1}$$

By $\xi^n \in \{1, \ldots, m\}$ we denote the identity of the firm defaulting at time T_n, i.e. $\xi^n = i$ if $\tau_i = T_n$. It will be convenient to have a succinct notation for flipping some coordinate of states in S. We therefore define for $\mathbf{y} \in S$ the flipped state $\mathbf{y}^i \in S$ by

$$y^i(i) := 1 - y(i) \text{ and } y^i(j) := y(j), \; j \in \{1, \ldots, m\} - \{i\}\,. \tag{2}$$

In order to model the dependence of defaults caused by fluctuations in the macroeconomic environment we introduce Markovian state variable process $\boldsymbol{\Psi} = (\boldsymbol{\Psi}_t)_{t \geq 0}$ with values in $\mathbb{R}^d$, representing the evolution of macroeconomic variables such as interest rates, broad share price indices or measures of economic activity. In applications $\boldsymbol{\Psi}$ is typically a (jump-)diffusion or a finite-state Markov chain. The overall state of our system is described by the process $\boldsymbol{\Gamma}$ with $\boldsymbol{\Gamma}_t := (\boldsymbol{\Psi}_t, \mathbf{Y}_t)$. The state space of $\boldsymbol{\Gamma}$ is denoted by $\overline{S} = \mathbb{R}^d \times S$; elements of $\overline{S}$ are denoted by $\boldsymbol{\gamma} = (\boldsymbol{\psi}, \mathbf{y})$.

The default intensity of a non-defaulted firm i at time t is modelled as a function $\lambda_i(\boldsymbol{\Psi}_t, \mathbf{Y}_t)$ of economic factors and of the default state of the portfolio. Hence the default intensity of a firm may change if there is a change in the default state of other firms in the portfolio, so that counterparty risk can be modelled. In mathematical terms we assume that for a given trajectory of $\boldsymbol{\Psi}$ the default indicator process $\mathbf{Y}$ follows a time-inhomogeneous continuous-time Markov chain on S with transitions to neighbouring states $\big(\mathbf{Y}_t\big)^i$ which occur with transition rate $1_{\{Y_t(i)=0\}}\lambda_i(\boldsymbol{\Psi}_t, \mathbf{Y}_t)$. An explicit probabilistic model is introduced below.

The mathematical model. It will be convenient for the analysis of the limiting behaviour of the model as $m \to \infty$ to construct the process $\boldsymbol{\Gamma}$ on a probability space which has a product structure. Denote by $\mathbf{D}([0,\infty), E)$ the Skorohod space of all RCLL functions from $[0,\infty)$ into some Polish space E. Put $\Omega_1 := \mathbf{D}([0,\infty), \mathbb{R}^d)$ and $\Omega_2 := \mathbf{D}\left([0,\infty), S\right)$ and denote by $\mathcal{F}^i$ the Borel σ-field on Ω_i. Our underlying measurable space is given by $(\Omega, \mathcal{F}) := (\Omega_1 \times \Omega_2, \mathcal{F}^1 \times \mathcal{F}^2)$; elements in Ω will be written as $\omega = (\omega_1, \omega_2)$. The coordinate process on Ω_1 is denoted by $\boldsymbol{\Psi}$ (i.e. $\boldsymbol{\Psi}_t(\omega_1) = \omega_1(t)$ for $t \geq 0$); it represents the economic factor process. The coordinate process on Ω_2, denoted by $\mathbf{Y}$, models the default indicator process. For $t \in [0,\infty)$ we define $\mathcal{F}_t^1 := \sigma(\boldsymbol{\Psi}_s : s \leq t)\,,\, \mathcal{F}_t^2 := \sigma(\mathbf{Y}_s : \, s \leq t)$ and $\mathcal{F}_t := \mathcal{F}_t^1 \vee \mathcal{F}_t^2$; moreover, we define the filtration $\{\mathcal{G}_t\}$ by $\mathcal{G}_t := \mathcal{F}_\infty^1 \vee \mathcal{F}_t^2$. We assume that investors have access to $\{\mathcal{F}_t\}$ (information about the default history and the economic factor process up to time t), whereas the larger filtration $\{\mathcal{G}_t\}$ (information about the default history up to time t and information about the entire path $(\boldsymbol{\Psi}_s(\omega_1))_{s\geq 0}$ of the economic factor process) serves mainly theoretical purposes.

We consider a family of probability measures $(P_{\boldsymbol{\gamma}})_{\boldsymbol{\gamma}\in\overline{S}}$ on $(\Omega, \mathcal{F})$, where each measure is of the form $P_{\boldsymbol{\gamma}} = \mu_{\boldsymbol{\psi}} \times K_{\mathbf{y}}(\omega_1, d\omega_2)$. Here $\mu_{\boldsymbol{\psi}}$ is a probability measure on Ω_1 which gives the law of $\boldsymbol{\Psi}$; $K_{\mathbf{y}}$ is a transition kernel from $(\Omega_1, \mathcal{F}^1)$ to $(\Omega_2, \mathcal{F}^2)$, which models the conditional distribution of the default indicator process $\mathbf{Y}$ for a given trajectory of $\boldsymbol{\Psi}$.

Assumption 2.1.

(i) Under $\mu_{\boldsymbol{\psi}}$ the process $\boldsymbol{\Psi}$ is a non-exploding $\{\mathcal{F}_t\}$-Markov process with generator $\mathcal{L}^{\Psi}$ and initial value $\boldsymbol{\psi}$.

(ii) Under $K_{\mathbf{y}}(\omega_1, d\omega_2)$ the process $\mathbf{Y}$ is a time-inhomogeneous Markov chain with state space S, initial value $\mathbf{y}$ and infinitesimal generator as follows: Define for continuous functions $\lambda_i : \overline{S} \to (0, \infty)$ and given $\boldsymbol{\psi} \in \mathbb{R}^d$ the operator $G_{[\boldsymbol{\psi}]}$ on the set of all functions from S to $\mathbb{R}$ by

$$G_{[\boldsymbol{\psi}]} f(\mathbf{x}) = \sum_{i=1}^{m} (1 - x(i)) \lambda_i(\boldsymbol{\psi}, \mathbf{x}) \left(f(\mathbf{x}^i) - f(\mathbf{x}) \right), \quad \mathbf{x} \in S . \tag{3}$$

Then the infinitesimal generator of $\mathbf{Y}$ under $K_{\mathbf{y}}(\omega_1, d\omega_2)$ at time t is given by $G_{[\boldsymbol{\Psi}_t(\omega_1)]}$.

If there is no ambiguity we simply write P, μ, and K and drop the reference to the initial values to ease the notation. Moreover, unless explicitly stated otherwise, we will always assume that $\mathbf{Y}_0 = \mathbf{0} \in S$.

Comments. 1) An intuitive picture of the dynamics of the default indicator process $\mathbf{Y}$ implied by the generator $G_{[\boldsymbol{\psi}]}$ in (3) is as follows. Suppose that $\boldsymbol{\Gamma}_t = (\boldsymbol{\psi}, \mathbf{x})$. Then $\mathbf{Y}$ can jump only to the neighbouring states $\mathbf{x}^i$, $1 \leq i \leq m$. As these states differ from $\mathbf{x}$ in exactly one component, there are no simultaneous defaults. If firm i has survived up to time t (i.e. $x(i) = 0$), the probability of a jump in the small time interval $(t, t+h]$ to the neighbouring state $\mathbf{x}^i$, where firm i is default, is approximately equal to $h\lambda_i(\boldsymbol{\psi}, \mathbf{x})$. If firm i has defaulted in $[0, t]$ (i.e. $x(i) = 1$), the probability of a jump to $\mathbf{x}^i$ is equal to zero, so that default is an absorbing state.
2) For an explicit construction of a conditional Markov chain or equivalently of a family of kernels $K_{\mathbf{y}}(\omega_1, d\omega_2)$ satisfying Assumption 2.1 (ii) we refer to the literature; see for instance Chapter 11.3 of Bielecki & Rutkowski (2002) or Chapter 2 of Davis (1993). There are alternative ways to construct a model with interacting intensities. A construction via a change of measure using the Girsanov theorem for point processes is given in Kusuoka (1999) or Bielecki & Rutkowski (2002). Yu (2004) uses the general hazard rate construction from survival analysis as developed by Norros (1986) and Shaked & Shanthikumar (1987).

Markov property and default intensities. We now discuss some mathematical implications of Assumption 2.1 related to the Markov property. To begin with, note that for every bounded random variable $F(\boldsymbol{\Psi}, \mathbf{Y}) : \Omega \to \mathbb{R}$

$$E(F(\boldsymbol{\Psi}, \mathbf{Y}) \mid \mathcal{G}_t)(\omega_1, \omega_2) = E^{K(\omega_1, \cdot)}(F(\boldsymbol{\Psi}(\omega_1), \mathbf{Y}) \mid \mathcal{F}_t^2)(\omega_2) . \tag{4}$$

Relation (4) is easily shown for $F = F_1(\mathbf{\Psi})F_2(\mathbf{Y})$; the extension to general F is done via a monotone class argument.

Now we turn to the Markov property of $\mathbf{Y}$. Define for $t \in [0, \infty)$ and an arbitrary Polish space E the shift operator $\theta_t : \mathbf{D}([0, \infty), E) \to \mathbf{D}([0, \infty), E)$, $\theta_t \omega\,(s) := \omega(t + s)$. Let $F : \Omega \to \mathbb{R}$ be bounded and measurable. Using relation (4) and the fact that $\mathbf{Y}$ is a time-inhomogenous Markov chain under $K(\omega_1, d\omega_2)$ we get for $t \geq 0$

$$\begin{aligned} E\big(F(\mathbf{\Psi}, \mathbf{Y} \circ \theta_t) \mid \mathcal{G}_t\big)(\omega_1, \omega_2) &= E^{K(\omega_1,\cdot)}\left(F(\mathbf{\Psi}(\omega_1), \mathbf{Y} \circ \theta_t) \mid \mathcal{F}_t^2\right)(\omega_2) \\ &= E^{K_{\mathbf{Y}_t(\omega_2)}(\theta_t\omega_1,\cdot)}\left(F(\mathbf{\Psi}(\omega_1), \mathbf{Y})\right). \end{aligned} \tag{5}$$

In the sequel we refer to relation (5) as *conditional Markov property* of $\mathbf{Y}$. Since $\mathbf{\Psi}$ is an $\{\mathcal{F}_t\}$-Markov process the conditional Markov property implies that the process $\mathbf{\Gamma}$ is Markov wrt $\{\mathcal{F}_t\}$, as we now show. Define the random variable

$$H : \Omega_1 \times S \to \mathbb{R}, \quad H(\omega_1, \mathbf{x}) = E^{K_{\mathbf{x}}(\omega_1,\cdot)}\left(F(\mathbf{\Psi}(\omega_1), \mathbf{Y})\right).$$

Using the law of iterated expectations, the conditional Markov property of $\mathbf{Y}$, the definition of H and the $\{\mathcal{F}_t\}$-Markov property of $\mathbf{\Psi}$ we obtain

$$\begin{aligned} E\left(F(\mathbf{\Psi} \circ \theta_t, \mathbf{Y} \circ \theta_t) \mid \mathcal{F}_t\right)(\omega_1, \omega_2) &= E\left(E(F(\mathbf{\Psi} \circ \theta_t, \mathbf{Y} \circ \theta_t) \mid \mathcal{G}_t) \mid \mathcal{F}_t\right)(\omega_1, \omega_2) \\ &= E\left(H(\mathbf{\Psi} \circ \theta_t, \mathbf{Y}_t) \mid \mathcal{F}_t\right)(\omega_1, \omega_2) \\ &= \int_{\Omega_1} H(u, \mathbf{Y}_t(\omega_2))\, \mu_{\mathbf{\Psi}_t(\omega_1)}(du). \end{aligned}$$

By definition of H this equals $E_{\mathbf{\Gamma}_t(\omega)}\big(F(\mathbf{\Psi}, \mathbf{Y})\big)$, which yields the Markov property of $\mathbf{\Gamma}$.

It is intuitively clear that $\lambda_i(\mathbf{\Psi}_t, \mathbf{Y}_t)$ is the default intensity of company i. Using the conditional Markov property we can give a formal proof of this fact. According to (5), $\mathbf{Y}$ forms an time-inhomogeneous Markov chain wrt $\{\mathcal{G}_t\}$ under P. The process $M_t(i) := Y_t(i) - \int_0^{t\wedge\tau_i} \lambda_i(\mathbf{\Psi}_s, \mathbf{Y}_s)\, ds$ is therefore a $\{\mathcal{G}_t\}$-martingale by the Dynkin formula, and hence an $\{\mathcal{F}_t\}$-martingale, as $M_t(i)$ is $\{\mathcal{F}_t\}$-adapted.

Remark 2.2 (Computation of expectations). Suppose that we want to compute a conditional expectation of the form $E\big(h(\mathbf{\Psi}_T, \mathbf{Y}_T) \mid \mathcal{F}_t\big)$ for some $h : \overline{S} \to \mathbb{R}$. By the Markov property of $\mathbf{\Gamma}$ the conditional expectation is given by $H(t, \mathbf{\Psi}_t, \mathbf{Y}_t)$ for a suitable function $H : [0, T] \times \overline{S} \to \mathbb{R}$. Now we have various approaches for computing $H(t, \boldsymbol{\psi}, \mathbf{y})$. First we can try to solve directly the backward PDE for the Markov process $\mathbf{\Gamma}$ given by

$$\frac{\partial}{\partial t} H(t, \boldsymbol{\psi}, \mathbf{y}) + \mathcal{L}^{\Psi} H(t, \boldsymbol{\psi}, \mathbf{y}) + \mathcal{G}_{[\boldsymbol{\psi}]} H(t, \boldsymbol{\psi}, \mathbf{y}) = 0, \quad H(T, \boldsymbol{\psi}, \mathbf{y}) = h(\boldsymbol{\psi}, \mathbf{y}).$$

In case that $\mathbf{\Psi}$ follows a diffusion this leads to a linear reaction-diffusion equation; existence results suitable for financial applications are for instance given in Becherer & Schweizer (2004). Alternatively, we may use a two-step approach, which uses only the Kolmogorov equations for the conditional transition probability. Here we get

$$H(t, \boldsymbol{\psi}, \mathbf{y}) = \int_{\Omega_1} E^{K_{\mathbf{y}}(\omega_1,\cdot)}\big(h(\mathbf{\Psi}_{T-t}(\omega_1), \mathbf{Y}_{T-t})\big)\, \mu_{\boldsymbol{\psi}}(d\omega_1).$$

Now the inner expectation can often be computed using techniques for Markov chains such as the Kolmogorov forward and backward equations discussed below; the integral over Ω_1 can then be computed in a second step, typically via Monte Carlo simulation. This approach can be advantageous, if the direct numerical solution of the backward equation for $\mathbf{\Gamma}$ is infeasible, because the dimension of the problem is too high.

Conditional transition functions and the Kolmogorov equations. Next we introduce the conditional transition probabilities of the chain $\mathbf{Y}$ under $K(\omega_1, d\omega_2)$. Define for $0 \le t \le s < \infty$ and $\mathbf{x}, \mathbf{y} \in S$

$$p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1) := E^{K(\omega_1, d\omega_2)}(\mathbf{Y}_s = \mathbf{y} \mid \mathbf{Y}_t = \mathbf{x}). \tag{6}$$

It is well-known that for ω_1 fixed the function $p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)$ satisfies the Kolmogorov forward and backward equations. These equations will be very useful numerical tools in our analysis of the model. The backward equation is a system of ODE's for the function $(t, \mathbf{x}) \to p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)$, $0 \le t \le s$; s and $\mathbf{y}$ are considered as parameters. In its general form the equation is

$$\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial t} + G_{[\mathbf{\Psi}_t(\omega_1)]} p(t, s, \mathbf{x}, \mathbf{y}) = 0\,, \quad p(s, s, \mathbf{x}, \mathbf{y}) = 1_{\{\mathbf{y}\}}(\mathbf{x})\,. \tag{7}$$

In our model this leads to the following system of ODE's

$$\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial t} + \sum_{k=1}^{m} (1 - x(k))\, \lambda_k(\mathbf{\Psi}_t(\omega_1), \mathbf{x})(p(t, s, \mathbf{x}^k, \mathbf{y} \mid \omega_1) - p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)) = 0. \tag{8}$$

The forward-equation is an ODE-System for the function $(s, \mathbf{y}) \to p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)$, $s \ge t$. Denote by $G^*_{[\boldsymbol{\psi}]}$ the adjoint operator to $G_{[\boldsymbol{\psi}]}$, operating again on functions from S to $\mathbb{R}$. In its general form the forward equation reads

$$\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial s} = G^*_{[\mathbf{\Psi}_t(\omega_1)]} p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1), \quad p(t, t, \mathbf{x}, \mathbf{y} \mid \omega_1) = 1_{\{\mathbf{x}\}}(\mathbf{y}). \tag{9}$$

An explicit form is given in the following lemma.

Lemma 2.3. *Under Assumption 2.1 (ii) the forward equation for the conditional transition rates is*

$$\begin{aligned}\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial s} =& \sum_{k=1}^{m} y(k) \lambda_k(\mathbf{\Psi}_s(\omega_1), \mathbf{y}^k) p(t, s, \mathbf{x}, \mathbf{y}^k \mid \omega_1) \\ &- \sum_{k=1}^{m} (1 - y(k)) \lambda_k(\mathbf{\Psi}_s(\omega_1), \mathbf{y}) p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1).\end{aligned} \tag{10}$$

The proof is given in Appendix A.2.

For small m (8) and (10) are easily solved numerically. Note however, that the cardinality of the state space equals $|S| = 2^m$ so that for m large the Kolmogorov equations are no longer useful. In that case one could either reduce the dimension of the state space, for instance by considering a model with a homogeneous group structure as in Section 3 below, or one has to resort to simulation approaches. Fortunately, the model introduced in Assumption 2.1 is quite easy to simulate from using the standard simulation approach for continuous-time Markov chains; in particular, simulation is no more costly (in terms of computing time) than simulating a standard reduced form model with conditionally independent defaults. We give a detailed description of the simulation algorithm in Appendix A.1.

Some models for the default intensity. We begin with the default intensities considered in Jarrow & Yu (2001). These authors study a special form of interacting intensities, which they call *primary-secondary framework*. In this framework firms are divided into two classes, primary and secondary firms. The default intensity of primary firms depends only on the factor process $\mathbf{\Psi}$; default intensities of secondary firms depend on $\mathbf{\Psi}$ and on the default state of the primary firms. This simplifying assumption allows Jarrow and Yu to use Cox-process techniques for the analysis of their model. For concreteness we now present a specific example from their paper. We let $m = 2$ and $d = 1$ and identify the economic factors with the short rate of interest r_t, which follows an extended Vasicek-model. The default intensities are then given by

$$\lambda_1(r_t, \mathbf{Y}_t) = \lambda_{1,0} + \lambda_{1,1} r_t \quad \text{and} \quad \lambda_2(r_t, \mathbf{Y}_t) = \lambda_{2,0} + \lambda_{2,1} r_t + \lambda_{2,2} 1_{\{Y_t(1)=1\}};$$

hence company one is a primary firm and company two is a secondary firm.

The primary-secondary framework is typical for a model with local interaction, i.e. a model, where for all $i \in \{1, \ldots, m\}$ the default-intensity of firm i depends on the default state of some small set $N(i)$ of neighboring firms such as business partners or direct competitors. Alternatively, one can introduce some global or macroeconomic interaction in the sense that individual default intensities depend on the empirical distribution $\rho(\mathbf{Y}_t, \cdot) = \frac{1}{m}\sum_{i=1}^m \delta_{Y_t(i)}(\cdot) \in \mathcal{M}_1(S)$ of the default indicators at time t. In our simple model, where each firm can be in only two states, $\rho(\mathbf{Y}_t, \cdot)$ is obviously characterized by the proportion of defaulted firms in the portfolio at time t, and we will work with that description in the sequel.

3 Models with Mean-Field Interaction

3.1 A Mean-Field Model with Homogeneous Groups

The model. Assume that we can divide our portfolio of m firms into k groups (typically $k \ll m$), within which risks are exchangeable. These groups might for instance correspond to firms with identical credit rating or to firms from the same industries. Let $\kappa(i) \in \{1, \ldots, k\}$ give the group membership of firm i, $m_\kappa = \sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}}$ the number of firms in group κ, and denote for a given $\mathbf{y} \in S$ by $\rho_\kappa(\mathbf{y}, \cdot) = \frac{1}{m_\kappa}\sum_{i=1}^m 1_{\{\kappa(i)=\kappa\}}\delta_{y(i)}(\cdot)$ the empirical distribution of firms in group κ. Define for $\kappa \in \{1, \ldots, k\}$ the functions $\overline{M}_\kappa(\mathbf{y}) := \rho_\kappa(\mathbf{y}, \{1\})$, put $\overline{\mathbf{M}}(\mathbf{y}) = \big(\overline{M}_1(\mathbf{y}), \ldots, \overline{M}_k(\mathbf{y})\big)$, and define the process $\overline{\mathbf{M}}$ by $\overline{\mathbf{M}}_t = \overline{\mathbf{M}}(\mathbf{Y}_t)$; obviously, $\overline{M}_{t,\kappa}$ gives the proportion of firms in group κ which have defaulted by time t. The state space of $\overline{\mathbf{M}}_t$ is given by

$$S^{\overline{M}} := \left\{ \bar{\boldsymbol{l}} = \left(\tfrac{l_1}{m_1}, \ldots, \tfrac{l_k}{m_k}\right) : l_\kappa \in \{0, \ldots, m_\kappa\}, 1 \le \kappa \le k \right\}.$$

Assumption 3.1 (Mean-field model with homogeneous groups). The default intensities of firms in our portfolio belonging to the same group are identical and of the form $\lambda_i(\boldsymbol{\psi}, \mathbf{y}) = h_{\kappa(i)}\big(\boldsymbol{\psi}, \overline{\mathbf{M}}(\mathbf{y})\big)$ for continuous functions $h_\kappa : \mathbb{R}^d \times S^{\overline{M}} \to \mathbb{R}^+$, $1 \le \kappa \le k$.

Comments. 1) As discussed in the introduction, this type of interaction makes immediate sense in the context of portfolio credit risk.
2) Assumption 3.1 implies that for all κ the default indicator processes $\{Y_t(i) : 1 \le i \le m,\ \kappa(i) = \kappa\}$ of firms belonging to the same group are exchangeable, a fact which we will

exploit frequently below. Conversely, consider an arbitrary portfolio of m counterparties with default indicators satisfying Assumtion 2.1, and suppose that the portfolio can be split in $k < m$ homogeneous groups. Homogeneity implies that a) the default intensities are invariant under permutations π of $\{1, \dots, m\}$, which leave the group structure invariant, i.e. $\lambda_i(\boldsymbol{\psi}, \mathbf{y}) = \lambda_i(\boldsymbol{\psi}, \pi(\mathbf{y}))$ for all i and all permutations π with $\kappa(\pi(j)) = \kappa(j)$ for all $1 \leq j \leq m$, and b) that default intensities of different firms from the same group are identical. Condition a) immediately yields that $\lambda_i(\boldsymbol{\psi}, \mathbf{y}) = h_i(\boldsymbol{\psi}, \mathbf{y})$ for some $h_i : \mathbb{R}^d \times S^{\overline{M}} \to \mathbb{R}^+$ and hence a model of mean-field type; together with condition b) this implies that the default intensities satisfy Assumption 3.1. Hence the mean-field model is the natural counterparty-risk model for portfolios consisting of homogeneous groups.

Example 3.2 (An affine model with counterparty risk). Often we will assume that the default intensities depend only on the overall proportion of defaulted companies given by $\sum_{\kappa=1}^{k} \frac{m_\kappa}{m} \overline{M}_{t,\kappa}$. A useful example is provided by the following (nearly) affine model with counterparty risk. Given for every group κ nonnegative constants $\lambda_{\kappa,j}$, $j = 0, \dots, d+1$ and an expected default intensity $\bar{\lambda}_\kappa$ we put

$$h_\kappa(t, \boldsymbol{\psi}, \bar{\boldsymbol{l}}) = \Big[\lambda_{\kappa,0} + \sum_{j=1}^{d} \lambda_{\kappa,j}\psi_j + \lambda_{\kappa,d+1}\Big(\sum_{j=1}^{k} \frac{m_j}{m}\bar{l}_j - \sum_{j=1}^{k} \frac{m_j}{m}\big(1 - e^{-\bar{\lambda}_j t}\big)\Big)\Big]^+ . \tag{11}$$

These default intensities have the following intepretation. The number $1 - e^{-\bar{\lambda}_j t}$ measures the expected proportion of defaulted firms in group j at time t. In case that $\lambda_{\kappa,d+1} > 0$ the default intensity of non-defaulted companies is increased (decreased), if the overall proportion of defaulted companies is higher (lower) than the overall expected proportion $\sum_{j=1}^{k} \frac{m_j}{m}\big(1 - e^{-\bar{\lambda}_j t}\big)$; in particular we have counterparty risk. If $\lambda_{\kappa,d+1} = 0$ for all κ we are in a standard Cox-process framework as studied for instance by Duffie & Singleton (1999). Following the latter paper we assume that the factor process follows a square-root diffusion model with independent components, i.e.

$$d\Psi_{t,j} = \bar{\kappa}_j(\theta_j - \Psi_{t,j})dt + \sigma_j\sqrt{\Psi_{t,j}}\, dW_{t,j} \tag{12}$$

for a standard Brownian motion $\mathbf{W}_t = (W_{t,1}, \dots, W_{t,d})$ and constants $\bar{\kappa}_j$, θ_j, $\sigma_j > 0$.

Example 3.3. This example is proposed by Yu (2004) as a model for similar firms in a concentrated industry. Yu works with default-intensities of the form

$$\lambda_t^i = a_0 + a_1 1_{\{T_1 \leq t\}} = a_0 + a_1 1_{\{\overline{M}_t > 0\}}, \quad i \in \{1, \dots, m\},\ a_0, a_1 > 0\,,$$

i.e. at the first default time T_1 of a firm in the portfolio the default intensities of the surviving firms jump from a_0 to $a_0 + a_1$. Yu suggests that for a portfolio of high-quality credits a reasonable order of magnitude for the model parameters is $a_0 \approx 1\%$ and $a_1 \approx 0.1\%$. Simulation studies reported in his paper indicate, that the model might be able to explain certain features of credit spreads in the market for European telecom bonds.

The next lemma shows that the process $\overline{\mathbf{M}}_t$ is itself conditionally Markov and gives the form of the generator.

Lemma 3.4. *Assume that the default intensities satisfy Assumption 3.1. Then under $K(\omega_1, d\omega_2)$ the process $\overline{\mathbf{M}}_t$ follows a time-inhomogeneous Markov chain with state space*

$S^{\overline{M}}$. The generator of this chain equals $G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_t(\omega_1)]}$, where the operator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$ is given by

$$G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]} f\ (\bar{\boldsymbol{l}}) = \sum_{\kappa=1}^{k} m_\kappa (1-\bar{l}_\kappa) h_\kappa\left(\boldsymbol{\psi}, \bar{\boldsymbol{l}}\right) \left(f\big(\bar{\boldsymbol{l}} + \tfrac{1}{m_\kappa}\mathbf{e}_\kappa\big) - f\big(\bar{\boldsymbol{l}}\big)\right). \tag{13}$$

Here $\bar{\boldsymbol{l}} = (\bar{l}_1, \ldots, \bar{l}_k) \in S^{\overline{M}}$ and $\mathbf{e}_\kappa \in \mathbb{R}^k$ denotes the κ-th unit vector.

Proof. Suppose that $\overline{\mathbf{M}}_t = \left(\frac{l_1}{m_1}, \ldots, \frac{l_k}{m_k}\right)$. Obviously, the component $\overline{M}_{t,\kappa}$ can increase only in steps of size $(m_\kappa)^{-1}$, so that the support of the jump-distribution equals $\{\overline{\mathbf{M}}_t + {m_\kappa}^{-1} e_\kappa : 1 \leq \kappa \leq k,\ \overline{M}_{t,\kappa} < 1\}$. Now $\overline{\mathbf{M}}_t$ jumps to $\overline{\mathbf{M}}_t + {m_\kappa}^{-1}\mathbf{e}_\kappa$ if and only if the next defaulting firm belongs to group κ. Hence the transition rate from $\overline{\mathbf{M}}_t$ to $\overline{\mathbf{M}}_t + {m_\kappa}^{-1}\mathbf{e}_\kappa$ equals

$$\begin{aligned} \sum_{i=1}^{m} 1_{\{\kappa(i)=\kappa\}}\left(1 - Y_t(i)\right)\, \lambda_i(\mathbf{\Psi}_t, \mathbf{Y}_t) &= h_\kappa(\mathbf{\Psi}_t, \overline{\mathbf{M}}_t) \sum_{i=1}^{m} 1_{\{\kappa(i)=\kappa\}}\left(1 - Y_t(i)\right) \\ &= h_\kappa(\mathbf{\Psi}_t, \overline{\mathbf{M}}_t)\, m_\kappa \left(1 - \overline{M}_{t,\kappa}\right). \end{aligned}$$

The claim follows, as this transition-rate depends on $\mathbf{Y}_t$ only via $\overline{\mathbf{M}}_t$, which shows that $\overline{\mathbf{M}}$ is Markov with respect to $\{\mathcal{G}_t\}$. The form of $G^{\overline{\mathbf{M}}}_{[\mathbf{\Psi}_t(\omega_1)]}$ is obvious. □

In our analysis of the mean-field model introduced in Assumption 3.1 we will frequently use the Kolmogorov equations for the conditional Markov chain $\overline{\mathbf{M}}$. The form of the backward equation follows immediately from the definition of the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$; the ODE-system for the forward equation can be computed analogously to Lemma 2.3; see Lemma A.1 in the appendix for the precise form of the equation. Note that the size of the state space of $\overline{\mathbf{M}}$ equals $\left|S^{\overline{M}}\right| := (m_1+1)\cdots(m_k+1)$. For k fixed $\left|S^{\overline{M}}\right|$ grows at most at rate $(m/k)^k$ in m, whereas $|S|$ grows exponentially in m. Hence for k small the conditional distribution of $\overline{\mathbf{M}}_T$ can be inferred using the Kolmogorov equations for $\overline{\mathbf{M}}$ even for m relatively large.

Implications of exchangeability. We can infer individual default probabilities as well as within-group and between-group default correlations from the distribution of the random vector $\overline{\mathbf{M}}_T$ using the fact that within a given group κ the random variables $\{Y_T(i) : \kappa(i) = \kappa\}$ are exchangeable under $K(\omega_1, d\omega_2)$ and therefore also under P. Hence we get that $P\left(Y_T(i) = 1 \mid \overline{M}_{T,\kappa(i)}\right) = \overline{M}_{T,\kappa(i)}$, and for two firms i, j belonging to the same group κ

$$P\left(Y_T(i) = 1, Y_T(j) = 1 \mid \overline{M}_{T,\kappa} = \frac{M}{m_\kappa}\right) = \frac{\binom{m_\kappa-2}{M-2}}{\binom{m_\kappa}{M}} = \frac{M(M-1)}{m_\kappa(m_\kappa-1)}, \tag{14}$$

provided that m_κ and $M \geq 2$; otherwise the left hand side of (14) is obviously equal to zero. Finally, we have for obligors i, j belonging to different groups κ_1 and κ_2

$$P\left(Y_T(i) = 1, Y_T(j) = 1 \mid \overline{M}_{T,\kappa_1}, \overline{M}_{T,\kappa_2}\right) = \overline{M}_{T,\kappa_1}\, \overline{M}_{T,\kappa_2}\,.$$

Hence we get for oligors i, j in group κ

$$P\left(Y_T(i) = 1\right) = E\left(P(Y_T(i) = 1 \mid \overline{M}_{T,\kappa})\right) = E(\overline{M}_{T,\kappa}), \tag{15}$$

$$P\left(Y_T(i) = 1, Y_T(j) = 1\right) = E\left(\overline{M}_{T,\kappa}\, \frac{m_\kappa \overline{M}_{T,\kappa} - 1}{m_\kappa - 1}\right), \tag{16}$$

and finally for obligors i, j from different groups κ_1 and κ_2

$$P\left(Y_T(i)=1, Y_T(j)=1\right)=E\left(\overline{M}_{T,\kappa_1}\,\overline{M}_{T,\kappa_2}\right). \tag{17}$$

Of course, expressions similar to (16) can also be obtained for higher order default probabilities. More generally, we can even express the probability $P(\mathbf{Y}_T=\overline{\mathbf{y}})$ for some $\overline{\mathbf{y}} \in S$ in terms of the distribution of $\overline{\mathbf{M}}_T$. As the distribution of $\mathbf{Y}_T$ is invariant under permutations of $\{1,\dots,m\}$, which respect the homogeneous group structure, we have with $\bar{\boldsymbol{l}} := \overline{\mathbf{M}}(\overline{\mathbf{y}})$

$$P(\overline{\mathbf{M}}_T=\bar{\boldsymbol{l}})=\left|\{\mathbf{y}\in S:\overline{\mathbf{M}}(\mathbf{y})=\bar{\boldsymbol{l}}\}\right|\,P(\mathbf{Y}_T=\overline{\mathbf{y}})=\prod_{\kappa=1}^{k}\binom{m_\kappa}{m_\kappa\bar{l}_\kappa}P(\mathbf{Y}_T=\overline{\mathbf{y}}). \tag{18}$$

Of course, since the relations above depend only on the exchangeability of the default indicator processes of firms belonging to the same group, they hold also under the kernel $K(\omega_1, d\omega_2)$.

3.2 Limits for Large Portfolios

We now consider the limit (in the sense of convergence in distribution) of the model with k homogeneous groups as the size m of the portfolio tends to infinity, assuming that k remains fixed. It will turn out that in the limit the evolution of $\overline{\mathbf{M}}$ becomes deterministic given the evolution of the economic factor process $\boldsymbol{\Psi}$.

Our setup is as follows. Denote by $\Omega^{(m)}=\mathbf{D}\left([0,\infty),\mathbb{R}^d\right)\times\mathbf{D}([0,\infty),S^{(m)})$ the probability space in model m and define the filtrations $\{\mathcal{F}_t^m\}$, $\{\mathcal{F}_t^{i,m}\}$, $i=1,2$, and $\{\mathcal{G}_t^m\}$ in the obvious way. We assume that for each m the probability measure $P^{(m)}=\mu\times K^{(m)}$ satisfies Assumption 2.1; moreover, μ is assumed to be identical for all m. Denote by $m_\kappa^{(m)}$ the number of obligors in group κ of model m, define the process $\overline{\mathbf{M}}^{(m)}$ by $\overline{\mathbf{M}}_t^{(m)}=\overline{\mathbf{M}}(\mathbf{Y}_t^{(m)})$, and assume that for all m the transition rates have the group structure as in Assumption 3.1; in particular the default intensity of company i in model m equals

$$\lambda_i^{(m)}\left(\boldsymbol{\psi},\mathbf{y}^{(m)}\right)=h_{\kappa(i)}^{(m)}\left(\boldsymbol{\psi},\overline{\mathbf{M}}(\mathbf{y}^{(m)})\right).$$

According to Lemma 3.4, for given ω_1 the process $\overline{\mathbf{M}}_t^{(m)}$ is Markov under the measure $K^{(m)}(\omega_1,d\omega_2)$. Put $\widetilde{\Omega}_2:=\mathbf{D}([0,\infty),[0,1]^k)$, and denote by $\widetilde{K}^{(m)}(\omega_1,d\widetilde{\omega}_2)$ the distribution of $\overline{\mathbf{M}}_t^{(m)}$ on $\widetilde{\Omega}_2$ under $K^{(m)}(\omega_1,d\omega_2)$.

Next we describe the limiting distribution of $\overline{\mathbf{M}}^{(m)}$. Suppose that for all $\kappa=1,\dots,k$ the function $h_\kappa^{(m)}$ converges uniformly on compacts to some locally Lipschitz function $h_\kappa^{(\infty)}:\mathbb{R}^d\times[0,1]^k\to\mathbb{R}^+$. Denote by $\overline{\mathbf{M}}_t^{(\infty)}(\omega_1)=\left(\overline{M}_{t,1}^{(\infty)}(\omega_1),\dots,\overline{M}_{t,k}^{(\infty)}(\omega_1)\right)'$ the solution of the following system of ODE's with random coefficients

$$\frac{d}{dt}\,\overline{M}_{t,\kappa}^{(\infty)}(\omega_1)=\left(1-\overline{M}_{t,\kappa}^{(\infty)}(\omega_1)\right)h_\kappa^{(\infty)}\left(\boldsymbol{\Psi}_t(\omega_1),\overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\right), \tag{19}$$

with initial value $\overline{\mathbf{M}}_0^{(\infty)}=\bar{\boldsymbol{l}}\in[0,1]^k$. Note that for fixed $\omega_1\in\Omega_1$ and $T>0$ the rhs of (19) is Lipschitz in the second argument, since $[0,1]^k$ is compact and $h_\kappa^{(\infty)}$ is locally Lipschitz; hence a solution of (19) exists. For every ω_1 the trajectory $\left[t\mapsto\overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\right]$ is an element of $\widetilde{\Omega}_2$. Denote by $\delta\left(\overline{\mathbf{M}}^{(\infty)}(\omega_1),\,d\widetilde{\omega}_2\right)$ the Dirac measure on $\widetilde{\Omega}_2$ in the point

$\big[t \mapsto \overline{\mathbf{M}}_t^{(\infty)}(\omega_1)\big]$, and define a transition kernel $\widetilde{K}^{(\infty)}$ from Ω_1 to $\widetilde{\Omega_2}$ by $\widetilde{K}^{(\infty)}(\omega_1, d\widetilde{\omega}_2) := \delta\big(\overline{\mathbf{M}}^{(\infty)}(\omega_1),\, d\widetilde{\omega}_2\big)$. Now we have

Proposition 3.5. *Given a sequence of models as above, suppose that* $\lim_{m\to\infty} m_\kappa^{(m)} = \infty$ *for all* $\kappa = 1, \ldots, k$ *and that* $\lim_{m\to\infty} \overline{\mathbf{M}}_0^{(m)} = \bar{\boldsymbol{l}}$. *Then for all* ω_1 *the measure* $\widetilde{K}^{(m)}(\omega_1, d\widetilde{\omega}_2)$ *converges weakly to* $\widetilde{K}_{\bar{\boldsymbol{l}}}^{(\infty)}(\omega_1, d\widetilde{\omega}_2)$.

Proof. Denote by $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}}$ the generator of $\overline{\mathbf{M}}^{(m)}$, and define for $f \in \mathcal{C}^1\left([0,1]^k\right)$ an operator

$$G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}} f(\bar{\boldsymbol{l}}) = \sum_{\kappa=1}^{k} \left(1 - \bar{l}_\kappa\right) h_\kappa^{(\infty)}(\boldsymbol{\psi}, \bar{\boldsymbol{l}}) \frac{\partial}{\partial \bar{l}_\kappa} f(\bar{\boldsymbol{l}}). \tag{20}$$

Note that $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}}$ is the generator of the process $\overline{\mathbf{M}}^{(\infty)}$ defined in (19). It follows from the Lipschitz continuity of $h_\kappa^{(\infty)}$ and the form of $G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}}$ (see (13)), that for all $f \in \mathcal{C}^1\left([0,1]^k\right)$ and every compact set $K \subset \mathbb{R}^d$

$$\lim_{m\to\infty} \sup\left\{ \left| G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(m)}} f(\bar{\boldsymbol{l}}) \;-\; G_{[\boldsymbol{\psi}]}^{\overline{\mathbf{M}}^{(\infty)}} f(\bar{\boldsymbol{l}}) \right| \,:\, \boldsymbol{\psi} \in K, \bar{\boldsymbol{l}} \in [0,1]^k \right\} = 0\,.$$

This implies that μ almost all ω_1 the transition semigroup of $\overline{\mathbf{M}}^{(m)}$ converges to the semigroup of $\overline{\mathbf{M}}^{(\infty)}$ by Ethier & Kurtz (1986), Theorem 1.6.1, so that the claim follows from Ethier & Kurtz (1986), Theorem 4.2.5. □

Note that the solution of (19) is deterministic given the trajectory $(\boldsymbol{\Psi}_t(\omega_1))_{t\geq 0}$. This shows that for $m \to \infty$ the proportion of defaulted companies is fully determined by the evolution of the economic factors. A similar result has been obtained among others by Frey & McNeil (2003) in the much simpler context of static Bernoulli mixture models for portfolio credit risk. Next we show that the pair of processes $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)})$ converges in distribution to $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(\infty)})$.

Corollary 3.6. *Suppose that the hypothesises of Proposition 3.5 hold. Then the sequence* $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)})$ *converges in distribution to* $(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(\infty)})$, *i.e. we have for every bounded and continuous function* $F : \mathbf{D}\left([0,\infty), \mathbb{R}^d\right) \times \mathbf{D}\left([0,\infty), [0,1]^k\right) \to \mathbb{R}$

$$\lim_{m\to\infty} E^{(m)}\left(F\big(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)}\big)\right) = \int_{\Omega_1} F\big(\boldsymbol{\Psi}(\omega_1), \overline{\mathbf{M}}^{(\infty)}(\omega_1)\big)\mu(d\omega_1)\,.$$

Proof. Denote by $\widetilde{\mathbf{Y}}$ the coordinate process on $\widetilde{\Omega}_2$. We have

$$E^{(m)}\left(F\big(\boldsymbol{\Psi}, \overline{\mathbf{M}}^{(m)}\big)\right) = \int_{\Omega_1}\int_{\widetilde{\Omega}_2} F\big(\boldsymbol{\Psi}(\omega_1), \widetilde{\mathbf{Y}}(\widetilde{\omega}_2)\big)\widetilde{K}^{(m)}(\omega_1, d\widetilde{\omega}_2)\mu(d\omega_1).$$

Now the inner integral on the rhs converges for μ almost all ω_1 to

$$\int_{\widetilde{\Omega}_2} F\left(\boldsymbol{\Psi}(\omega_1), \widetilde{\mathbf{Y}}(\widetilde{\omega}_2)\right) \widetilde{K}^{(\infty)}(\omega_1, d\widetilde{\omega}_2) = F\big(\boldsymbol{\Psi}(\omega_1), \overline{\mathbf{M}}^{(\infty)}(\omega_1)\big)$$

by Proposition 3.5. Hence the claim follows from the dominated convergence theorem. □

Example 3.7. Consider the affine model with counterparty risk introduced in Example 3.2. In order to apply Proposition 3.5, we assume that for all κ the proportion $m_\kappa^{(m)}/m$ of firms in group κ converges to some $\gamma_\kappa \in [0,1]$ as $m \to \infty$. This yields

$$h_\kappa^{(\infty)}(\boldsymbol{\psi}, \bar{\boldsymbol{l}}) = \Big[\lambda_{\kappa,0} + \sum_{j=1}^{d} \lambda_{\kappa,j}\psi_j + \lambda_{\kappa,d+1} \sum_{r=1}^{k} \gamma_r \big(\bar{l}_r - (1 - e^{-\bar{\lambda}_r t})\big)\Big]^+ ,$$

and $\overline{\mathbf{M}}^{(\infty)}$ solves the ODE-system

$$\frac{d}{dt}\,\overline{M}_{t,\kappa}^{(\infty)} = \big(1 - \overline{M}_{t,\kappa}^{(\infty)}\big) \Big[\lambda_{\kappa,0} + \sum_{j=1}^{d} \lambda_{\kappa,j}\Psi_{t,j} + \lambda_{\kappa,d+1} \sum_{r=1}^{k} \gamma_r \big(\overline{M}_{t,r}^{(\infty)} - (1 - e^{-\bar{\lambda}_r t})\big)\Big]^+ , \quad (21)$$

$1 \leq \kappa \leq k$. Note that counterparty risk (a positive $\lambda_{\kappa,d+1}$) implies that deviations of $\sum_{r=1}^{k} \gamma_r \overline{M}_{t,r}^{(\infty)}$ from the expected level $\sum_{r=1}^{k} \gamma_r (1 - e^{-\bar{\lambda}_r t})$ will have a positive feedback effect on default intensities. Hence the fluctuations in the number of defaults caused by the random evolution of the economic factors are intensified by counterparty risk, so that we should expect heavier tails of the distribution of $\overline{M}_{t,\kappa}^{(\infty)}$. This is illustrated further in simulations in the next section.

3.3 Default Correlation and Quantiles

Here we present a number of simulations, which illustrate the impact of counterparty risk on default correlations and quantiles of $\overline{\mathbf{M}}$ in the affine mean-field model with counterparty risk specified in Example 3.2. In all simulations we consider a homogeneous portfolio with only one group. The economic factor process is modelled as one-dimensional square-root diffusion with parameters $\kappa = 0.03$, $\theta = 0.005$, $\sigma = 0.016$ and initial value $\psi_0 = \theta$; these values have been taken from the empirical study by Driessen (2002). The default intensity equals

$$h(t, \psi, \bar{l}) = \big[\alpha(0.004 + 5.707\psi) + \lambda_2(\bar{l} - (1 - e^{-\bar{\lambda} t}))\big]^+ \text{ with } \bar{\lambda} = 0.03251.$$

The value for $\bar{\lambda}$ has been chosen so that $1 - e^{-\bar{\lambda}}$ corresponds to the one-year default probability without interaction, i.e. for $\lambda_2 = 0$. We take the horizon to be $T = 1$ year. In our simulations we increase the parameter λ_2, which controls the strength of the interaction, from 0 to 3 and adjust α in order to ensure that the one-year default probabilities $P(Y_1(i) = 1)$ remain unchanged as we vary λ_2. We consider portfolios of size $m = 100$, $m = 500$ and, using the results from Section 3.2, the case $m = \infty$. The distribution of $\overline{M}_1$ is evaluated in two steps: first we simulate 5000 trajectories of the economic factor process ψ; second we evaluate for each trajectory the conditional distribution of $\overline{M}_1$ by solving numerically the Kolmogorov forward equation using a Runge-Kutta method. The simulation results are presented in Table 1 below. Inspection of the table yields the following observations.

- Quantiles and (except for $m = \infty$) default correlations $\rho_Y = \operatorname{corr}(Y_1(i), Y_1(j))$, $i \neq j$ are increasing in λ_2.
- The increase is more pronounced for smaller portfolios. For instance, for $m = 100$ the 99% quantile of $\overline{M}_1$ is increased by a factor of almost 4.75 as λ_2 increases from 0 to 3; for $m = \infty$ the factor is only about 1.64.

Both findings make perfect economic sense. In our counterparty risk model a higher (lower) than usual number of defaults in the portfolio leads to an increase (decrease) of the default intensity of the remaining firms in the portfolio and thus to a further increase (decrease) in the ratio of realized versus expected defaults, so that the resulting distribution of $\overline{M}_T$ will have more mass in the tails. Now in our model there are two reasons why the number of defaults should be higher than its theoretical value in the first place: a) we might have a high realization of Ψ; b) for a given trajectory of Ψ we might have a realization of the Markov chain with unusually many defaults. As the limit results from Section 3.2 show, for $m \to \infty$ reason b) becomes less and less important, which explains, why the effect of mean-field interaction is more pronounced for small portfolios. Note finally that for $m = \infty$ default correlations seem to vary only very little as λ_2 increases whereas quantiles change a lot, so that default probabilities and default correlations alone do not determine high quantiles of the distribution of $\overline{M}_T$. This is interesting, as it contrasts results of Frey & McNeil (2003) in the context of standard static credit risk models.

100 firms						
λ_2	$P(Y_1(i) = 1)$	ρ_Y	Quantile			
			90%	95%	97.5%	99%
0	0.031987	0.000416	0.06	0.06	0.07	0.08
1	0.031989	0.020918	0.07	0.09	0.11	0.13
3	0.031997	0.19118	0.12	0.21	0.29	0.38

500 firms						
λ_2	$P(Y_1(i) = 1)$	ρ_Y	Quantile			
			90%	95%	97.5%	99%
0	0.03199	0.00041579	0.044	0.046	0.05	0.054
1	0.03198	0.0050753	0.052	0.058	0.066	0.072
3	0.03199	0.058283	0.096	0.128	0.156	0.19

The case $m = \infty$						
λ_2	$P(Y_1(i) = 1)$	ρ_Y	Quantile			
			90%	95%	97.5%	99%
0	0.03199	0.00042	0.0367	0.0380	0.0393	0.0408
1	0.03199	0.00041	0.0390	0.0409	0.0429	0.0452
3	0.031982	0.00043	0.0503	0.0554	0.0611	0.0669

Table 1: Default correlation and quantiles in the mean-field model for varying m.

4 Pricing of Credit Derivatives

In this chapter we discuss the pricing of credit risky securities in our model with interacting intensities. Our main interest is in portfolio-related credit derivatives such as k-th to default swaps and synthetic CDOs, whose payoff distribution is particularly sensitive with respect to dependence between defaults.

4.1 Generalities

Our setup. We use the martingale modelling approach and specify asset price dynamics and default intensities directly under a risk neutral pricing measure, which we denote again by P. Since credit derivatives are usually priced relative to traded credit products such as corporate bonds or single-name Credit Default Swaps, martingale modelling is standard practice in the literature. We assume that under P the default indicators satisfy Assumption 2.1 with only time-dependent default intensities $\lambda_i(t, \mathbf{y})$. Models with factor-independent default intensities are practically relevant as dependence between defaults can be introduced via the interaction between default intensities. In fact, the literature on pricing credit derivatives in the popular copula models focuses almost exclusively on models with factor-independent default intensities. Moreover, our results are easily extended to default intensities which depend on some stochastic background process using the two-step approach sketched in Remark 2.2.

We assume that the default-free interest rate is deterministic and given by $r(t) \geq 0$; $B(t) = \exp(\int_0^t r(s)ds$ denotes the default-free savings account. Assuming deterministic interest rates is natural when working with deterministic marginal hazard rates; moreover, given the great sensitivity of most portfolio-related credit derivatives with respect to fluctuations in risk neutral default correlations and the huge degree of uncertainty surrounding every approach to calibrating these numbers, the additional complexity of stochastic interest rates is simply not warranted.

The assumption of deterministic default intensities and interest rates allows us to simplify the notation. The distribution of the time-inhomogeneous Markov chain $\mathbf{Y}_t$ starting at time t in state $\mathbf{y}$ will be denoted by $P_{(t,\mathbf{y})}$, and the underlying filtration is simply given by $\mathcal{F}_t = \sigma(\mathbf{Y}_s : s \leq t)$. Since in this section portfolio size and composition are considered fixed we work directly with the absolute number of defaults within the portfolio or within a particular group. In particular, if the model has the homogeneous group structure of Assumption 3.1, the default intensity is denoted by $h_\kappa(t, \boldsymbol{l})$, where l_κ gives the absolute number of defaults in group κ. In that case the distribution of the Markov chain $\mathbf{M}_t$ starting at time t in state $\boldsymbol{l}$ is denoted by $P_{(t,\boldsymbol{l})}$. The function $M(\mathbf{y}) := \sum_{i=1}^m y(i)$ gives the number of defaults for a given $\mathbf{y} \in S$. The following sets of states from S

$$A_0(l, j) := \{\mathbf{y} : M(\mathbf{y}) = l, y(j) = 0\} \text{ and } A_1(l, j) := \{\mathbf{y} : M(\mathbf{y}) = l, y(j) = 1\} \tag{22}$$

will appear frequently below. Finally we recall the notation $\mathbf{y}^i$ for flipping a portfolio state introduced in (2).

Conditional expectations. In the sequel we derive analytical expressions for certain conditional expectations with respect to the σ-field generated by a particular default time τ_{i_0}, which will come in handy in the pricing of basket credit derivatives and CDOs.

Proposition 4.1. *For $i_0 \in \{1, \ldots, m\}$ the density of τ_{i_0} equals*

$$P(\tau_{i_0} \in dt) = \sum\nolimits_{\mathbf{y}: y(i_0)=0} \lambda_{i_0}(t, \mathbf{y}) P(\mathbf{Y}_t = \mathbf{y}) \,. \tag{23}$$

Moreover, we have for $\mathbf{y} \in S$

$$P\big(\mathbf{Y}_t = \mathbf{y} \big| \tau_{i_0} = t\big) = y(i_0) P(\tau_{i_0} \in dt)^{-1} \lambda_{i_0}(t, \mathbf{y}^{i_0}) P(\mathbf{Y}_t = \mathbf{y}^{i_0}) \,. \tag{24}$$

Proof. We first show that for $\mathbf{y} \in S$ with $y(i_0) = 1$ we have

$$\lim_{\epsilon \to 0+} \frac{1}{\epsilon} \, P\big(\mathbf{Y}_t = \mathbf{y}, \; \tau_{i_0} \in (t-\epsilon, t]\big) = \lambda_{i_0}(t, \mathbf{y}^{i_0}) P\big(\mathbf{Y}_t = \mathbf{y}^{i_0}\big). \tag{25}$$

To verify (25) we argue as follows. The probability to have more than one default in $(t-\epsilon, t]$ is of order $o(\epsilon)$. Thus we have $P\big(\mathbf{Y}_t = \mathbf{y}, \; \tau_{i_0} \in (t-\epsilon, t]\big) = P\big(\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}, \; \tau_{i_0} \in (t-\epsilon, t]\big) + o(\epsilon)$. Now we get

$$\begin{aligned} P\big(\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}, \; \tau_{i_0} \in (t-\epsilon, t]\big) \; &= \; E\Big(E\big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}\}} \mathbf{1}_{\{\tau_{i_0} \in (t-\epsilon, t]\}} \big| \mathcal{F}_{t-\epsilon}\big)\Big) \\ &= \; E\Big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}\}} \mathbf{1}_{\{\tau_{i_0} > t-\epsilon\}} P\big(\tau_{i_0} \circ \theta_{t-\epsilon}(\omega) \le \epsilon \Big| \mathcal{F}_{t-\epsilon}\big)\Big). \end{aligned} \tag{26}$$

By the Markov property of $\mathbf{Y}$ we have

$$P\big(\tau_{i_0} \circ \theta_{t-\epsilon}(\omega) \le \epsilon \big| \mathcal{F}_{t-\epsilon}\big) = P_{(t-\epsilon, \mathbf{Y}_{t-\epsilon})}(\tau_{i_0} \le \epsilon).$$

Moreover, $P_{(t-\epsilon, \mathbf{y}^{i_0})}(\tau_{i_0} \le \epsilon) = \epsilon \lambda_{i_0}(t-\epsilon, \mathbf{y}^{i_0}) + o(\epsilon)$. Hence and as $\tau_{i_0} > t - \epsilon$ on $\{\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}\}$, (26) equals $E\Big(\mathbf{1}_{\{\mathbf{Y}_{t-\epsilon} = \mathbf{y}^{i_0}\}} \epsilon \lambda_{i_0}(t-\epsilon, \mathbf{y}^{i_0})\Big) + o(\epsilon)$, and (25) follows.

The proof of the proposition is now straightforward. Relation (23) follows from (25) and the fact that $P\big(\tau_{i_0} \in (t-\epsilon, t]\big) = \sum_{\mathbf{y}: y(i_0)=1} P\big(\mathbf{Y}_t = \mathbf{y}, \; \tau_{i_0} \in (t-\epsilon, t]\big)$; relation (24) follows from (25), the definition of the elementary conditional expectation and a standard limit argument. □

Remark 4.2. In principle, it is possible to write down the density for $(\tau_1, \dots, \tau_m)$ in closed form and to determine the marginal density τ_{i_0} by integrating out the other default times. However, as shown in Yu (2004), the resulting expressions become quite cumbersome already for $m = 3$, so that even for medium-sized portfolios this approach is infeasible.

For a homogeneous model the results of Proposition 4.1 simplify further.

Corollary 4.3. *Consider a homogeneous mean-field model with only one group and default intensity $h(t, l)$. We have for $l, i_0 \in \{1, \dots, m\}$*

$$\begin{aligned} P(\tau_{i_0} \in dt) \; &= \; m^{-1} \sum_{k=0}^{m-1} h(t,k) P(M_t = k)(m-k) \;\; \textit{and} \end{aligned} \tag{27}$$

$$P\big(M_t = l \,\big|\, \tau_{i_0} = t\big) \; = \; \frac{(m-l+1) h(t, l-1) P(M_t = l-1)}{\sum_{k=0}^{m-1} (m-k) h(t,k) P(M_t = k)}. \tag{28}$$

Proof. We have $P\big(M_t = l, \; \tau_{i_0} \in (t-\epsilon, t]\big) = \sum_{\mathbf{y} \in A_1(l, i_0)} P\big(\mathbf{Y}_t = \mathbf{y}, \; \tau_{i_0} \in (t-\epsilon, t]\big)$. Using (25) we get

$$\begin{aligned} \lim_{\epsilon \to 0+} \frac{1}{\epsilon} P\big(M_t = l, \; \tau_{i_0} \in (t-\epsilon, t]\big) \; &= \; \sum_{\mathbf{y} \in A_1(l, i_0)} h(t, l-1) P\big(\mathbf{Y}_t = \mathbf{y}^{i_0}\big) \\ &= \; \sum_{\mathbf{y} \in A_0(l-1, i_0)} h(t, l-1) P\big(\mathbf{Y}_t = \mathbf{y}\big). \end{aligned} \tag{29}$$

Now note that $\big|A_0(l-1, i_0)\big| = \binom{m-1}{l-1}$ and $\big|\{\mathbf{y} \in S : M(\mathbf{y}) = l-1\}\big| = \binom{m}{l-1}$. Since $\binom{m-1}{l-1} / \binom{m}{l-1} = \frac{m-(l-1)}{m}$, expression (29) equals

$$\frac{m-(l-1)}{m} \sum_{\{\mathbf{y}: M(\mathbf{y}) = l-1\}} h(t, l-1) P\big(\mathbf{Y}_t = \mathbf{y}\big) = \frac{m-l+1}{m} \, h(t, l-1) P\big(M_t = l-1\big).$$

Now (27) follows as $P\big(\tau_{i_0} \in dt\big) = \sum_{l=1}^m \lim_{\epsilon\to 0+} \frac{1}{\epsilon} P\big(\tau_{i_0} \in (t-\epsilon, t],\ M_t = l\big)$; relation (28) follows as in the proof of Proposition 4.1 from the definition of elementary conditional expectation and a standard limit argument. □

Some simple building blocks. As a simple first example we consider the pricing of a *terminal value claim* with payoff $H = g(\mathbf{Y}_T)$ for some function $g : S \to \mathbb{R}$. A prime example is a defaultable zero-coupon bond issued by firm i with zero recovery or more generally with recovery of treasury in the sense of Jarrow & Turnbull (1995) and deterministic recovery rate δ, where $g(\mathbf{y}) = (1-\delta)1_{\{y(i)=0\}} + \delta$. Using the Markov-property of $\mathbf{Y}$ we get for the price of a terminal value claim in $t < T$

$$H_t = \exp\Big(-\int_0^{T-t} r(s)ds\Big) E_{(t,\mathbf{Y}_t)}\left(g(\mathbf{Y}_T)\right)\,, \quad t \le T\,,$$

which is easily computed using the Kolmogorov backward equation (8). In the context of the mean-field model of Assumption 3.1 further simplifications are possible. For example, if the payment is contingent on the survival of a particular firm i_0 from group κ_0, i.e. if $g(\mathbf{y}) = 1_{\{y(i_0)=0\}}$, we obtain by an analogous argument as in (15)

$$P_{(t,\mathbf{Y}_t)}(Y_T(i_0) = 0) = 1_{\{Y_t(i_0)=0\}} \left(1 - E_{(t,\mathbf{M}_t)} \left(\frac{M_{T,\kappa_0} - M_{t,\kappa_0}}{m_{\kappa_0} - M_{t,\kappa_0}} \right) \right),$$

and the expectation on the right hand side can be computed using the backward equation for $\mathbf{M}_t$, which leads to a substantial reduction in the size of the ODE-system to be solved. Of course, there are alternative ways to compute prices of defaultable zero-coupon bonds in models with interacting intensities. In particular, as shown by Collin-Dufresne et al. (2003), for m small analytical expressions for prices of zero-coupon bonds can be derived using a change of measure.

Next we turn to the pricing of *recovery payments*. A recovery payment with deterministic payoff δ and maturity T on firm i_0 is a claim which pays δ at the default time τ_{i_0} if $\tau_{i_0} \le T$; otherwise there is no payment. The price of this claim at $t = 0$ equals

$$\delta E\Big(B(\tau_{i_0})^{-1}\Big) = \delta \int_0^T B(t)^{-1} P(\tau_{i_0} \ \in dt)\, dt\,, \tag{30}$$

which can be evaluated numerically using (23) or (27). Using our pricing formulas for terminal value claims and recovery payments it is straightforward to compute the price of a standard single-name CDS, at least if we neglect the possibility that the protection seller may default. This is important for calibrating the model to given CDS spreads. The more general case with default of the protection seller can be dealt with using similar arguments as in the pricing of k-th-to-default swaps in the next subsection.

4.2 Pricing of k-th-to-default swaps

Payoff description. We consider a portfolio of m names with nominal N_i and deterministic recovery rate δ_i, $1 \le i \le m$. If the k-th default time T_k is smaller than the maturity T of the swap, the protection buyer in a k-th-to-default swap on this portfolio receives at time T_k the loss of the portfolio incurred at the k-th default given by $(1 - \delta_{\xi^k})N_{\xi^k}$ (the default payment leg of the swap). Note that the size of the default premium is random as it depends on the identity ξ^k of the k-th defaulting firm.

As a compensation the protection buyer pays to the protection seller a fixed premium $X^{k\text{th}}$ at fixed dates $t_1, t_2, \dots, t_N = T$ until T_k; after T_k the regular premium payments stop. Moreover, at T_k the protection seller gets an *accrued premium payment* of size $\sum_{n=1}^N 1_{\{t_{n-1}<T_k\le t_n\}} X^{k\text{th}} \frac{T_k - t_{n-1}}{t_n - t_{n-1}}$ (the premium payment leg).

The default payment leg. Under the above assumptions the value of the default payment leg at $t=0$ can be written as

$$V^{\text{def}} := \sum_{j=1}^m (LGD)_j \, E\Big(B^{-1}(\tau_j)\mathbf{1}_{\{\tau_j\le T\}}\mathbf{1}_{\{M_{\tau_j}=k\}}\Big),$$

where $(LGD)_j := (1-\delta_j)N_j$. Now we obtain by iterated conditional expectations

$$\begin{aligned} E\Big(B^{-1}(\tau_j)\mathbf{1}_{\{\tau_j\le T\}}\mathbf{1}_{\{M_{\tau_j}=k\}}\Big) &= E\left(E\Big(B^{-1}(\tau_j)\mathbf{1}_{\{\tau_j\le T\}}\mathbf{1}_{\{M_{\tau_j}=k\}}\Big|\tau_j\Big)\right) \\ &= \int_0^T B^{-1}(t) P\big(M_t = k \big| \tau_j = t\big) P(\tau_j \in dt)\, dt. \quad (31) \end{aligned}$$

Using Corollary 4.3 we get in the model with mean-field interaction and one homogeneous group

$$V^{\text{def}} := \sum_{j=1}^m (LGD)_j \int_0^T B^{-1}(t) \frac{h(t,k-1)P(M_t = k-1)(m-k+1)}{m}\, dt. \qquad (32)$$

To compute (32) we only need the distribution of M_t, which is easily obtained from the Kolmogorov forward equation, and a one-dimensional numerical integration. In the general model we have $P\big(M_t = k \mid \tau_j = t\big) = \sum_{\mathbf{y}\in A_1(k,j)} P(\mathbf{Y}_t = \mathbf{y} \mid \tau_j = t)$; hence we get from (24) and (31)

$$V^{\text{def}} = \sum_{j=1}^m (LGD)_j \sum_{\mathbf{y}\in A_1(k,j)} \int_0^T B^{-1}(t)\lambda_j(t,\mathbf{y}^{(j)}) P(\mathbf{Y}_t = \mathbf{y}^{(j)})\, dt, \qquad (33)$$

which can be computed analytically for m small.

The premium payment leg. The premium payment leg consists of the sum of the value of the regular premium payments and the accrued premium payment; since $\{T_k \le t\} = \{M_t \ge k\}$ its value at $t=0$ for an arbitrary spread X can be written as

$$V^{\text{prem}} := X \sum_{n=1}^N \left[B^{-1}(t_n) P\Big(M_{t_n} < k\Big) + E\Big(B^{-1}(T_k)\frac{T_k - t_{n-1}}{t_n - t_{n-1}}\mathbf{1}_{\{t_{n-1}<T_k\le t_n\}}\Big)\right]. \qquad (34)$$

Using iterated conditional expectations we get for the second term

$$E\Big(B^{-1}(T_k)\frac{T_k - t_{n-1}}{t_n - t_{n-1}} 1_{\{t_{n-1}<T_k\le t_n\}}\Big) = \frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} B^{-1}(t)(t-t_{n-1}) P(T_k \in dt) dt.$$

Using partial integration we can write this as

$$\frac{1}{t_n - t_{n-1}}\left(B^{-1}(t_n)(t_n - t_{n-1}) P(M_{t_n} \ge k) - \int_{t_{n-1}}^{t_n} B^{-1}(t)(1 - r(t)(t - t_{n-1})) P(M_t \ge k) dt \right).$$

Hence we get from (34)

$$V^{\text{prem}} = X \sum_{n=1}^{N} \left(B^{-1}(t_n) - \frac{1}{t_n - t_{n-1}} \int_{t_{n-1}}^{t_n} B^{-1}(t)(1 - r(t)(t - t_{n-1}))P(M_t \geq k)dt \right), \tag{35}$$

which is easy to compute given the distribution of M_t. By equating the value premium payment leg and the default payment leg we finally obtain the fair spread $X^{k\text{th}}$ of the k-th-to-default swap.

A simple numerical example. In the following example we want to illustrate the effect of increasing interaction for the fair spread of k-th-to-default swaps. We have considered a portfolio of 5 names each with a recovery rate of $\delta = 40\%$. We calibrated the model to single-name CDS spreads. For simplicity we have assumed that for all 5 names the CDS-spreads are independent of the maturity and equal to 0.8%, 0.9%, 1.0%, 1.1% and 1.2%, leading 5-year risk-neutral default probabilities of 6.25%, 7.00%, 7.74%, 8.45% and 9.21%. The riskless interest rate was taken constant and equal to $r = 5\%$. We considered a general Markovian model with default intensities

$$\lambda_i(t, \mathbf{y}) = \max \left\{ \lambda_{i,0} \cdot \left(1 + \frac{\lambda_{i,1}}{5} \sum_{j=1}^{5} \left(y(j) - (1 - e^{-\lambda_{i,0} t}) \right) \right), \frac{\lambda_{i,0}}{2} \right\}, \ i = 1, \ldots, 5.$$

We computed the fair spread of a k-th-to-default swap ($k = 1, \ldots, 5$) with nominal $N_i = 1$ for all firms. We considered three cases with increasing interaction parameter $\lambda_{i,1} = 3, 6, 10$ (identical for all firms). The parameters $\lambda_{i,0}$ were calibrated to the given 5-year default probabilities. The parameters and the default correlations are given in Table 4 in the Appendix. The fair swap spreads were computed from (33) and (35) using the Kolmogorov forward equation (10) for the general model.

We have compared the results of the Markov model with a one factor Gaussian copula model (see for instance Laurent & Gregory (2003)), which is the industry standard for pricing such claims. The copula model was calibrated to the same risk-neutral 5-year default probabilities and default correlations as the Markov model. Fair swap spreads were computed using Monte Carlo simulation.

The fair spreads $X^{k\text{th}}$ for k-th-to-default swaps we obtained in both models are documented in Table 2. As expected, in both models the spread of the first to default swap decreases with increasing interaction parameter $\lambda_{i,1}$ and hence increasing default correlation, whereas the spreads of the higher order swaps increase with increasing interaction. Moreover, the Markov model and the copula model generate nearly identical results except for the extreme case $k = 5$, where the copula model leads to higher spreads. This seems to indicate that for given marginal default probabilities and given pairwise default correlation the distribution of the default times is to a large extent determined, independently of the particular model used. While this observation is in line with findings for static credit portfolio models (see for instance Frey & McNeil (2003)), further research is needed before such a statement can reliably be made.

k	without interaction	Case 1 ($\lambda_{i,1} = 3$) Markov	Case 1 ($\lambda_{i,1} = 3$) Copula	Case 2 ($\lambda_{i,1} = 6$) Markov	Case 2 ($\lambda_{i,1} = 6$) Copula	Case 3 ($\lambda_{i,1} = 10$) Markov	Case 3 ($\lambda_{i,1} = 10$) Copula
1	4.96%	4.55%	4.54%	4.13%	4.12%	3.65%	3.64%
2	0.61%	0.84%	0.83%	1.04%	1.01%	1.18%	1.14%
3	0.05%	0.13%	0.14%	0.25%	0.26%	0.40%	0.40%
4	0.002%	0.014%	0.016%	0.044%	0.054%	0.11%	0.12%
5	0.00003%	0.00077%	0.00105%	0.00431%	0.00714%	0.01610%	0.02566%

Table 2: Fair spreads of k-th-to-default swaps

4.3 Pricing of synthetic CDOs

Payoff description. A synthetic CDO is based on a portfolio of m single-name CDSs with nominal N_i and (possibly random) recovery rate δ_i, $1 \le i \le m$. The default losses of the portfolio are allocated to K tranches. Each of this tranches is determined by a fixed lower boundary l_k and upper boundary u_k, $k = 1, \ldots, K$, where $0 = u_K < l_K = u_{K-1} < \cdots = u_1 < l_1 = \sum_{i=1}^m N_i$. The maximum loss of tranche k is $l_k - u_k$. If firm i defaults before the maturity T of the contract, the holder of the lowest tranche K pays at time τ_i the loss of this default given by $LGD_i := N_i(1 - \delta_i)$ until he has reached his maximum loss; after that the holder of tranche $K - 1$ pays the loss and so on. Denote by $L_t = \sum_{i=1}^m (LGD)_i \mathbf{1}_{\{\tau_i \le t\}}$ the total loss of the portfolio at time t. For tranche k with boundaries l_k and u_k we define the function v_k by

$$v_k(x) = (x - l_k)1_{\{x \in [l_k, u_k]\}} + (l_k - u_k)1_{\{x > u_k\}} ,$$

so that the accumulated loss of tranche k up to time t equals $v_k(L_t)$. As a compensation for making the default payments the holder of a tranche gets a premium at fixed dates $t_1 < t_2 < \ldots < t_N = T$, whose size is based on the nominal of the tranche in the last period. If a default occurs the holder of a tranche moreover receives an accrued margin payment on the change in the value of his nominal between last regular payment time and default time. We denote by s^k the spread of tranche k. Then the regular payment on tranche k at time t_n equals $s^k(t_n - t_{n-1})((u_k - l_k) - v(L_{t_n}))$; in case name j defaults at time $\tau_j \in (t_{n-1}, t_n]$ the accrued margin payment equals $s^k(\tau_j - t_{n-1})(v(L_{\tau_j}) - v(L_{\tau_j -}))$, where L_{t-} is the left-hand limit of L_t in t, so that $v(L_{\tau_j}) - v(L_{\tau_j -})$ gives the the loss of the tranche due to the default of name j.

General pricing results. Using partial integration we obtain for the value of the default payments of tranche k

$$V^{\text{def}} := E\left(\int_0^T B^{-1}(t)\, dv_k(L_t)\right) = B^{-1}(T)E\big(v_k(L_T)\big) + \int_0^T r(t)B^{-1}(t)E\big(v_k(L_t)\big)dt \,;$$

see Laurent & Gregory (2003) for details. This is easily computed once we know the distribution of the total loss. The premium payment leg consists of the regular payment and of the accrued margin payments. With deterministic interest rates the value of the regular payments at $t = 0$ can be written as $s^k \cdot \sum_{n=1}^N B^{-1}(t_n) \left[\big(u_k - l_k - E\left(v_k(L_{t_n})\right)\big)(t_n - t_{n-1})\right]$. For the value of the accrued margin payments in $t = 0$ we obtain

$$s^k \cdot \sum_{n=1}^N \sum_{j=1}^m E\left(B^{-1}(\tau_j)\left(v_k(L_{\tau_j}) - v_k(L_{(\tau_j)-})\right)(\tau_j - t_{n-1})1_{\{t_{n-1} < \tau_j \le t_n\}}\right) .$$

If we condition on τ_j and use iterated conditional expectation we can write a single term of this sum as

$$\int_{t_{n-1}}^{t_n} B^{-1}(s)(s-t_{n-1})E\big(v_k(L_s)-v_k(L_{s-})\big|\tau_j=s\big)P(\tau_j\in ds)\,ds.$$

Thus in order to compute the premium payment leg we need the distribution of the loss of the tranche, the conditional distribution of the loss of the tranche given the default times and the density of the default times. For deterministic recovery rates all these quantities can in principle be determined using Proposition 4.1 and the Kolmogorov equations; without further homogeneity assumptions this may however become infeasible for m moderately large, so that one has to resort to simulations. Next we consider the extreme case of a completely homogeneous portfolio.

Results in the homogeneous mean-field model. Consider the mean-field model with one homogeneous group and identical nominals $N_1=\cdots=N_m=N$ and identical deterministic recovery rates $\delta_1=\cdots=\delta_m=\delta$. Then we get for the distribution of the total loss $P(L_t=x)=P\left(M_t=x/(N(1-\delta))\right)$ and for the expected loss of tranche k $E(v_k(L_t))=\sum_{i=0}^m v_k(iN(1-\delta))P(M_t=i)$. For the conditional expectations we get

$$E\Big(v_k(L_t)\Big|\tau_j=t\Big) = \sum_{i=1}^{m} v_k\big(iN(1-\delta)\big)P(M_t=i|\tau_j=t) \tag{36}$$

$$E\Big(v_k(L_{t-})\Big|\tau_j=t\Big) = \sum_{i=1}^{m-1} v_k\big(iN(1-\delta)\big)P(M_t=i+1|\tau_j=t)\,. \tag{37}$$

Defining $E_{t,k}(j):=mP(\tau_j\in dt)E\big(v_k(L_t)-v_k(L_{t-})\big|\tau_j=t\big)$ we get using (36), (37) and Corollary 4.3

$$\begin{aligned}
E_{t,k}(j) &= mP(\tau_j\in dt)\Bigg[\sum_{i=1}^{m-1} v_k\Big(iN(1-\delta)\Big)\Big(P(M_t=i\,|\,\tau_j=t)-P(M_t=i+1\,|\,\tau_j=t)\Big) \\
&\qquad + v_k\Big(mN(1-\delta)\Big)P(M_t=m\,|\,\tau_j=t)\Bigg] \\
&= \sum_{i=1}^{m-1} v_k\Big(iN(1-\delta)\Big)h(t,i-1)P(M_t=i-1)(m-i+1)-h(t,i)P(M_t=i)(m-i) \\
&\qquad + v_k\Big(mN(1-\delta)\Big)h(t,m-1)P(M_t=m-1).
\end{aligned}$$

Since $E_{t,k}(j)$ is independent of j, the overall value of the premium payments in $t=0$ is given by

$$s^k\cdot\sum_{n=1}^{N} B^{-1}(t_n)\Big(u_k-l_k-E\big(v_k(L_{t_n})\big)\Big)(t_n-t_{n-1})+\int_{t_{n-1}}^{t_n} B^{-1}(t)(t-t_{n-1})E_{t,k}(1)\,dt\,.$$

The case of stochastic recovery rates can be dealt with using Fourier inversion techniques; see for instance Laurent & Gregory (2003) for a discussion in the context of copula models.

A numerical example. Consider the simple example of pricing a synthetic CDO in the homogeneous mean-field model. The portfolio consists of 100 names with identical nominal $N_i = 1$ and deterministic recovery rate $\delta_i = 50\%$. The maturity of the CDO is taken to be $T = 5$ years and the premium payments are due at $t_n = 1, \ldots, 5$ years. The CDO has 3 tranches, equity 3%, mezzanine 7% and senior 90%, i.e. we have the boundaries $0 = u_3 < l_3 = 3 = u_2 < l_2 = 10 = u_1 < l_1 = 100$. We assume a risk free short rate of r= 3%. We model the default intensity as

$$h(t,l) = \max\left\{\lambda_0 \cdot \left(1 + \lambda_1 \left(\frac{l}{m} - (1 - e^{-\bar{\lambda}t})\right)\right),\ \lambda_0/2\right\},$$

where we set $\bar{\lambda}$ so, that we have $P(Y_1(i) = 1) = 1 - e^{-\bar{\lambda}}$ in the case without interaction. We assume that the one-year risk neutral default probability for each firm equals $P(Y_1(i) = 1) = 3.246\%$. We increase the interaction parameter λ_1 and calibrate the other parameter λ_0 such that the 5-year default probability is unchanged.

In Table 3 we show the behaviour of the annual default probability (for a constant 5-year default probability), the one- and 5-year default correlation and the fair spreads of the three tranches. As expected the spread of the first tranche decreases whereas the spreads of the other tranches increase as we increase λ_1 and thus the dependence between defaults. This behaviour of CDO-spreads is well-known from other studies such as Duffie & Garleanu (2001).

λ_1	λ_0	annual default probability %	annual default correlation %	5year default correlation %	fair spread of tranche [0,3] %	[3,10] %	[10,100] %
0	0.03300	3.246	0.0000	0.0000	93.16	16.23	0.02
10	0.03304	3.244	0.4005	3.8474	78.11	14.03	0.17
20	0.03072	2.934	0.9130	11.948	60.62	10.13	0.39
30	0.02811	2.577	1.3724	22.233	49.60	7.41	0.57

Table 3: Fair spreads of different tranches of a synthetic CDO for different interaction levels and a constant 5-year default probability $P(Y_5(i) = 1) = 15.21\%$.

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A Appendix

A.1 Simulation

We now describe an approach for simulating a trajectory of the process $\mathbf{\Gamma}$ with dynamics as in Assumption 2.1 and initial values $\mathbf{\Gamma}_0 = (\boldsymbol{\psi}, \mathbf{y})$ up to some finite horizon T. The approach follows the standard construction of conditional continuous time Markov chains. First we simulate a trajectory of $\mathbf{\Psi}$. Depending on the specific model for $\mathbf{\Psi}$ various approaches can be used; see for instance Glasserman (2003). Next we have to simulate the first default time T_1. It is well-known that T_1 has hazard-rate process $\lambda_t^{(1)} = \sum_{i=1}^m \big(1 - y(i)\big)\lambda_i(\mathbf{\Psi}_t, \mathbf{y})$. Hence we simply simulate a unit exponential random variable θ_1 independent of $\mathbf{\Psi}$ and put $T_1 = \inf\big\{t \geq 0 : \int_0^t \lambda_s^{(1)} ds \geq \theta_1\big\}$. Next we determine the identity ξ^1 of the first defaulting firm. It is shown for instance in Bielecki & Rutkowski (2002) that

$$P_{(\boldsymbol{\psi},\mathbf{y})}(\xi^1 = i \mid T_1 = t) = \frac{\big(1 - y(i)\big)\lambda_i(\mathbf{\Psi}_t, \mathbf{y})}{\sum_{j=1}^m \big(1 - y(j)\big)\, \lambda_j(\mathbf{\Psi}_t, \mathbf{y})} =: p_i^{(1)}\,;$$

Hence ξ^1 can be simulated as realisation of a random variable ξ with $P(\xi = i) = p_i^{(1)}$ for $1 \leq i \leq m$.

In case that $T_1 \geq T$ we have accomplished our task and stop. Else we define the vector $\mathbf{y}^{(1)} := \mathbf{y}^{\xi^1}$ (recall the notational convention (2)), and for $t \geq T_1$ the process $\lambda_t^{(2)} = \sum_{j=1}^m (1 - y^{(1)}(j))\lambda_j(\mathbf{\Psi}_t, \mathbf{y}^{(1)})$. In analogy to the previous step we put $T_2 = \inf\big\{t \geq T_1 : \int_{T_1}^t \lambda_s^{(2)} ds \geq \theta_2\big\}$, where θ_2 is again a unit exponential rv independent of all other variables. ξ^2 is determined as before, using the identity

$$P(\xi^{(2)} = i \mid T_2 = t, \xi^1) = \Big(\lambda_t^{(2)}\Big)^{-1} \Big(1 - y^{(1)}(i)\Big)\, \lambda_i(\mathbf{\Psi}_t, \mathbf{y}^{(1)}).$$

The algorithm proceeds this way until we have reached some j with $T_j \geq T$ or until all companies are default.

A.2 Forward equations

Proof of Lemma 2.3. We identify $G_{[\boldsymbol{\psi}]}$ with an $|S| \times |S|$ matrix $(\Lambda_{ij}(t \mid \omega_1))$; $G^*_{[\boldsymbol{\psi}]}$ corresponds then to the transpose matrix. For this we choose a bijection $I : \{1, \ldots, |S|\} \to S$, $i \mapsto \mathbf{y}_i$. By definition of the generator of $\mathbf{Y}$ we have for $i \neq j$

$$\Lambda_{ij}(t \mid \omega_1) = \begin{cases} (1 - y_i(k))\lambda_k(\boldsymbol{\Psi}_t(\omega_1), \mathbf{y}_i), & \text{if } \mathbf{y}_j = \mathbf{y}_i^k \text{ for some } k \in \{1, \ldots, m\}, \\ 0 \text{ else}. \end{cases} \tag{38}$$

For $i = j$ we put $\Lambda_{ii}(t \mid \omega_1) = -\sum_{j \neq i} \Lambda_{ij}(t \mid \omega_1)$, yielding $\Lambda_{ii}(t \mid \omega_1) = -\sum_{k=1}^m (1 - y_i(k))\lambda_k(\boldsymbol{\Psi}_t(\omega_1), \mathbf{y}_i)$. Now fix $\mathbf{y} = I(j_0) \in S$. Since $G^*_{[\boldsymbol{\Psi}_t(\omega_1)]}$ corresponds to multiplication with the transpose matrix $(\Lambda^*_{ij}(t \mid \omega_1))$, the forward equation becomes

$$\frac{\partial p(t, s, \mathbf{x}, \mathbf{y} \mid \omega_1)}{\partial s} = \sum_{i=1}^{|S|} \Lambda_{ij_0}(s \mid \omega_1) p(t, s, \mathbf{x}, \mathbf{y}_i \mid \omega_1).$$

Using the definition of $\Lambda_{ij}(s \mid \omega_1)$ and the relation $(1 - y^k(k)) = y(k)$ we obtain the final version (10) of the forward equation. □

Next we consider forward equations for $\overline{\mathbf{M}}_t$. We have

Lemma A.1. *Assume that the default intensities satisfy Assumption 3.1. Then the adjoint operator $G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]}$ to the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}_t(\omega_1)]}$ of $\overline{\mathbf{M}}_t$ is given by*

$$\begin{aligned} G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]} f\,(\bar{\boldsymbol{l}}) = & \sum_{\kappa=1}^{k} 1_{\{\bar{l}_\kappa > 0\}} \big(1 + m_\kappa(1 - \bar{l}_\kappa)\big) h_\kappa\big(\boldsymbol{\psi}, \bar{\boldsymbol{l}} - \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) f\big(\bar{\boldsymbol{l}} - \frac{1}{m_\kappa}\mathbf{e}_\kappa\big) \\ & - \sum_{\kappa=1}^{k} m_\kappa(1 - \bar{l}_\kappa) h_\kappa\left(\boldsymbol{\psi}, \bar{\boldsymbol{l}}\right) f(\bar{\boldsymbol{l}}). \end{aligned} \tag{39}$$

Sketch of proof. As in the proof of Lemma 2.3 we define a $\left|S^{\overline{M}}\right| \times \left|S^{\overline{M}}\right|$ matrix $(\Lambda_{ij}(t|\omega_1))$ and identify the generator $G^{\overline{\mathbf{M}}}_{[\boldsymbol{\psi}]}$ with the matrix through a bijection $I : \{1, 2, \ldots, \left|S^{\overline{M}}\right|\} \to S^{\overline{M}}$, $I(i) = \bar{\boldsymbol{l}}^{(i)}$. According to Lemma 3.4 we have for $i \neq j$

$$\Lambda_{ij}(t \mid \omega_1) = m_\kappa(1 - \bar{l}^{(i)}_\kappa) h_\kappa(\boldsymbol{\Psi}_t, \bar{\boldsymbol{l}}^{(i)}), \tag{40}$$

if there is a $\kappa \in \{1, \cdots, k\}$ with $\bar{l}^{(j)}_\kappa = \bar{l}^{(i)}_\kappa + \frac{1}{m_\kappa}$ and $\bar{l}^{(j)}_\gamma = \bar{l}^{(i)}_\gamma$ for $\gamma \neq \kappa$, and $\Lambda_{ij}(t \mid \omega_1) = 0$ else; for $i = j$ we obtain

$$\Lambda_{ii}(t \mid \omega_1) = -\sum_{j=1, j \neq i}^{\left|S^{\overline{M}}\right|} \Lambda_{ij}(t \mid \omega_1) = -\sum_{\kappa=1}^{k} m_\kappa(1 - \bar{l}^{(i)}_\kappa) h_\kappa(\boldsymbol{\Psi}_t, \bar{\boldsymbol{l}}^{(i)}).$$

The result follows as the adjoint operator $G^{*\overline{\mathbf{M}}}_{[\boldsymbol{\Psi}]}$ corresponds to multiplication with the transpose matrix $(\Lambda^*_{ij}(t|\omega_1))$. □

A.3 Complementary numerical results

Case 1: Lower Interaction, $\lambda_{i,1} = 3$, $i = 1, \ldots, 5$					
$\lambda_{1,0} = 0.013235$	100%	4.32%	4.52%	4.72%	4.89%
$\lambda_{2,0} = 0.014919$		100%	4.77%	4.97%	5.16%
$\lambda_{3,0} = 0.016605$			100%	5.21%	5.41%
$\lambda_{4,0} = 0.018300$				100%	5.64%
$\lambda_{5,0} = 0.019999$					100%

Case 2: Medium Interaction, $\lambda_{i,1} = 6$, $i = 1, \ldots, 5$					
$\lambda_{1,0} = 0.013773$	100%	9.81%	10.25%	10.66%	11.04%
$\lambda_{2,0} = 0.015584$		100%	10.78%	11.21%	11.61%
$\lambda_{3,0} = 0.017410$			100%	11.72%	12.14%
$\lambda_{4,0} = 0.019256$				100%	12.62%
$\lambda_{5,0} = 0.021119$					100%

Case 3: Higher Interaction, $\lambda_{i,1} = 10$, $i = 1, \ldots, 5$					
$\lambda_{1,0} = 0.013876$	100%	16.76%	17.44%	18.07%	18.65%
$\lambda_{2,0} = 0.015783$		100%	18.28%	19.45%	19.55%
$\lambda_{3,0} = 0.017727$			100%	19.74%	20.38%
$\lambda_{4,0} = 0.019711$				100%	21.13%
$\lambda_{5,0} = 0.021733$					100%

Table 4: Parameter $\lambda_{i,0}$ of the Markov model and the resulting 5-year default correlation in 3 cases with increasing interaction. The parameters $\lambda_{i,0}$ are calibrated to the following 5-year default probabilities $P(Y_5(1) = 1) = 6.25\%$, $P(Y_5(2) = 1) = 7.00\%$, $P(Y_5(3) = 1) = 7.74\%$, $P(Y_5(4) = 1) = 8.45\%$ and $P(Y_5(5) = 1) = 9.21\%$.