The Effect of GARCH-Type Volatilities on the Prices and Payoff Distributions of Nonlinear Derivative Assets - a Simulation Study

Rüdiger Frey[1]
ETH Zürich

Paul K. Michaud[2]
UBS Zürich

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1. Introduction

There is a considerable amount of empirical evidence suggesting that the classical Black-Scholes model with constant volatility does not describe the statistical properties of the price process of typical assets very well. For instance, researchers have found fat tails, volatility clustering and a strong amount of correlation between volatility innovations and asset returns in many different (stock) return series.[3] Moreover, the ARCH/GARCH-models of Engle (1982), Bollerslev (1986) and their successors have been applied with great success to the modeling of financial time series. It is therefore of interest to study derivative asset analysis in models where the underlying asset follows a process with GARCH-type volatilities, as this yields important insights on the robustness of the Black-Scholes model and indicates when there is a need for using econometrically more refined models.

In this paper we fit different GARCH-type models to time series data of the CRSP-index (a major US stock-price index). For each of these estimated models we use Monte Carlo techniques to simulate future asset price trajectories. This allows us to assess the effect of different specifications of mean and volatility of the return process on price and payoff-distribution of derivatives. Adding to the existing literature in the fields we study options with path-dependent payoff where the effects of different model specifications are even more pronounced.

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[1] Rüdiger Frey, Department of Mathematics, ETH Zentrum, CH-8092 Zürich, Switzerland, email: frey@math.ethz.ch
[2] Paul K. Michaud, UBS Zürich, Bahnhofsstrasse 45, CH-8021 Zürich, Switzerland, email: Paul.Michaud@ubs.com
[3] A more intense discussion of these and other empirical violations of the Black-Scholes model can for instance be found in Frey (1997).

It is well known that the pricing of derivatives in GARCH-models poses conceptual difficulties. This stems from the fact that these models are incomplete, meaning that the payoff of a typical derivative can no longer be perfectly replicated by a dynamic trading strategy. In the present paper we develop a pricing theory for derivatives under GARCH type volatilities along the lines of Amin and Ng (1993, 1994) who use a state price density to represent asset prices. The incompleteness of our models implies that the dynamics of the underlying asset do not uniquely determine this state price density so that we have to justify our choice by additional arguments.

We go on and study the asset price distributions implied by our models. We find that the distributions corresponding to the models allowing for an asymmetric reaction of volatility to return shocks have considerably more mass in the lower tail than those of the symmetric models. This effect is even stronger if we consider models where the mean of the conditional return distribution is non-constant. These differences are reflected in prices and payoff distributions of derivatives: The implied volatility curves corresponding to the asymmetric GARCH-models exhibit a strong "skew pattern" and we also observe sizable effects on the prices and payoff distributions of options with path-dependent payoff.

The paper is organized as follows: In Section 2 we introduce the GARCH-models used in this paper and discuss the pricing of derivatives in GARCH-models. Section 3 contains our results on the distribution of the underlying asset. In Section 4 we present our empirical results on derivative securities, Section 5 finally concludes.

2. GARCH-Models and Derivative Pricing

2.1 GARCH-Models

Formally all the GARCH-models we will use in this paper can be described as follows. Assume that a risky asset S is traded at discrete equidistant points in time t_k and denote its price at time t_k by S_{t_k}. Define the return process by $R_k := \ln S_{t_k} - \ln S_{t_{k-1}}$. In all GARCH-models we consider in this paper the dynamics of R are of the following form:

$$R_k = \mu_k + \sigma_k \varepsilon_k .$$

Here ε_k is an i.i.d. sequence of standard normal random variables; μ_k and σ_k are assumed to be measurable with respect to F_{k-1}, the information available up to time t_{k-1}. Hence the conditional distribution of R_k given information up to time t_{k-1} is normal with mean μ_k and variance σ_k^2. We will work with two different models for the mean:

- constant mean model: $\mu_k = \overline{\mu}$ for a constant $\overline{\mu}$.
- AR(1)-Model: $\mu_k = \overline{\mu} + \rho R_{k-1}$ for constants $\overline{\mu}$ and ρ

The existing GARCH-models mainly differ in their specification of the conditional standard deviation of the returns. The most popular GARCH-model is the GARCH (1,1) of Bollerslev (1986). Here the dynamics of σ are given by

$$\sigma_k^2 = \omega + \alpha \ \sigma_{k-1}^2 \varepsilon_{k-1}^2 + \beta \ \sigma_{k-1}^2 . \tag{1}$$

While this model is able to explain a good deal of the excess kurtosis we observe in most financial time series, it is not able to capture the asymmetric reaction of volatility to return shocks termed the "leverage effect" since Black (1976). It has been observed that - at least in equity markets -"good news" (large positive return shocks) tend to increase volatility less than "bad news" (large negative return shocks). Therefore a number of models allowing for an asymmetric response of volatility to return shocks have been proposed. A very general asymmetric GARCH-model is the AGARCH of Hentschel (1995), where we have the following dynamics for σ:

σ_{k-1}^d

$$\sigma_k^a = \omega + \alpha f^d(\varepsilon_{k-1}) + \beta \ \sigma_{k-1}^a . \tag{2}$$

Here a and d are given constants; $f(\varepsilon) := |\varepsilon - b| - c(\varepsilon - b)$ measures the impact of return shocks on volatility. Hentschel gives sufficient conditions for the solution of (2) to be positive and stationary. The main restriction on the parameters to be made in (2) is $|c| < 1$; this is sufficient to ensure the positivity of f. A positive b and a positive c imply that for negative ε volatility rises more than for equally large positive shocks. Hentschel (1995) contains a detailed discussion of the role of the parameters b and c.

The model (2) is very general and nests most of the existing GARCH-models: If we put $d = a = 2$ and $b = c = 0$ we get the GARCH(1,1)-model defined in (1). Another interesting GARCH-specification, the APARCH of Ding, Engle and Granger (1993), is obtained by putting $a = d$ and $b = 0$. The model of Hentschel nests more models than just GARCH or AGARCH. For instance for $a \to 0$ and $d = 1$ we obtain the EGARCH of Nelson (1991), see again Hentschel (1995) for details.

We will use a constant volatility model with $\sigma_k = \varpi$ for some constant ϖ and the previously introduced GARCH, APARCH and AGARCH specifications in this paper; each of these four volatility models will be considered with constant and AR(1) mean, respectively.

2.2 Derivative Pricing under GARCH

Now we discuss the pricing of derivative securities in models where the price of the underlying security follows a GARCH-type process. We consider a market, where a non-dividend-paying risky asset S (some stock or stock index) with return dynamics given by one of the previously mentioned GARCH-models and some riskless asset B is traded. We assume that at time t_k the price of B equals $\exp(rk\Delta t)$ where r represents the continuously compounded interest rate and where $\Delta t := t_k - t_{k-1}$ represents the time between two observations of the asset price.[4].

Following Constantinides (1992) and Amin and Ng (1993, 1994) we model directly the dynamics of a strictly positive integrable *state price density process* $(\xi_k)_{k \in N}$. In terms of this state price density process the price p_{k-1} at time t_{k-1} of any contingent claim with payoff C_k at time t_k is given by

(3)
$$p_{k-1} = E\left[C_k \frac{\xi_k}{\xi_{k-1}} \middle| F_{k-1} \right],$$

in particular we have the pricing formula $p_0 = E[C_k \xi_k]$. In a representative agent economy of the type considered for instance by Rubinstein (1976) ξ_k can be interpreted as marginal rate of

[4] As usual we are measuring times in years. As we are working with daily observations and as there are approximately 252 trading days per year we took $\Delta t = 1/252$ in our study.

substitution between consumption at date 0 and date t_k. However, we need not assume the existence of a representative consumer maximizing expected utility: As shown by Dalang, Morton and Willinger (1989) the existence of a state price density is equivalent to absence of arbitrage.

While the existence of a state price density may therefore safely be assumed, we will now explain why ξ is not uniquely determined from the asset price dynamics in our model. Applying (3) to our riskless asset B where C_k is deterministic and equal to $\exp(rk\Delta t)$ we get the relation

(4) $$\xi_{k-1} \exp(-r\Delta t) = E\left[\xi_k \middle| F_{k-1}\right],$$

and by plugging the price process of the risky asset into (3) we get that

(5) $$S_{k-1} = E\left[S_k \frac{\xi_k}{\xi_{k-1}} \middle| F_{k-1} \right].$$

Now every strictly positive integrable process ξ satisfying (4) and (5) qualifies as a state price density process. It is easy to construct different processes ξ satisfying (4) and (5) such that in our framework there exist infinitely many state price densities. By the second fundamental theorem of asset pricing this reflects the incompleteness of our models.

The nonuniqueness of the state price density leaves us with a modeling choice. Following Amin and Ng we chose the following functional form for ξ

(6) $$\xi_k = \exp\left(-rk\Delta t + \sum_{j=1}^{k} \left(\lambda_j \varepsilon_j - \frac{1}{2}\lambda_j^2 \right) \right)$$

for a F_{k-1}-measurable finite random variable λ_k. It is immediately checked that for a state price density as in (6) relation (4) holds for every F_{k-1}-measurable finite random variable λ_k; substituting (6) into (5) and evaluating the expectation we see that relation (5) holds if and only if

$$(7) \qquad \lambda_k = \frac{r\Delta t - \left(\mu_k + \frac{1}{2}\sigma_k^2\right)}{\sigma_k}.$$

A detailed derivation of these formulae is given in Appendix 7.1.

A number of arguments can be put forth to motivate the - inevitably somewhat ad hoc - choice of the functional form of our state price density. In our model the incompleteness of the market is due to the fact that we are working in a model with discrete trading but with an infinite state space at every step (as the support of the distribution of our error terms is the whole real line). Now, if the time between two trading dates is small as in our study one might look at some limit of our discrete time setup. Arguments that justify the choice of a certain state price density process in the limit model then yield via a convergence result a motivation for the choice of ξ in our discrete-time framework. Now there are two ways of taking a continuous-time limit of a GARCH-type model.

- First work by Nelson (1990) and Duan (1996) has shown that under suitable rescaling many GARCH-models converge to a continuous-time stochastic volatility model. In that case our state price density defined in (6) converges to the continuous-time state price density corresponding to *minimal martingale measure* of Föllmer and Schweizer. A definition of this measure and arguments in favour of its use as a pricing measure can be found in Hofmannet. al. (1992).
- Kallsen and Taqqu (1995) construct continuous-time models with piecewise constant volatility that "interpolate" the discrete-time GARCH-models. The resulting models are complete such that prices for derivatives are uniquely determined. Kallsen and Taqqu show that at the discrete times t_k their prices coincide with the prices obtained from the pricing formula (3) with a state price density of the form (6).

Moreover, in our benchmark case with constant volatility ($\sigma_k = \varpi$) the choice of ξ ensures that the prices of derivatives obtained from the pricing formula (3) coincide with the prices one obtains in the standard continuous time Black-Scholes model with constant volatility ϖ [5]. Finally we remark that Duan (1995) has developed an interesting equilibrium model leading to a state price density of the form (6).

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[5]This is literally true only for path-independent options as certain path dependent derivatives may require some continuity correction, see e.g. Broadie et.al. (1997).

3. Empirical Results - Underlying Asset

3.1 Data

We used the daily index data collected by the Center for Research in Security Prices (CRSP) for our analysis. The data are return data for a value weighted index of US equities and include dividends. We used 1000 trading dates (approximately four years) prior to 31 December 1992, i.e. our data sample starts short after the stock market crash of October 1987. Most academic studies use far more than 1000 data points for the estimation of GARCH-models. We restricted ourselves to a relatively short time period because we believe that the structural properties of asset price time series are likely to change over time due to institutional changes or changes in trading behavior. Moreover, for many individual stocks there are only 500 to 1000 data points available. Hence if our approach is to be used in practice it should be applicable even if the available data sample is smaller than the samples that are usually used in the academic literature.

3.2 Estimation Results

We used maximum likelihood to estimate the coefficients of our models. The results are displayed in Table 1 in the appendix. The following table explains the model-naming conventions we will use throughout the rest of this paper.

	Constant Vol	GARCH	APARCH	AGARCH
Constant Mean	1,1	1,2	1,3	1,4
AR(1)-Mean	2,1	2,2	2,3	2,4

While the focus of our analysis is not so much on econometrics a few remarks concerning our estimation results are in order as they help to interpret our findings on derivative prices.

- We observe that our time series displays strong negative asymmetry, i.e. negative return shocks tend to increase volatility more than positive return shocks, as can be seen from the large positive values for c in the models (1,3), (2,3), (1,4) and (2,4) which allow for nonzero values for this parameter. This appears reasonable, as the period after October 1987 was marked by strong "crash-fears" among equity investors. These crash fears are often quoted as intuitive explanation for the asymmetry in stock price volatility. The linear GARCH-models (models (1,2) and (2,2)), which cannot account for this asymmetry, yielded a far worse fit than the other models. We will see below that asset price distributions and hence also prices and payoff distributions of derivatives in the linear GARCH-models differ markedly from the asset price distributions in the models allowing for asymmetry.

- We conducted likelihood ratio tests to check if the improvement in likelihood due to the introduction of additional parameters is statistically significant. We found that the introduction of additional parameters in the dynamics of σ was always a significant improvement with the exception of the transition from APARCH models ((1,3) or (2,3)) to AGARCH ((1,4) or (2,4)). Therefore we will not always present results for the (2,4) model. Allowing for an AR-component in the mean turned out to be always a significant improvement over the respective model with constant mean at a 95% confidence level.
- In all models with non-constant volatility the value of σ_k at the end of the estimation period was relatively low compared to its average or to historical volatility computed over the whole sample. This will be important when it comes to interpreting the pricing differences between the constant volatility model and the models with GARCH-type volatilities.

3.3 Distribution of the Price of the Underlying Asset

In figures 1 and 2 we have graphed the inverse cumulative distribution[6] of the asset price for a current price of 1000 in 132 days (approx. 6 months) from now, once for models with constant mean and once for models with AR-mean. We immediately see that the models with asymmetric volatility dynamics have a much fatter lower tail. Comparing figures 1 and 2 we also see that the asset price distributions in the models with AR-mean have fatter tails than those in the respective models with constant mean. These findings are confirmed by table 2, where we have the "VAR" (lower 5% quantile) of the asset price distribution implied by the different values for a time horizon of 132 days.

[6] The inverse cumulative distribution function G is defined by $G(p) = \inf\{a > 0,\ P[S_T \le a] > p\}$.

4. Empirical Results - Derivatives

4.1 Computation of Derivative Prices

Even for simple European call and put options analytic expressions for option prices as given by (3) are no longer available if volatility follows a GARCH-process. We therefore used Monte-Carlo-simulation to compute derivative prices. This allows us to include path-dependent derivatives into our analysis without additional difficulties.[7] In contrast to Amin and Ng (1994) or Duan (1995) who are working with the dynamics of the return process under some equivalent martingale measure we simulate price paths under the "real world" probability measure, i.e. we use directly the coefficients coming out of our estimation procedure. Derivative prices are computed using (3).[8] This gives us information about derivative prices and at the same time about the payoff distribution under the real world probability measure. An antithetic variable technique was used for variance reduction[9]. We experimented with the number of path per simulation. We found that 50,000 paths are by far sufficient for pricing purposes; in order to obtain accurate estimates for the tails of the payoff distributions we used 1,000,000 paths per simulation.

4.2 Ordinary Options

To make our results on pricing of ordinary European call and put options[10] comparable accross strike prices and maturities we compute the *implied volatility* of the option prices. We obtained the following results which are illustrated by figure 3.

- As predicted by theory the implied volatilities of the option prices in the benchmark model with constant volatility are constant across strike prices and maturities.
- We observe that the implied volatilities in the asymmetric volatility models (model (1,3) and model (1,4)) display a strong skew pattern, i.e. they are a falling function of the exercice price. Looking at the symmetric GARCH-model (model (1,2)) we also see the typical smile pattern of implied volatility, i.e. the graph of the implied volatility function is

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[7] Note that at present we cannot price American options. A new approach to the computation of option prices in GARCH-models that can deal with American options see Ritchkin and Trevor (1997) or Duan (1997).

[8] We experienced numerical difficulties when looking at the AR(1)models, so we computed derivative prices only in the models with constant mean.

[9] For further information about Monte Carlo simulation in Finance see e.g. Boyle, Broadie and Glassermann (1995).

[10] In our model the Put-Call-Parity holds so that we need not distinguish between put- and call options here.

U-shaped.

- Comparing implied volatilities across maturities we observe a *mean-reverting* behavior. As mentioned before, the instantaneous volatility σ_k at the end of the sample period was low compared to its average value. Therefore the implied volatilities of options with a short time to maturity are relatively low. As time to maturity increases the implied volatilities rise, reflecting the fact that the values of the volatility tend to return to their "normal level".

Both, the smile and skew pattern and the term-structure effect are typical for the implied volatility of traded option contracts, see e.g. Rubinstein (1985). Using models with GARCH-type volatilities enables us to account for these observations in a systematic manner. The usual approach to option pricing in the presence of excess kurtosis and asymmetry in the time series of the underlying asset is to use the Black-Scholes formula and to account for excess kurtosis and asymmetry by choosing different volatilities. This is not only internally inconsistent, it is also only a crude way of reflecting the econometric information about the time series properties of the underlying asset.

A number of authors have studied the qualitative properties of prices of ordinary options in models with GARCH-type volatilities, see e.g. Duan (1995). However, at least to our knowledge no study on the properties of path-dependent option prices in models with GARCH-type volatilities has been published so far. We have run simulations for various types of exotic options (Asians, Binaries, Lookbacks) with varying characteristics. In order to save space we present only selected results for lookback options and double barrier binary options. The results we obtained for these selected payoffs are very representative for our findings.

4.3 Lookback Options

The payoff of a *lookback put* option with strike *K* and maturity *T* is given by

$$\left[K - \min_{0 \le t \le T} S_t\right]^+ .$$

One would expect that price and expected payoff of this derivative are very sensitive to the presence of asymmetry in the reaction of volatility to return shocks. This intuition is confirmed by our study. In table 3 and 4 we give numbers for price and expected payoff in case of a lookback put with exercice price 975. We clearly see that price and expected payoff of lookback puts are increased if a model that allows for negative asymmetry is used.

Looking at the expected payoff we see that allowing for a non-constant mean makes this effect even more pronounced. For instance the expected payoff in model (2,4) is more than the double of the expected payoff in model (1,2). This difference is remarkable as both models were fitted to the same time series.

4.4 Double Barrier Binary Options

Let there be given positive numbers b_1, b_2 and V with $b_1 > S_0 > b_2$ where S_0 is the current price of the underlying asset. The payoff of a double barrier binary option on S with maturity T, upper barrier b_1, lower barrier b_2 and face value V equals V if the underlying asset does not leave the interval $[b_2, b_1]$ in the time $[0, T]$; otherwise the payoff is zero. This payoff is a discontinuous function of the price process of the underlying asset; we therefore expect its price and its payoff distribution[11] to be very sensitive to the modeling of the dynamics of the underlying asset. We found that price and in particular payoff distribution of double barrier binary options were strongly influenced by the introduction of asymmetry into the volatility dynamics. Also the introduction of non-constant mean had a sizable effect on the probability that the option finishes in the money. These effects are particularly pronounced if the lower barrier is close to the current price of the underlying. These findings are illustrated by figures 4 and5.

4.5 Comparison across Option Types

In figure 6 we have graphed the relative pricing difference to the constant volatility model given by the ratio of the price in model $(1,i)$ over the price in model $(1,1)$ for $i = 1,\dots 4$ and for various option types. We see that the relative pricing differences for derivatives which are "in the money" with a rather high probability is relatively low compared to the relative pricing differences of "out of the money" options. Moreover, we see that the relative pricing differences of the "exotic options" tend to be larger than those of the standard European options. This effect is as expected; it shows that the use of refined pricing methods can be particularly useful for books with exotic options.

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[11] Note that the payoff of a double barrier binary option is a two-valued random variable which takes the value V with probability p (the probability that the option finishes in the money) and the value 0 with probability $(1-p)$

5. Conclusions

In this paper we fitted GARCH-type models to time series data of the CRSP-index. We used Monte Carlo techniques to simulate future asset price trajectories and hence the payoff-distribution of various derivatives. Using a state price density model for derivatives we computed prices for derivative contracts. This allowed us to assess the effect of different specifications of mean and volatility of the return process on price and payoff-distribution of derivatives. We found that the data sample we considered was marked by the presence of a strong asymmetry in the reaction of volatility to return shocks. This asymmetry was reflected in prices and payoff distribution of the derivative contracts we considered. We also found that the introduction of non-constant models for the conditional mean of the return distribution had sizeable effects on the payoff distribution of the derivatives under consideration. This effect has been overlooked by academic studies so far, as standard derivative pricing theory focuses mostly on payoff distributions under a risk-neutral measure.

Our study showed that by combining econometric analysis with simulation pricing techniques for derivatives it is possible to account for the empirical violations of the Black-Scholes assumption of constant volatility in a flexible and systematic manner. In particular this approach avoids the need of making "educated guesses" when specifying volatilities. We therefore think that these techniques are a useful tool for pricing and risk-management of derivatives.

6. References

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7. Appendix

7.1 Complements to Section 2.2

We first show that (4) is satisfied for any state price density of the form (6). As $\xi_k = \xi_{k-1} \exp(-r\Delta t)\exp(\lambda_k \varepsilon_k - \frac{1}{2}\lambda_k^2)$ equation (4) is equivalent to

(A1) $$E\left[\exp(\lambda_k \varepsilon_k - \frac{1}{2}\lambda_k^2) \middle| F_{k-1}\right] = 1$$

As ε_k is normally distributed conditional on F_{k-1} and as λ_k is F_{k-1}- measurable we get that

(A2) $$\begin{aligned} E\left[\exp(\lambda_k \varepsilon_k - \frac{1}{2}\lambda_k^2) \middle| F_{k-1}\right] &= (2\pi)^{-1}\int_{\mathrm{R}} \exp(\lambda_k x - \frac{1}{2}\lambda_k^2)\exp(-\frac{x^2}{2})dx \\ &= (2\pi)^{-1}\int_{\mathrm{R}} \exp(-\frac{(x-\lambda_k)^2}{2})dx \\ &= 1 \end{aligned}$$

To derive formula (7) note that

(A3) $$\frac{\xi_k}{\xi_{k-1}} = S_{k-1}\exp\left((\mu_k - r\Delta t - \frac{1}{2}\lambda_k^2) + (\lambda_k + \sigma_k)\varepsilon_k\right)$$

Now it follows from the same type of calculations as in (A2) that the conditional expectation of the right hand side of (A3) equals S_{k-1} if and only if

(A4) $$(\mu_k - r\Delta t - \frac{1}{2}\lambda_k^2) = -\frac{1}{2}(\lambda_k + \sigma_k)^2$$

which is equivalent to formula (7).

7.2 Tables and Figures

Table 1: Model Coefficients

	Model Coefficients - Constant Mean			
	Model 11	Model 12	Model 13	Model 14
mu	0.05512	0.05955	0.05396	0.05289
omega	0.57195	0.04553	0.14247	0.19281
alpha	0.00000	0.04545	0.09369	0.00902
beta	0.00000	0.87453	0.73823	0.79746
a	2.00000	2.00000	1.02769	0.08549
b	0.00000	0.00000	0.00000	-0.06471
c	0.00000	0.00000	0.70154	0.76448
d	2.00000	2.00000	1.02769	0.91239

	Model Coefficients - AR(1) Mean			
	Model 21	Model 22	Model 23	Model 24
mu	0.05010	0.05070	0.03980	0.04250
rho ~~lambda~~	0.09120	0.08650	0.10700	0.11000
omega	0.56700	0.00741	0.15100	0.20400
alpha	0.00000	0.01620	0.08770	0.01150
beta	0.00000	0.97000	0.72800	0.77700
a	2.00000	2.00000	1.05000	0.15800
b	0.00000	0.00000	0.00000	0.22000
c	0.00000	0.00000	0.86300	0.70100
d	2.00000	2.00000	1.05000	1.19000

Table 2: 5% VAR for the asset over 132 days (approx. 6 month) period

	Constant Vol	GARCH	APARCH	AGARCH
Constant Mean	67.885	59.970	76.591	77.4022
AR(1)-Mean	80.969	65.321	103.828	

Table 3: Price of a Lookback-Put with strike price 975 and maturity 132

	Constant Vol	GARCH	APARCH	AGARCH
Constant Mean	31.259	30.344	35.227	35.216

Table 4: Expected payoff of a Lookback-Put with strike 975 and maturity 132

	Constant Vol	GARCH	APARCH
Constant Mean	18.003	15.663	21.292
AR-1 Mean	36.347	16.357	30.924

Figure 1:

Figure 1: Inverse Cumulative Distributions for constant mean

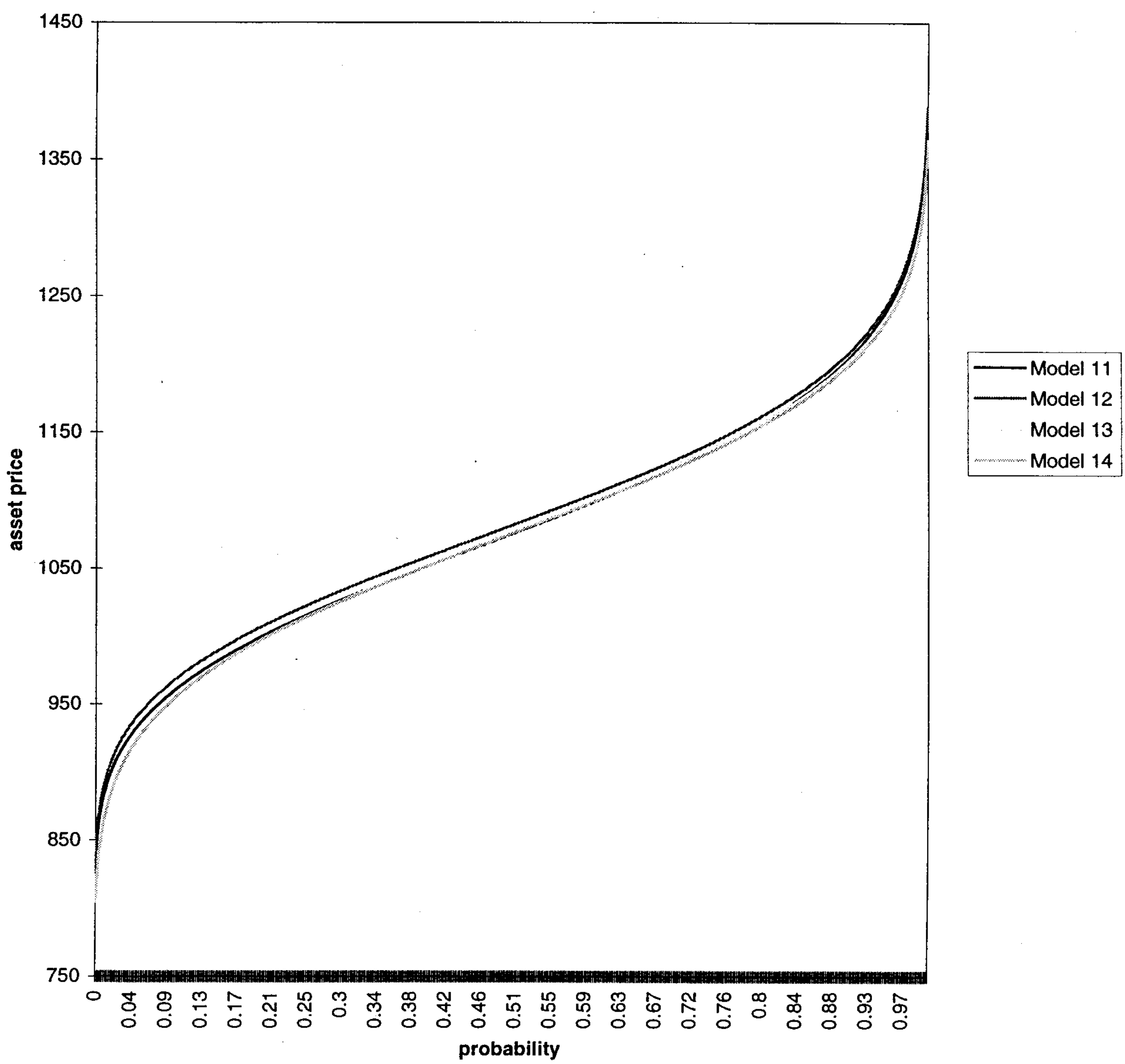


Figure 2:

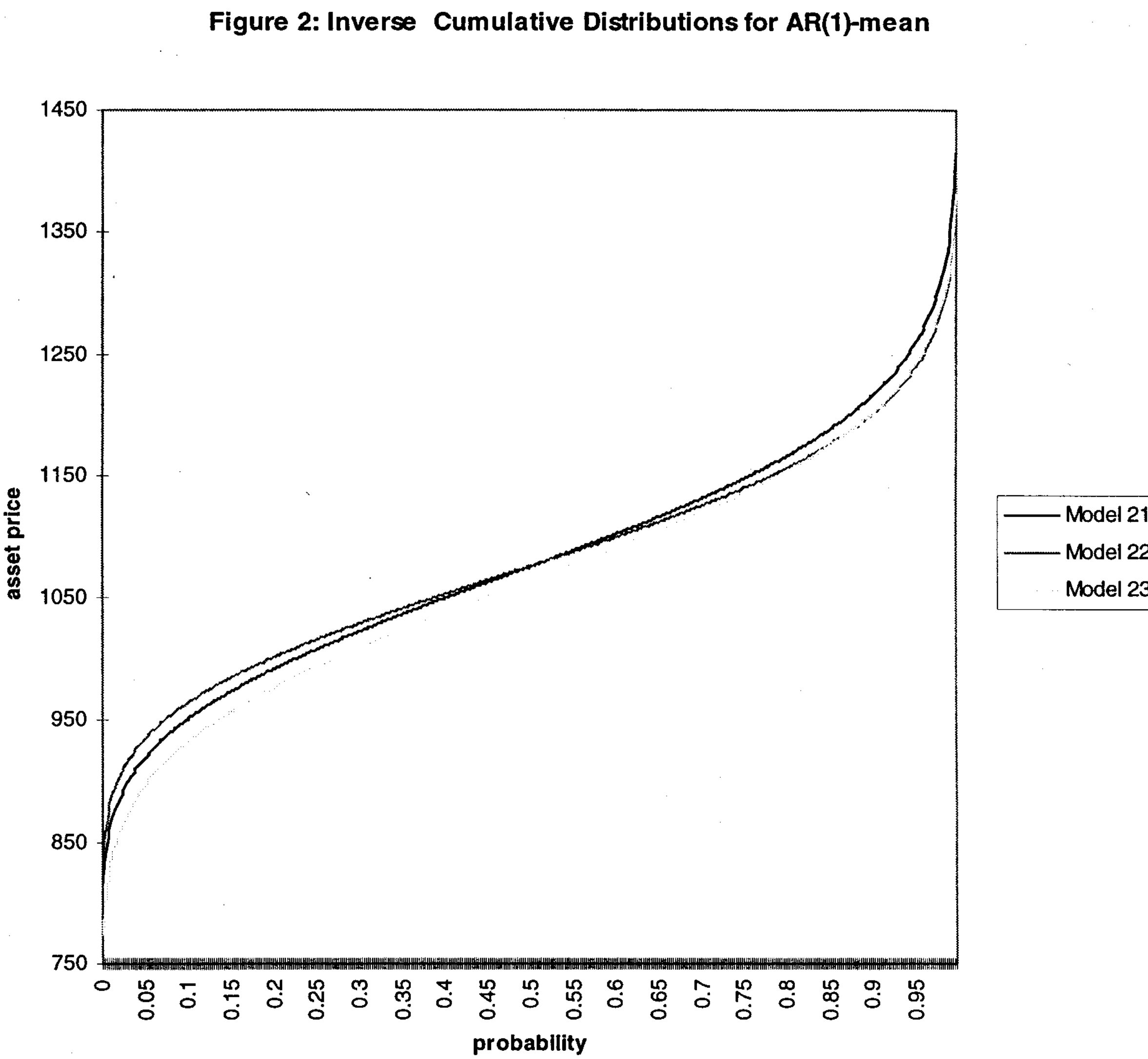
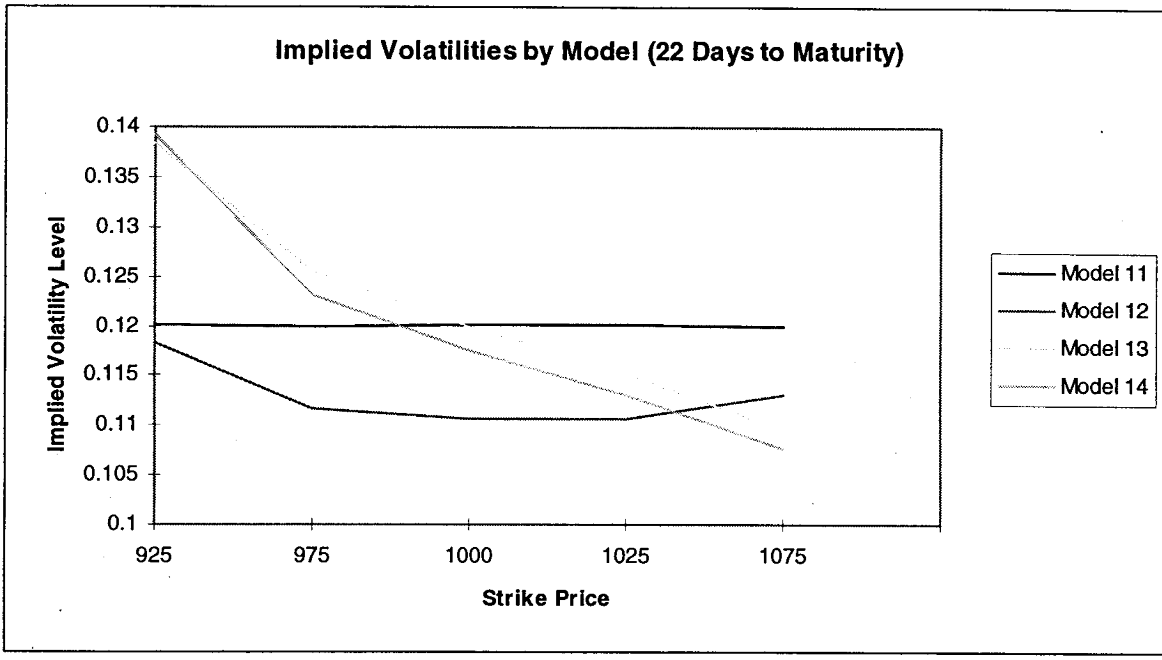
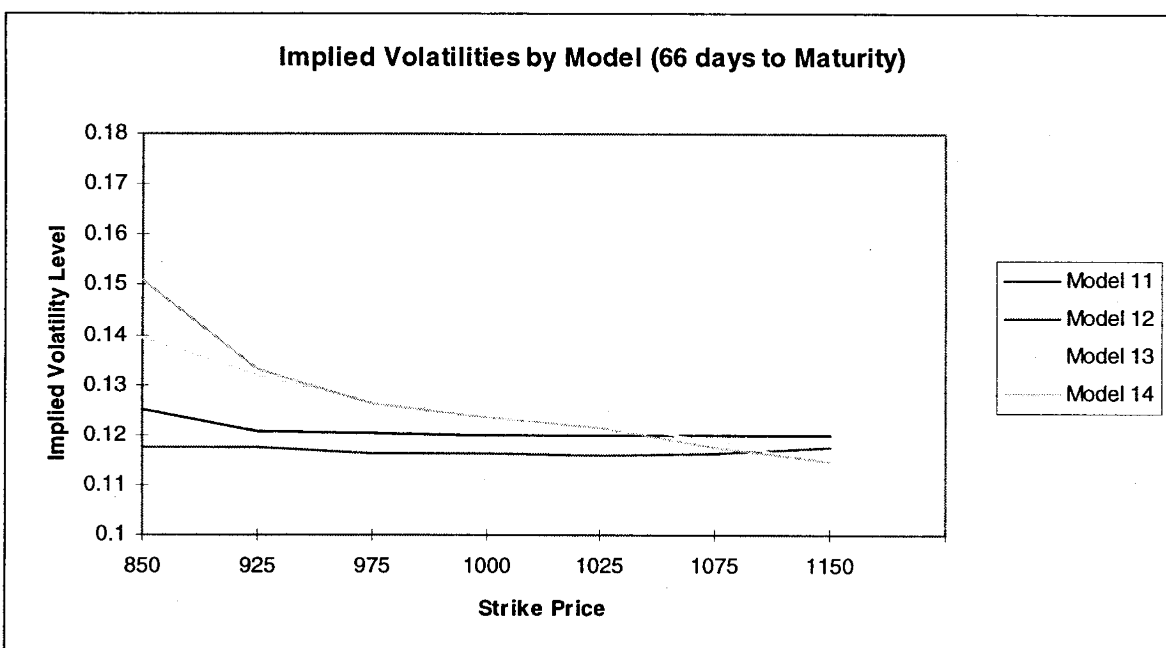
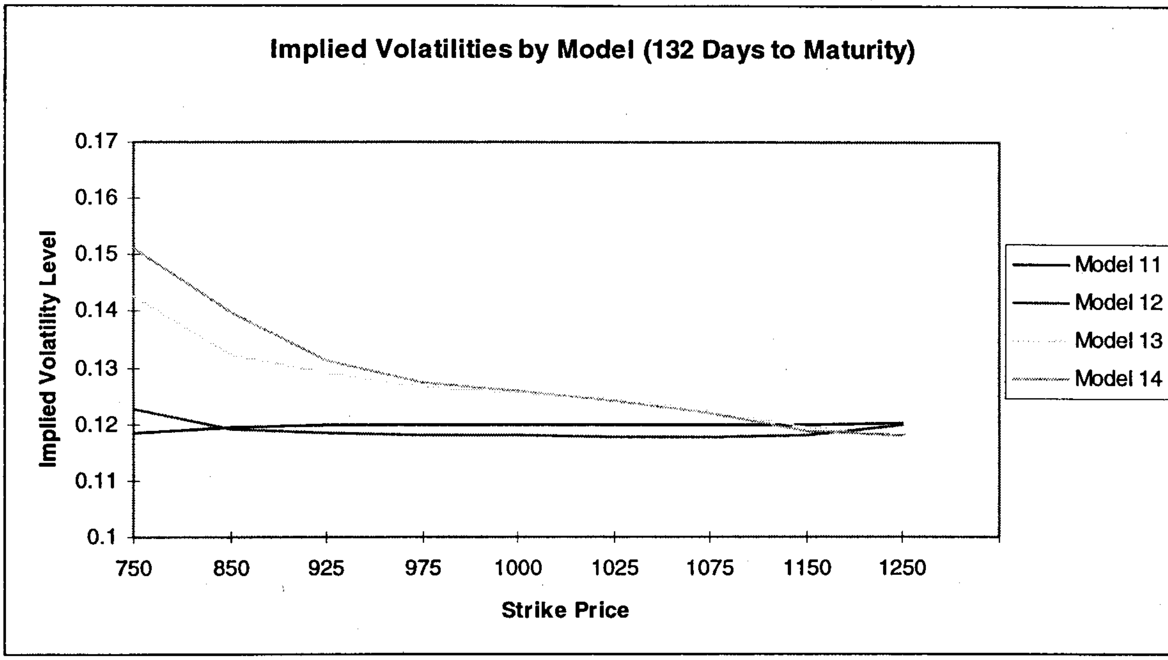


Figure 2: Inverse Cumulative Distributions for AR(1)-mean
1450
1350
1250
1150
1050
950
850
750
asset price
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
0.75
0.8
0.85
0.9
0.95
probability
Model 21
Model 22
Model 23

Figure 3: Implied Volatilities

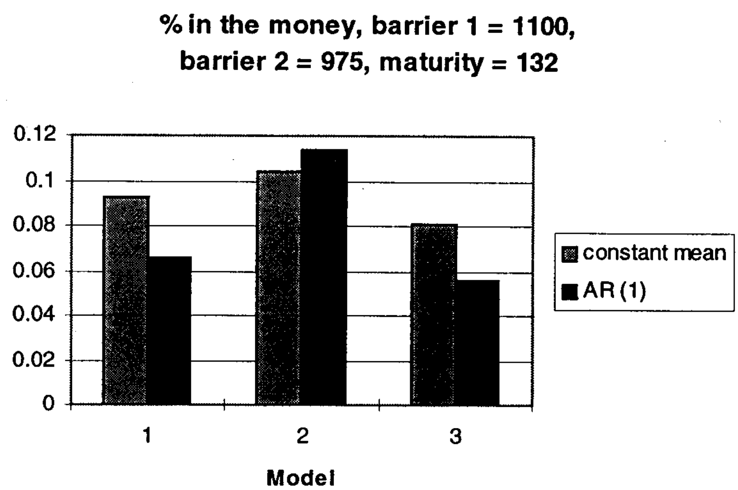






Implied Volatilities by Model (132 Days to Maturity)
0.17
0.16
0.15
0.14
0.13
0.12
0.11
0.1
Implied Volatility Level
750
850
925
975
1000
1025
1075
1150
1250
Strike Price
Model 11
Model 12
Model 13
Model 14

Figure 4: Double Barrier Binary Options, Percent in the Money



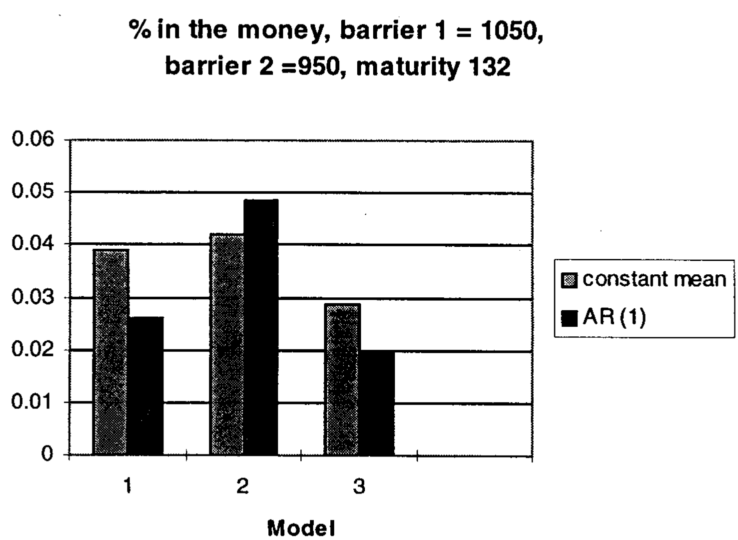
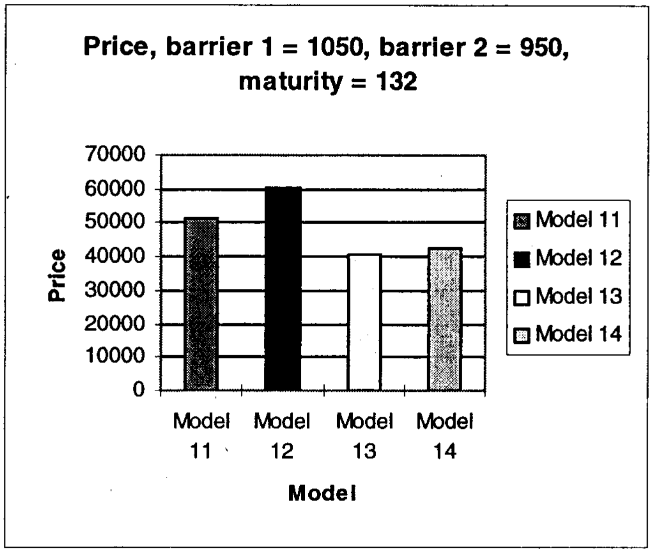


Figure 5: Double Barrier Binaries:, Prices



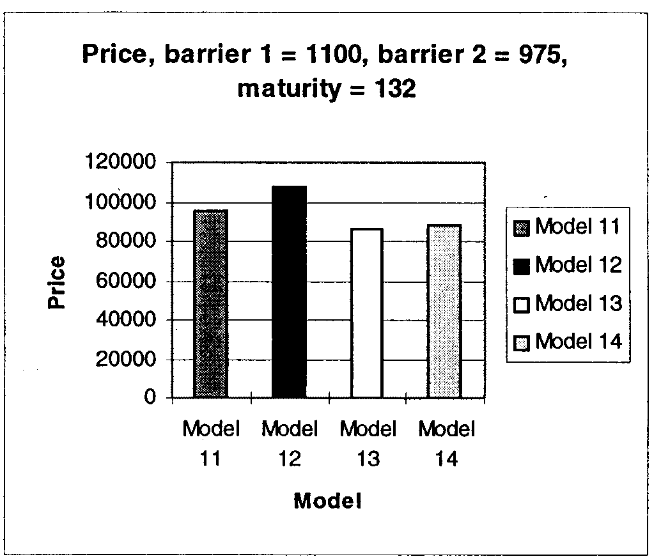


Figure 6: Relative Pricing Differences