Dynamic Hedging of Synthetic CDO Tranches with Spread Risk and Default Contagion

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Abstract

The paper is concerned with the hedging of credit derivatives, in particular synthetic CDO tranches, in a dynamic portfolio credit risk model with spread risk and default contagion. The model is constructed and studied via Markov-chain techniques. We discuss the immunization of a CDO tranche against spread- and event risk in the Markov-chain model and compare the results with market-standard hedge ratios obtained in a Gauss copula model. In the main part of the paper we derive model-based dynamic hedging strategies and study their properties in numerical experiments.

Keywords: Dynamic hedging, portfolio credit risk, credit derivatives, incomplete markets, default contagion.

1 Introduction

The risk management for books of synthetic CDO tranches has become an issue of high concern for many investors on credit markets. Typically an investor has taken a protection-seller position in one or several CDO tranches and tries to offset the ensuing risk by taking an opposite (protection-buyer) position in the single-name credit default swaps (CDSs) or in the CDS-index underlying the tranche. In practice, the size of the hedging positions is determined by a pragmatic approach, akin to the use of duration in interest rate risk management: in order to protect a CDO tranche against fluctuations in credit spreads the tranche is first priced via the Gauss copula model, using observed CDS spreads and implied-correlation methodology to determine the model parameters. Next one varies the swap spread of one of the underlying names, name k, say, and defines the so-called *spread delta* of that name as the ratio of the change in the market value of the CDO tranche and of a CDS on name k. The hedge ratios immunizing the tranche against a change in the index spread are determined in a similar way. Sometimes investors also seek to protect their position against defaults in the underlying reference pool (hedging of jump-to-default risk). The hedge ratio immunizing a CDO tranche against the default of firm k is known as *jump-to-default ratio*; it is computed as the ratio of the loss due to the default of that name in the tranche and in the CDS on name k. In computing these losses it is assumed that the credit spreads of the surviving firms are not affected by the default event. Further details on market-standard hedging practice can for instance be found in Neugebauer (2006).

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The market-standard approach has a number of problems: contagion effects (the fact that credit spreads of surviving firms often jump in reaction to default events) are neglected, and there is no theoretically consistent methodology supporting the definition of spread deltas and jump-to-default ratios. These issues are clearly important also from a practical point of view. To begin with, the current financial crisis underlines the relevance of default contagion on credit markets - just think of the events surrounding the default of Lehman Brothers - and neglecting contagion effects may lead to inappropriate hedge ratios. Moreover, it is well-known from markets for other types of derivatives that ad-hoc hedging strategies frequently lead to unaccounted drift- and time-decay effects (see for instance El Karoui, Jeanblanc-Picqué & Shreve (1998)). The lack of a sound hedging methodology for portfolio credit derivatives is of course closely related to the fact that the market-standard copula models are static, so that theoretically consistent dynamic hedging strategies cannot be derived in copula models. Note that this deficiency is inherent in the copula framework; it cannot be rectified by using more sophisticated copulas than the Gauss copula.

In this paper we make an attempt to address these issues. In Section 2, we propose a dynamic credit risk model which allows for the explicit modelling of default contagion and spread risk, and which is therefore an ideal workbench for analyzing the hedging of CDO tranches. The model belongs to the class of models with interacting default intensities such as Jarrow & Yu (2001), Davis & Lo (2001), Giesecke & Weber (2006), Bielecki & Vidozzi (2008), or Herbertsson (2008); it is closely related to the Markov-chain models within the so-called top-down approach to credit portfolio modelling studied by Arnsdorf & Halperin (2007), Lopatin & Misirpashaev (2007) or Cont & Minca (2008). In Section 3 we give a formal description of the cash-flow dynamics of CDSs and CDOs. In Section 4 we compute jump-to-default ratios and spread deltas for the Markov-chain model and compare the results with the market-standard values from a Gauss copula model. It turns out that in many cases the hedge ratios differ substantially, mainly because of contagion effects. In Section 5 we study the dynamic replication of CDO tranches using martingale representation results for marked point processes. We find that in the special case where credit spreads evolve deterministically between default times (pure jump-to-default risk) the market is complete; the dynamic replication strategy coincides with the jump-to-default ratio for the Markov-chain model. With spread risk and jump-to-default risk on the other hand markets are typically incomplete, so that we resort to the concept of risk-minimization introduced by Föllmer & Sondermann (1986). Numerical experiments further illustrate certain properties of risk-minimizing hedging strategies. It is shown that risk-minimizing hedging strategies interpolate between the hedging of spread- and jump-to-default risk and that deviations from the popular assumption of a homogeneous portfolio can have a sizeable impact on the form and on the performance of hedging strategies. At this point it is worth mentioning that while in the present paper we concentrate on CDO tranches, the theoretical results we obtain apply to many other credit derivatives such as index spread options or basket swaps, often with merely notational changes.

The dynamic hedging of credit risky securities is studied among others by Bielecki, Jeanblanc & Rutkowski (2004), Elouerkhaoui (2006), Bielecki, Jeanblanc & Rutkowski (2007) and Laurent, Cousin & Fermanian (2008). The latter two papers are closely related to our contribution. Laurent et al. (2008) study the hedging of CDO tranches via dynamic trading in CDS indices in the Markov-chain model of Frey & Backhaus (2008). However, they concentrate on the case without spread risk and hence on complete markets. Bielecki et al. (2007) derive interesting theoretical results on the hedging of basket swaps in a generic dynamic credit portfolio model without spread risk. Since the hedging against random fluctuations of credit spreads is an issue of high concern for practitioners, we believe that the inclusion of spread risk and the application of incomplete-market methodology is an important extension over these papers. Rosen & Saunders (2009) derive interesting results on *static* hedging strategies for CDOs.

2 The model

Notation. Fix some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$. The σ-field $\mathcal{F}_t$ represents the information available to investors at time t; all processes introduced below will be $(\mathcal{F}_t)$-adapted. We consider a fixed portfolio of m firms, indexed by $i \in \{1, \ldots, m\}$. The $(\mathcal{F}_t)$-stopping time τ_i with values in $(0, \infty)$ represents the default time of firm i. The default state of the portfolio is thus described by the *default indicator process* $Y = (Y_{t,1}, \ldots, Y_{t,m})_{t \geq 0}$ where $Y_{t,i} = 1_{\{\tau_i \leq t\}}$; note that $Y_t \in \{0,1\}^m$. For simplicity we assume that the that the exposure of each firm is normalized to one. Denoting the percentage loss given default (LGD) of firm i at time t by the predictable random variable $\delta_{t,i} \in (0, 1]$, the *loss state* $L_t = (L_{t,1}, \ldots, L_{t,m})$ of the portfolio and the *aggregate portfolio loss* $\overline{L}_t$ at time t are given by

$$L_{t,i} = \int_0^t \delta_{s,i} \, dY_{s,i} \,, \; 1 \leq i \leq m, \quad \text{ and } \overline{L}_t = \sum_{i=1}^m L_{t,i} \,. \tag{1}$$

Moreover, Y_t can be recovered from L_t via $Y_{t,i} = 1_{\{L_{t,i} > 0\}}$, $1 \leq i \leq m$. Since we consider only models without simultaneous defaults, we can define the ordered default times $T_0 < T_1 < \ldots < T_m$ recursively by

$$T_0 = 0 \text{ and } T_n = \min\{\tau_i : \, 1 \leq i \leq m, \, \tau_i > T_{n-1}\} \,, 1 \leq n \leq m \,.$$

By $\xi_n \in \{1, \ldots, m\}$ we denote the identity of the firm defaulting at time T_n, i.e. $\xi_n = i$ if $T_n = \tau_i$. We use the following notation for flipping the ith coordinate of a default state: given $y \in \{0,1\}^m$ we define $y^i \in \{0,1\}^m$ by $y^i_i := 1 - y_i$ and $y^i_j := y_j, \; j \in \{1, \ldots, m\} \setminus \{i\}$.

General setup. Following the literature we assume throughout the paper that the default-free interest rate is deterministic and equal to $r(t) \geq 0$; $p_0(t,T) = e^{-\int_t^T r(s) ds}$ denotes the price of the default-free zero-coupon bond with maturity T. Moreover, it is assumed that the measure Q represents the risk-neutral measure used for pricing, so that the price of any $\mathcal{F}_T$-measurable claim H is given by

$$H_t := p_0(t,T) E^Q(H \mid \mathcal{F}_t) \,, \quad t \leq T \,. \tag{2}$$

It is standard practice in the literature to construct portfolio credit risk models directly under a risk-neutral measure Q (martingale modelling), because pricing and calibration is done under Q anyhow. We come back to this issue in Remark 5.1.

Next we turn to modelling the Q-dynamics of the default indicator process. For this we introduce an $(\mathcal{F}_t)$-adapted factor process Ψ representing for instance the macroeconomic environment; the overall state of the economic system is described by the pair process $\Gamma^Y := (Y_t, \Psi_t)_{t \geq 0}$. For tractability reasons Ψ is modelled as a finite-state Markov chain with state-space $S^\Psi = \{\psi_1, \ldots, \psi_K\}$ and generator matrix $\boldsymbol{q}^\Psi = \big(q^\Psi(\psi_i, \psi_j)\big)_{1 \leq i,j \leq K}$; this can be viewed as an approximation to the more standard jump diffusion dynamics for factor processes. We assume that the default intensity of firm i is given by some nonnegative bounded function $\lambda_i(t, \Psi_t, Y_t)$. In this way fluctuations in Ψ will lead to random fluctuations in credit spreads between default times (spread risk). Moreover, since λ_i depends on the current portfolio state Y_t, the default intensity of a firm may change if there is a change in the default state of the portfolio, so that default contagion can be modelled explicitly. In addition, we assume the LGD is *state-dependent* and of the form $\delta_{t,k} = \delta_k\big(Y_{t-}\big)$ for some function $\delta\colon \{0,1\}^m \to (0,1]$.

The mathematical properties of Γ^Y are summarized in the next assumption.

Assumption 2.1. The process Γ^Y is a Markov chain on $(\Omega, \mathcal{F}, (\mathcal{F}_t), Q)$ with state space $S^{\Gamma^Y} := \{0,1\}^m \times S^\Psi$ and transition intensities given by

$$q_t^{\Gamma^Y}\big((y,\psi),(\tilde{y},\tilde{\psi})\big) := \begin{cases} q^\Psi(\psi,\tilde{\psi}), & \text{if } \tilde{y} = y \text{ and } \tilde{\psi} \neq \psi \ , \\ (1-y_i)\,\lambda_i(t,\psi,y), & \text{if } \tilde{\psi} = \psi \text{ and } \tilde{y} = y^i \text{ for some } 1 \le i \le m, \\ 0, & \text{else.} \end{cases} \tag{3}$$

We discuss some of the implications of Assumption 2.1. First, simultaneous defaults of several firms are excluded by assumption. Second, the probability that firm i defaults in the small time interval $[t, t+h)$ corresponds to the probability that $\Gamma_t^Y = (Y_t, \Psi_t)$ jumps to the new state $\big(Y_t^i, \Psi_t\big)$ in this time period. Such a transition occurs with rate $\lambda_i(t, \Psi_t, Y_t)$, so that this quantity is in fact the default intensity of firm i at time t; a formal argument is given in Frey & Backhaus (2008). Third, the form of the transition intensities in (3) implies that the process Ψ is individually Markov with generator matrix $\boldsymbol{q}^\Psi$ and that there are no joint jumps of Y and Ψ. From a mathematical point of view this assumption could be relaxed without major difficulties; it reflects the fact that we interpret Ψ as an exogenous factor process whose dynamics are not affected by defaults in the portfolio.

For further use we note that due to the assumption of a state-dependent LGD, under Assumption 2.1 the pair process (Ψ, L) is Markov as well; the generator of Γ^L is given by

$$\begin{aligned} \mathcal{L}_{\Gamma^L} f(\psi, l) = &\sum_{\tilde{\psi} \neq \psi} q^\Psi(\psi,\tilde{\psi})\big(f(\tilde{\psi}, l) - f(\psi, l)\big) \\ &+ \sum_{i=1}^m 1_{\{l_i = 0\}} \lambda_i(\psi, y)\big\{f\big(\psi, (l_1, \ldots, \delta_i(y), \ldots, l_m)\big) - f(\psi, l)\big\}, \end{aligned} \tag{4}$$

where of course $y = y(l) = \big(1_{\{l_1>0\}}, \ldots, 1_{\{l_m>0\}}\big)$.

Homogeneous-portfolio models. Homogeneous models with exchangeable loss processes are an important special case of Assumption 2.1. Assuming exchangeable loss processes is admittedly somewhat unrealistic for many credit portfolios such as the pool of names underlying the i-Traxx index. However, homogeneity drastically simplifies the numerical treatment of the model and is therefore frequently assumed in the literature. As discussed in Frey & Backhaus (2008), with exchangeable loss processes default intensities and loss given default take the form

$$\lambda_i(t,\psi,y) = h\big(t,\psi,\sum_{i=1}^m y_i\big) \ \text{ and } \ \delta_i(y) = \delta\big(\sum_{i=1}^m y_i\big)\,, \tag{5}$$

for functions $h\colon [0,\infty) \times S^\Psi \times \{0,\ldots,m\} \to (0,\infty)$ and $\delta\colon \{0,\ldots,m-1\} \to (0,1]$. It is well known that the introduction of an increasing LGD-function improves the fit of CDO-pricing models to observed CDO spreads, see for instance Frey & Backhaus (2008) or Andersen & Sidenius (2004). Concerning the modelling of default intensities, we mostly use the following parametric form for the function h, labelled *convex counterparty risk model* (Frey & Backhaus (2008)),

$$h(t,\psi,n)=\lambda_0\psi+\tfrac{\lambda_1}{\lambda_2}\Big(\exp\big(\lambda_2\tfrac{[n-\mu(t)]^+}{m}\big)-1\Big)\,,\quad \lambda_0>0,\,\lambda_1,\lambda_2\geq 0\,. \tag{6}$$

Here $\mu(t)$ is some deterministic threshold measuring the expected number of defaults up to time t, typically obtained by calibration to an observed CDS index spread. The first summand of the intensity function, $\lambda_0\psi$, gives the linear dependency on the factor process; the parameter λ_0 mainly determines the credit quality of firms in the portfolio. The second term models contagion effects: for $n>\mu(t)$ the default intensity $h(t,\psi,n)$ is larger than $\lambda_0\psi$. The parameter λ_1 gives the slope of function $l\mapsto h(t,\psi,n)$ for $l\downarrow\mu(t)$, so that λ_1 models the strength of the default contagion for a 'normal' number of defaults. Note finally that h is convex in n with the degree of convexity being controlled by λ_2. Convexity of h implies that a large number of defaults leads to very high values of default intensities, thus triggering a cascade of further defaults. We stress at this point that other parametric forms of $h(\cdot)$ such as the model proposed by Herbertsson (2008) are well within the reach of our analysis.

Denote by $M_t:=\sum_{i=1}^m Y_{t,i}$ the number of defaulted firms by time t. With default intensities of the form (5), the process $\Gamma^M=(M_t,\Psi_t)_{t\geq 0}$ is a Markov chain with state space $S^{\Gamma^M}:=\{0,\dots,m\}\times S^{\Psi}$; see Lemma 2.4 of Frey & Backhaus (2008). Moreover, in the homogeneous case we have $\overline{L}_t=\sum_{i=1}^{M_t}\delta(i-1)$ so that there is a one-to-one relation between M_t and the aggregate portfolio loss $\overline{L}_t$.[2] The Markov chain Γ^M thus provides a self-consistent model for the dynamics $\overline{L}$, and the homogeneous model can be embedded in the so-called *top-down-approach* to credit portfolio modelling. In this approach the the aggregate loss $\overline{L}$ is the modelling primitive and information related to credit quality and default state of individual names is ignored; see for instance Giesecke & Goldberg (2007), Schönbucher (2006), Arnsdorf & Halperin (2007), Lopatin & Misirpashaev (2007) or Cont & Minca (2008). In fact, from a mathematical viewpoint the homogeneous version of our model is essentially equivalent to the Markov-chain models considered in the latter three papers.

Numerical treatment and calibration. Semi-analytic and numeric methods for computing prices of credit derivatives in Markov-chain models are discussed among others in Frey & Backhaus (2008) and in Herbertsson (2008). It turns out that for relatively homogeneous portfolios analytic approaches based on the Kolmogorov equations or on matrix exponentials work quite well. For large heterogeneous portfolios on the other hand one has to resort to simulation approaches as in (Crépey & Carmona 2008); this is due to the fact that $\left|S^{\Gamma^Y}\right|=2^m\cdot\left|S^{\Psi}\right|$ which is prohibitively large unless m is a small number. This curse of dimensionality plagues most dynamic credit portfolio models with default contagion and is one of the reasons for the popularity of top-down approaches.

In principle a heterogeneous model should be calibrated to single-name CDS spreads and - if available - to observed spreads of portfolio products such as CDOs or basket swaps. While heterogeneous models are typically sufficiently rich in parameters for such a calibration exercise, the ensuing computations are very challenging from a numerical point of view, unless the portfolio is small or has a homogeneous-group structure as in the example considered in Section 5.4. Homogeneous models on the other hand imply identical CDS spreads for all names in the portfolio by definition, so that these models can be calibrated only to observed index- and tranche

[2] Note that in the heterogeneous case on the other hand the current loss state L_t cannot be inferred from the current default state Y_t; nonetheless with state-dependent recovery rates L is of course adapted to the default history (F_t^Y).

spreads. This is in line with the modelling philosophy of the top-down approach where information related to individual firms is ignored. Within the top-down approach various sophisticated calibration methodologies have been developed (see in particular Arnsdorf & Halperin (2007) and Cont & Minca (2008)); these techniques can be applied to to the homogeneous-portfolio version of our model in a straightforward way.

The natural area of application of heterogeneous models is thus the hedging of basket swaps where the underlying portfolio is small and where one uses single name CDSs as hedging instrument. For applications to CDO tranches on the other hand one has to resort to fairly homogeneous Markov-chain models; in that case the natural hedging instrument is the index on the underlying portfolio. In the subsequent analysis we nevertheless derive hedging strategies for the general model introduced in Assumption 2.1, using single-name CDSs as hedging instrument. Specializing these results to the homogeneous case we then obtain the hedging strategy in the corresponding index; this is also the hedging strategy in a top-down model where the portfolio-loss dynamics are given by the Markov-chain Γ^M. Working in the inhomogeneous setup permits us in particular to study the impact of portfolio-heterogeneity on the form and the performance of the model-based hedge ratios (see Section 5.4.) This sheds some light on the robustness of our results with respect to the somewhat unrealistic homogeneous-portfolio assumption and illustrates certain limitations of the top-down approach for hedging purposes. Moreover, in this way the hedging of basket credit derivatives is included in our theoretical results.

Example 2.2. In the numerical experiments we mostly work with the following parametrization of the homogeneous-portfolio model: the riskless interest rate is equal to $r = 0$; default intensities are given by the convex counterparty risk model (6); the LGD was taken to be $\delta \equiv 0.6$. We have chosen four different grids of different coarseness for S^Ψ, generating different levels of spread-volatility: S_0^Ψ corresponds to a constant factor process, S_1^Ψ corresponds a low spread volatility, S_2^Ψ to a medium volatility and S_3^Ψ to a relatively high volatility of credit spreads. Throughout we take $\left|S_i^\Psi\right| = 11$ and $\Psi_0 = 0.005$; moreover, it is assumed that Ψ can only jump to neighboring states with transition intensity $\nu^\Psi = 5.0$, i.e. the generator matrix $\boldsymbol{q}^\Psi$ of Ψ is tridiagonal with off-diagonal elements equal to ν^Ψ. In order to obtain reasonable parameter values the model was calibrated to iTraxx tranche spreads from January 2006; see Table 8 in the appendix for the exact values. These spreads were representative for the iTraxx before the credit crisis[3]; during the credit crisis iTraxx spreads have changed dramatically. Unfortunately we were not able to repeat the entire numerical analysis of the paper with new data. However, in Section 4 we briefly indicate how the change in market spreads affects the hedge ratios generated by the Markov chain model.

The sets $S_0^\Psi, \dots, S_3^\Psi$ and the corresponding parameters of the convex counterparty risk model (6) are given in Table 1. Note that with increasing spread volatility the contagion parameter λ_1 is reduced in the calibration procedure, as a large part of the dependence between defaults is generated by fluctuations in the common factor Ψ.

3 Credit Derivatives

In this section we discuss the payments, the market value and the gains process of CDSs and CDOs; this serves to set up the notation and is moreover a necessary prerequisite for studying

[3]The first version of the paper was written early in 2007

S^{Ψ}	λ_0	λ_1	λ_2
$S_0^{\Psi} = \{0.0050, 0.0050, \dots, 0.0050, \dots, 0.0050, 0.0050\}$	0.85910	0.18803	22.125
$S_1^{\Psi} = \{0.0045, 0.0046, \dots, 0.0050, \dots, 0, 0054, 0.0055\}$	0.85827	0.18789	22.126
$S_2^{\Psi} = \{0.0025, 0.0030, \dots, 0.0050, \dots, 0.0070, 0.0075\}$	0.87052	0.16363	24.372
$S_3^{\Psi} = \{0.0005, 0.0014, \dots, 0.0050, \dots, 0.0086, 0.0095\}$	0.89696	0.11306	30.538

Table 1: State space of Ψ and corresponding model parameters.

the dynamic hedging of portfolio credit derivatives. The gains process of a security/position is the sum of the current market value and of the cumulative cash-flows associated with the position (spread-, interest- and default payments). Since we are mainly interested in the case where an investor tries to hedge a protection-seller position in a CDO tranche by taking a protection-buyer position in the CDSs or in the CDS index underlying the transaction, we model the cash flows of the CDO tranche from the viewpoint of a protection seller and the cash flows of CDSs from the viewpoint of a protection buyer.

In the following we denote by T the maturity of all credit products considered. The spread payments of all contracts are scheduled at N payment dates $0 < z_1 < \cdots < z_N = T$. We define $z_0 := 0$, $\Delta z_n := z_n - z_{n-1}$, and, for $t \geq 0$,

$$n_t^z := \big|\{i = 1, \dots, N \colon z_i \leq t\}\big|; \tag{7}$$

note that n_t^z is the number of the spread payment dates up to time t. The LGD $\delta_{t,k} = \delta_k(Y_{t-})$ is assumed to be state dependent.

Single-name CDSs. The market value $V_{t,k}^{\text{CDS}}$ of a protection-buyer position in a CDS on firm k with fixed spread s^k is given by the difference between the value of the default payment and the value of the future premium payments (regular and accrued). Hence we get

$$\begin{aligned} V_{t,k}^{\text{CDS}} =& 1_{\{\tau_k > t\}} E^Q \Bigg(\delta_{\tau_k,k}\, p_0(t,\tau_k) 1_{\{\tau_k \leq T\}} \\ & - s^k \sum_{n=n_t^z+1}^{N} \Big((\Delta z_n) p_0(t, z_n) 1_{\{\tau_k > z_n\}} + (\tau_k - z_{n-1}) p_0(t,\tau_k) 1_{\{z_{n-1} < \tau_k \leq z_n\}} \Big) \Big| \mathcal{F}_t \Bigg). \end{aligned} \tag{8}$$

In the Markov-chain model the Markov-property of Γ^Y and the assumption of a state-dependent LGD imply that $V_{t,k}^{\text{CDS}} = v_k^{\text{CDS}}(t, Y_t, \Psi_t)$ for some function $\tilde{v}_k^{\text{CDS}} \colon [0,T] \times \{0,1\}^m \times S^{\Psi} \to \mathbb{R}$. Note that at a spread payment date $z_n < \tau_k$ there is a jump of size $s^k \Delta z_n > 0$ in the market value and that $V_{t,k}^{\text{CDS}} \equiv 0$ for $t \geq \tau_k$. The gains process G_k^{CDS} has dynamics

$$dG_{t,k}^{\text{CDS}} = -s^k \{(\Delta z_{n_t^z})(1 - Y_{t,k})\}\, dn_t^z + \Big(\delta_{t,k} - s^k(t - z_{n_t^z})\Big) dY_{t,k} + dV_{t,k}^{\text{CDS}}, \quad t \leq T\,. \tag{9}$$

For convenience premium payments are sometimes modelled by an absolutely-continuous payment stream with rate s^k. In that case (9) simplifies to

$$dG_{t,k}^{\text{CDS}} = -s^k(1 - Y_{t,k})dt + \delta_{t,k} dY_{t,k} + dV_{t,k}^{\text{CDS}}, \quad t \leq T. \tag{10}$$

CDS indices. The payoff of a CDS-index with fixed spread s^{Ind} on the reference pool equals the payoff of a portfolio consisting of one single-name CDS per name with identical spread $s^k = s^{\mathrm{Ind}}$, $k = 1, \dots, m$. Define the remaining notional of the index at time t as $N_t^{\mathrm{Ind}} := m - M_t$ (the number of surviving firms at time t). The cash-flow stream of the default-payment leg is given by the cumulative portfolio loss $\overline{L}$; the future regular premium payments can be written in the form $s^{\mathrm{Ind}} \sum_{n=n_t^z+1}^{N} (\Delta z_n) N_{z_n}^{\mathrm{Ind}}$ and the accrued premium payments are given by $s^{\mathrm{Ind}} \sum_{k=1}^{m} (T_k - z_{n-1}) 1_{\{z_{n-1} < T_k \le z_n\}}$. The market-value and the gains process can be computed as in (8), (9) or (10).

Remark 3.1. It will turn out that in a homogeneous portfolio the hedge ratios of a CDO tranche with respect to the individual CDSs are identical, $\theta_{t,k} \equiv \theta_t$ for all k. In that case a hedging strategy can be implemented by taking a protection-buyer position of size θ_t directly in the CDS index, which is much easier than running a dynamic portfolio strategy in, say, $m = 125$ single-name CDSs. Obviously θ_t is also the hedging strategy in a top-down model where the portfolio-loss dynamics are given by the Markov-chain Γ^M.

CDO Tranches. A synthetic CDO tranche on the reference portfolio is characterized by fixed lower and upper attachment points $0 \le l < u \le 1$. The tranche consists of a default payment leg and a premium payment leg. Define the cumulative tranche loss $L_t^{[l,u]}$ by

$$L_t^{[l,u]} := (\overline{L}_t - ml)^+ - (\overline{L}_t - mu)^+ \,, \tag{11}$$

i.e. the part of $\overline{L}_t$ falling in the layer $[l, u]$, and denote the remaining notional of the tranche by $N_t^{[l,u]} := m(u - l) - L_t^{[l,u]}$. At a default time $T_k \le T$ there is a default payment of size

$$\Delta L_{T_k}^{[l,u]} := L_{T_k}^{[l,u]} - L_{T_k-}^{[l,u]} \,.$$

The premium payment leg consists of regular and accrued premium payments. The regular premium payment at date z_n is given by $s^{[l,u]}(\Delta z_n) N_t^{[l,u]}$, $s^{[l,u]}$ the annualized tranche spread. The accrued payment at a default time $\tau \in (z_n, z_{n+1}]$ equals $s^{[l,u]}(\tau - z_n)\Delta L_\tau^{[l,u]}$. For the equity tranche there is moreover an upfront payment at $t = 0$, quoted in the form $s^{\mathrm{upf}} N_0^{[l,u]}$, s^{upf} the so-called upfront spread. The market value of a protection-seller position equals

$$\begin{aligned} V_t^{[l,u]} = & E^Q \Bigg(- \int_t^T p_0(t,s) dL_s^{[l,u]} + s^{[l,u]} \sum_{n=n_t^z+1}^{N} \Big\{ p_0(t, z_n)(\Delta z_n) N_{z_n}^{[l,u]} \\ & + \sum_{k=1}^{m} p_0(t, T_k)(T_k - z_{n-1}) \Delta L_{T_k}^{[l,u]} 1_{\{z_{n-1} < T_k \le z_n\}} \Big\} \mid \mathcal{F}_t \Bigg) . \end{aligned}$$

Note that by the Markovianity of Γ^L, in the Markov-chain model $V^{[l,u]} = v^{[l,u]}(t, L_t, \Psi_t)$ for some function $v^{[l,u]} \colon [0,T] \times (0,1]^m \times S^\Psi \to \mathbb{R}$. The gains process $G_t^{[l,u]}$ has dynamics

$$dG_t^{[l,u]} = s^{[l,u]}(\Delta z_{n_t^z})\, N_t^{[l,u]} \, dn_t^z + s^{[l,u]}(t - z_{n_t^z}) \, dL_t^{[l,u]} - dL_t^{[l,u]} + dV_t^{[l,u]}. \tag{12}$$

If spread payments are modeled as an absolutely continuous payment stream, (12) becomes

$$dG_t^{[l,u]} = s^{[l,u]} N_t^{[l,u]} dt - dL_t^{[l,u]} + dV_t^{[l,u]}. \tag{13}$$

Market values at a default time. At a default time T_k the market value of a CDO tranche changes for a number of reasons: first, the increase in $\overline{L}_t$ at T_k makes it more likely that a tranche with $ml > \overline{L}_t$ will be hit in the future; second, if $\Delta L_{T_k}^{[l,u]} > 0$, there is a change in the remaining nominal of the tranche affecting the size of future premium payments. Moreover, there may be an indirect contagion effect: with default contagion, the default event affects the default intensities of the surviving firms and thereby the market value of the tranche. This contagion effect also has an impact on the market value of a non-defaulted CDS. Table 2 shows that the indirect contagion effect can be quite substantial. In this table we compare the change in market value of a non-defaulted CDS and of various CDO tranches at a default time a) in a Gauss copula model and b) in the homogeneous Markov-chain model described in Example 2.2. In case a) we follow standard market practice and assume that the default event has no impact on the default probability of surviving firms. It turns out that the change in market value is much larger for the Markov-chain model with contagion effects than for the Gauss copula model. Moreover, the change in the market value decreases with increasing spread-volatility and hence decreasing interaction-parameter λ_1. It will turn out below that this has a substantial impact on the form of the ensuing hedge ratios.

Product	CDS	[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Gauss Copula	0.0000	-0.001	-0.103	-0.014	-0.0049	-0.0062
Markov chain, S_0^Ψ	0.0179	-0.369	-0.388	-0.163	-0.1101	-0.3018
Markov chain, S_3^Ψ	0.0114	-0.233	-0.252	-0.091	-0.0645	-0.1939

Table 2: Change in market value $\Delta V_t^{\mathrm{CDS}}$ and $\Delta V_t^{[l,u]}$ at $t = T_1$. Model parameters are given in Example 2.2. The opposite signs in $\Delta V_t^{\mathrm{CDS}}$ and $\Delta V_t^{[l,u]}$ are due to the fact that we consider a protection-buyer position in the CDS and a protection-seller position in the CDO tranches.

4 Sensitivity-based hedging with default contagion

In this section we consider the sensitivity-based hedging strategies used in practice (see for instance Neugebauer (2006)) in the context of the Markov-chain model with spread risk and default contagion.

Immunization against spread risk. Market practitioners frequently immunize a protection-seller position in a CDO tranche against fluctuations in the spread of the underlying CDS index. Following market practice (Neugebauer (2006)), we define the corresponding hedge ratio - the so-called index-spread delta of a CDO tranche - at a given spread-level s as

$$\Delta_t^{\mathrm{spread}} := -\frac{V_t^{[l,u]}|_{s^{\mathrm{Ind}}=s+1\mathrm{bp}} - V_t^{[l,u]}|_{s^{\mathrm{Ind}}=s}}{V_t^{\mathrm{Ind}}|_{s^{\mathrm{Ind}}=s+1\mathrm{bp}} - V_t^{\mathrm{Ind}}|_{s^{\mathrm{Ind}}=s}}\,; \tag{14}$$

spread deltas with respect to individual names are defined analogously. In the market standard base-correlation approach based on the Gauss copula model $V_t^{[l,u]}|_{s^{\mathrm{Ind}}=s+1\mathrm{bp}}$ is computed by calibrating the model to the index spread $s + 1\mathrm{bp}$, leaving the implied-correlation structure unchanged. In the same spirit, in the homogeneous Markov-chain model of Example 2.2 $V_t^{[l,u]}|_{s^{\mathrm{Ind}}=s+1\mathrm{bp}}$ is computed by calibrating the level-parameter λ_0 to the spread $s^{\mathrm{Ind}} = s + 1\mathrm{bp}$, leaving all other parameters unchanged. Numerical values for the Markov-chain model of Example 2.2 and for the homogeneous Gauss copula model with tranche correlations and default probabilities calibrated to the same iTraxx data are given in Table 3, row 1 and 3. It turns out that in this example the Gauss copula model leads to larger values for Δ^{spread} than the Markov-chain model, in particular for mezzanine tranches.

Immunization against jump-to-default risk. The jump-to-default ratio of a CDO tranche with respect to name i, denoted $J_{t,i}^{\text{def}}$, gives the number of CDSs on firm i one has to hold at time t in order to immunize the portfolio against the change-in-value occurring in the hypothetical scenario where name i defaults at time t. As just explained, in the presence of default contagion the default of firm i impacts the market value $V_{t,j}^{\text{CDS}}$, $j \neq i$. Assume for simplicity that $T_1 > t$ (no defaults in the portfolio up to time t). In that case the vector $(J_{t,1}^{\text{def}}, \dots, J_{t,m}^{\text{def}})$ thus has to solve the following system of m linear equations, indexed with $i \in \{1, \dots, m\}$

$$\begin{aligned} 0 &\overset{!}{=} \Delta G_t^{[l,u]}|_{\tau_i=t} + \sum_{k=1}^{m} J_{t,k}^{\text{def}} \, \Delta G_{t,k}^{\text{CDS}}|_{\tau_i=t} \\ &= \Delta G_t^{[l,u]}|_{\tau_i=t} + J_{t,i}^{\text{def}} \big(\delta_{t,i} - s_i^{\text{CDS}}(t - z_{n_t^z}) - V_{t,i}^{\text{CDS}} \big) + \sum_{k \neq i} J_{t,k}^{\text{def}} \, \Delta V_{t,k}^{\text{CDS}}|_{\tau_i=t} \, . \end{aligned} \tag{15}$$

If firm k has already defaulted one has $J_{t,k}^{\text{def}} = 0$ and the kth equation drops from the system. Note that by (12), $\Delta G_t^{[l,u]}|_{\tau_i=t}$ consists of the change in the market value

$$\Delta V^{[l,u]}|_{\tau_i=t} = v^{[l,u]}\big(t, (L_{t-,1}, \dots, \delta_i(Y_{t-}), \dots, L_{t-,m}), \Psi_t\big) - v^{[l,u]}\big(t, L_{t-}, \Psi_t\big) \, ,$$

and of the default payment and the accrued premium payment triggered by the default event. In a homogeneous portfolio J_t^{def} is identical for all firms; the system (15) reduces to

$$J_t^{\text{def}} = - \frac{\Delta G_t^{[l,u]}|_{\tau_i=t}}{(m - M_{t-} - 1) \Delta V_t^{\text{CDS}}|_{\tau_i=t} - V_t^{\text{CDS}} + \delta(M_{t-}) - s^{\text{CDS}}(t - z_{n_t^z})} = - \frac{\Delta G_t^{[l,u]}|_{\tau_i=t}}{\Delta G_t^{\text{Ind}}|_{\tau_i=t}} \, , \tag{16}$$

and the portfolio can be immunized by taking a position of size J_t^{def} directly in the underlying CDS index, see also Remark 3.1.

Numerical results. Next we compare the jump-to-default ratio in the Markov-chain model of Example 2.2 to the jump-to-default ratio in a homogeneous Gauss copula model with tranche correlations and default probabilities calibrated to the same iTraxx data. In the copula model $J_{t,i}^{\text{def}}$ is simply given by the ratio $-\Delta G_t^{[l,u]}|_{\tau_i=t} / \Delta G_{t,i}^{\text{CDS}}|_{\tau_i=t}$ where $\Delta G_t^{[l,u]}|_{\tau_i=t}$ is computed under the assumption that the fair swap spread of all non-defaulted firms remains unchanged. Numerical values for both models are given in Table 3, row 2 and 4. We see that there are sizeable differences between the jump-to-default ratios for the two models. These differences can be explained by the indirect contagion effect discussed in the previous section, which leads to a substantially higher change in the market value of CDS and CDO contracts for the Markov-chain model than for the copula model.

It is shown in Section 5.2 below that in the Markov-chain model the jump-to-default ratio is a perfect dynamic replication strategy, if the factor process is deterministic. Hence it is of interest to compare this model-based replicating strategy (row 4 of Table 3) with the market standard hedge, namely Δ^{spread} computed in the Gauss copula framework (row 1 of Table 3).

It turns out that for equity and mezzanine tranches the market-standard hedge ratio is larger than J^{def} whereas for the senior tranche this inequality is reversed. Qualitatively similar results are given in Laurent et al. (2008), Table 11.

Finally we also show numerical values for Δ^{spread} and J^{def} in the Markov chain model calibrated to spread data from 2009. We see that there are sizeable numerical differences to the results obtained before the credit crisis. In particular, J^{def} is now much higher for senior tranches because of the strong contagion effect necessary to explain the market quotes observed in 2009.

Tranche		[0,3]	[3,6]	[6,9]	[9,12]	[12,22]
Δ^{spread},	Gauss Copula	0.564	0.295	0.082	0.042	0.081
J^{def},	Gauss Copula	1.002	0.171	0.023	0.008	0.010
Δ^{spread},	Markov-chain model	0.535	0.141	0.044	0.027	0.070
J^{def},	Markov-chain model	0.344	0.138	0.058	0.039	0.107
Δ^{spread},	Markov-chain model, spread data from 2009	0.046	0.208	0.14	0.079	0.074
J^{def},	Markov-chain model, spread data from 2009	1.545	0.698	0.285	0.109	0.164

Table 3: Comparison of jump-to-default ratio J^{def} and of the spread-delta Δ^{spread} in the Markov-chain model and in the Gauss copula model.

5 Dynamic risk-minimizing hedging strategies

5.1 Risk-minimizing hedging strategies

In this section we study the hedging of credit derivatives for the Markov model introduced in Assumption 2.1, using model-based dynamic hedging strategies. We use one CDS per underlying name in the portfolio as hedging instrument, but the methodology obviously applies to other sets of hedging instruments as well. With jump-to-default risk and spread risk, that is for stochastic Ψ, we expect the market to be incomplete, so that a typical CDO tranche cannot be replicated perfectly by dynamic trading in one CDS per underlying name.[4] In order to deal with this problem we use the concept of risk minimization as introduced by Föllmer & Sondermann (1986). Denote by $\tilde{G}_t^{[l,u]}$ and by $\tilde{G}_{t,k}^{\text{CDS}}$, $1 \leq k \leq m$, the discounted gains processes of the CDO tranche and of the CDSs under consideration. A dynamic hedging strategy is a pair $(\theta_0, \boldsymbol{\theta})$, where $\boldsymbol{\theta} = (\theta_{t,1}, \ldots, \theta_{t,m})_{0 \leq t \leq T}$ is an $(\mathcal{F}_t)$-predictable process so that $\theta_{t,k}$ gives the number of CDS on name k in the portfolio at time t, and where the $(\mathcal{F}_t)$-adapted process θ_0 describes the cash position of the strategy. For any dynamic hedging strategy we have a representation of the form

$$0 = \tilde{G}_t^{[l,u]} - \tilde{G}_0^{[l,u]} + \sum_{k=1}^{m} \int_0^t \theta_{s,k} d\tilde{G}_{s,k}^{\text{CDS}} + G_t^{\perp}, \quad 0 \leq t \leq T. \tag{17}$$

Here the process $G^{\perp}$ - which obviously depends on the hedging strategy $\boldsymbol{\theta}$ - represents the hedging error of the strategy. According to Föllmer & Sondermann (1986), a strategy $\boldsymbol{\theta}$ is called *risk-minimizing* if the the so-called *remaining risk* (the conditional variance of the hedging

[4]Note that in certain cases market completeness can be restored if more than one hedging instrument per name is used; this depends on the form of the hedging instruments and on details of the dynamics of Ψ.

error), given by

$$E^Q\left((G^{\perp}_T(\boldsymbol{\theta}) - G^{\perp}_t(\boldsymbol{\theta}))^2 \,\Big|\, \mathcal{F}_t\right), \tag{18}$$

is minimized over all suitable strategies $\boldsymbol{\theta}$ simultaneously for all $0 \le t \le T$. It is well-known, that a risk-minimizing strategy exists and that it can be computed from the *Kunita-Watanabe decomposition* of the Q-martingale $\tilde{G}^{[l,u]}$ with respect to the Q-martingales $\tilde{G}^{\text{CDS}}_k$, $1 \le k \le m$; see Föllmer & Sondermann (1986) for details. In particular, the process $G^{\perp}$ must be orthogonal to the hedging instruments, i.e. $\langle G^{\perp}, \tilde{G}^{\text{CDS}}_k \rangle_t \equiv 0, \quad k = 1, \ldots, m,$. Using this result we can derive from (17) the system of equations

$$d\langle \tilde{G}^{[l,u]}, \tilde{G}^{\text{CDS}}_j \rangle_t = -\sum_{k=1}^m \theta_{t,k} d\langle \tilde{G}^{\text{CDS}}_k, \tilde{G}^{\text{CDS}}_j \rangle_t, \quad j = 1, \ldots, m, \tag{19}$$

from which the processes $\theta_1, \ldots, \theta_m$ can be determined. The cash-position θ_0 of the strategy is finally determined by the requirement that the discounted market value of the overall portfolio has to be equal to zero,

$$\theta_{t,0} + \sum_{k=1}^m \theta_{t,k} \tilde{V}^{\text{CDS}}_{t,k} + \tilde{V}^{[l,u]}_t \stackrel{!}{=} 0, \quad 0 \le t \le T\,.$$

Note that the risk-minimizing strategy $\boldsymbol{\theta}$ is in general not selffinancing; the cumulative injections or withdrawals of funds up to time t are given by $G^{\perp}_t(\boldsymbol{\theta}) - G^{\perp}_0(\boldsymbol{\theta})$, and a tranche can be replicated perfectly if and only if $G^{\perp}_t(\boldsymbol{\theta})$ is constant.

In the remainder of the paper we show how to compute the risk-minimizing strategy $\boldsymbol{\theta}$ and study some of its properties. We begin with the case where Ψ is deterministic and show that in that case the market is complete. In Section 5.3 we discuss the general case where Ψ follows a non-deterministic Markov chain.

Remark 5.1. Risk-minimization in the sense of Föllmer & Sondermann (1986) is well-suited for applications in credit risk, as the Kunita-Watanabe decomposition (19) is relatively easy to compute and as it suffices to know the risk-neutral dynamics of credit derivative prices. From a methodological point of view it might however be more natural to minimize the remaining risk (18) under the historical probability measure. This would lead to alternative quadratic-hedging approaches such as *local risk-minimization* (Föllmer & Schweizer (1991) or Colwell, El-Hassan & Kwon (2007)) or variance-minimizing hedging (Schweizer 2001). However, with discontinuous security prices - which arise naturally in the presence of jump-to-default risk - the computation of the corresponding strategies becomes a very challenging problem. Moreover, it is quite hard to determine the dynamics of CDS and CDO spreads under the historical measure as this requires the estimation of historical default intensities. For these reasons we prefer the simpler risk-minimization approach. Note also that in the complete-market case of Subsection 5.2 this issue does not arise, as the perfect-replication property is invariant with respect to an equivalent change of probability measures.

5.2 Constant factor process

If Ψ is constant, $\Psi_t \equiv \psi$, the gains process of all securities involved are adapted to the default history $(\mathcal{F}^Y_t)$, so that $(\mathcal{F}^Y_t)$ can be taken as the underlying filtration. Note that the assumption of state-dependent recovery rates enters crucially at this point. Moreover, the number of driving risk factors (the default indicator processes $Y_1, \ldots, Y_m$) is equal to the number of hedging instruments, and we expect the market to be complete. In this case there is a direct way for computing the hedging strategy $\boldsymbol{\theta}$ which is not based on the system (19). Define the compensated default indicator processes by

$$N_{t,i} = Y_{t,i} - \int_0^{\tau_i \wedge t} \lambda_i(s, \psi, Y_s) ds\,, \quad 1 \le i \le m\,. \tag{20}$$

Since there are no joint defaults in our model, the default history $(\mathcal{F}_t^Y)$ is generated by the marked point process $(T_n, \xi_n)_{1 \le n \le m}$ with mark space $\{1, \ldots, m\}$. By standard results from stochastic calculus - see for instance Brémaud (1981), Chapter VIII, Theorem T8 - every $(\mathcal{F}_t^Y)$-martingale can therefore be represented as stochastic integral with respect to the m martingales $N_{t,1}, \ldots, N_{t,m}$, i.e. there are predictable processes $\phi_{t,1}^{[l,u]}, \ldots, \phi_{t,m}^{[l,u]}$ and $\phi_{t,1}^k, \ldots, \phi_{t,m}^k$, $1 \le k \le m$, such that

$$d\tilde{G}_t^{[l,u]} = \sum_{i=1}^m \phi_{t,i}^{[l,u]} dN_{t,i}, \quad d\tilde{G}_{t,k}^{\mathrm{CDS}} = \sum_{i=1}^m \phi_{t,i}^k dN_{t,i}, \quad k = 1, \ldots, m\,. \tag{21}$$

In order to determine the hedging strategy $\boldsymbol{\theta}$ we argue as follows: From (17) we get, as $G^\perp \equiv 0$,

$$d\tilde{G}_t^{[l,u]} = -\sum_{k=1}^m \theta_{t,k}\, d\tilde{G}_{t,k}^{\mathrm{CDS}} = -\sum_{k=1}^m \theta_{t,k}\, d\Big(\sum_{i=1}^m \phi_{t,i}^k dN_{t,i}\Big) = -\sum_{i=1}^m \Big(\sum_{k=1}^m \theta_{t,k}\phi_{t,i}^k\Big)\, dN_{t,i}\,. \tag{22}$$

Denote by $A_t := \{i = 1, \ldots, m :\ Y_{t-,i} = 0\}$ the set of non-defaulted firms immediately prior to time t. Comparing (22) with the first equation in (21), it is immediately seen that a hedging strategy $\boldsymbol{\theta}$ exists if and only if the following system of equations has a solution:

$$\sum_{k \in A_t} \theta_{t,k}\phi_{t,i}^k = -\phi_{t,i}^{[l,u]}, \quad i \in A_t\,, 0 \le t \le T\,; \tag{23}$$

for $k \notin A_t$ we let $\theta_{t,k} = 0$. Note that (23) is a linear system of $|A_t|$ equations for $|A_t|$ unknowns with coefficient matrix $\boldsymbol{\Phi}_t := \big(\phi_{t,i}^j\big)_{i,j \in A_t}$.

It remains to determine the integrands $\phi_{t,i}^k$ and $\phi_{t,i}^{[l,u]}$ in the martingale representation (21). If (21) holds, we have $\Delta\tilde{G}_t^{[l,u]} = \sum_{i=1}^m \phi_{t,i}^{[l,u]}\Delta Y_{t,i}$ and $\Delta\tilde{G}_{t,k}^{\mathrm{CDS}} = \sum_{i=1}^m \phi_{t,i}^k \Delta Y_{t,i}$. Hence

$$\phi_{t,i}^{[l,u]} = (1 - Y_{t,i})\Delta\tilde{G}_t^{[l,u]}|_{\tau_i = t} \text{ and } \phi_{t,i}^k = (1 - Y_{t,i})\Delta\tilde{G}_{t,k}^{\mathrm{CDS}}|_{\tau_i = t}. \tag{24}$$

Summing up, we have the following result.

Proposition 5.2. *If the matrix $\boldsymbol{\Phi}_t$ has full rank for all $t \in [0, T]$, the gains process $G^{[l,u]}$ (and in fact every $\mathcal{F}_T^Y$-measurable claim H) can be replicated by dynamic trading in the savings account and the m CDSs. The trading strategy $\boldsymbol{\theta}$ is given as solution to the linear system* (23) *with coefficients given in* (24)*; the cash-position θ_0 is determined by the equation $\theta_{t,0} + \sum_{k=1}^m \theta_{t,k}\tilde{V}_{t,k}^{CDS} + \tilde{V}_t^{[l,u]} = 0$.*

If we plug the expressions (24) into the linear system (23), it is immediately seen that this system reduces to the system (15) for the jump-to-default ratio (J^{def}) in the Markov-chain model. Hence in the absence of spread risk the dynamic hedging strategy is given by $\boldsymbol{\theta} = (J_{t,1}^{\mathrm{def}}, \ldots, J_{t,m}^{\mathrm{def}})_{0 \le t \le T}$. This is quite intuitive as in that case the portfolio is only exposed to default risk.

In a homogeneous portfolio we obviously have $\theta_{t,j} \equiv \theta_t$, $\forall j = 1, \dots m$. Moreover, $\phi^k_{t,j} = \tilde{G}^{\text{CDS}}_{t,1}|_{\tau_2=t} - \tilde{G}^{\text{CDS}}_{t-,1}$, $\forall k \neq j$, $\phi^k_{t,k} = \tilde{G}^{\text{CDS}}_{t,1}|_{\tau_1=t} - \tilde{G}^{\text{CDS}}_{t-,1}$ and $\phi^{[l,u]}_{t,j} = \tilde{G}^{[l,u]}_t|_{\tau_1=t} - \tilde{G}^{[l,u]}_{t-}$, $j = 1, \dots, m$, and we obtain, assuming $t < T_1$ for notational simplicity,

$$\theta_t = -\frac{\Delta \tilde{G}^{[l,u]}_t|_{\tau_1=t}}{(m-1)(\tilde{G}^{\text{CDS}}_{t,1}|_{\tau_2=t} - \tilde{G}^{\text{CDS}}_{t-,1}) + (\tilde{G}^{\text{CDS}}_{t,1}|_{\tau_1=t} - \tilde{G}^{\text{CDS}}_{t-,1})} \,. \tag{25}$$

Note that the discount factor cancels, so that (25) is in fact equivalent to (16). The denominator in (25) can alternatively be viewed as change in the gains process of the CDS index, and we obtain the simple formula $\theta_t = \Delta \tilde{G}^{[l,u]}_t|_{\tau_1=t} \big/ \Delta \tilde{G}^{\text{Ind}}_t|_{\tau_1=t}$, see also Laurent et al. (2008).

Remark 5.3 (The full-rank condition)**.** Conditions ensuring that the full rank condition on $\mathbf{\Phi}_t$ holds (and hence market completeness) are discussed in Frey & Backhaus (2008). In particular, it is shown that $\mathbf{\Phi}_t$ is complete if the contagion effects are not too strong or if the time to maturity $T - t$ is not too large. A risk-minimizing strategy can of course be computed even if $\mathbf{\Phi}_t$ does not have full rank. The necessary computations are a special case of the arguments used in Section 5.3 below, and we omit the details.

5.3 Random factor process

If Ψ is random, the gains processes of all securities involved are adapted to the filtration $\mathcal{F}^Y_t \vee \mathcal{F}^\Psi_t$, and we may work under that filtration. With random Ψ we expect the market to be incomplete, and the derivation of the risk-minimizing hedging strategy is based on the system (19). The main task is to compute the quadratic covariations $\langle \tilde{G}^{[l,u]}, \tilde{G}^{\text{CDS}}_k \rangle_t$ and $\langle \tilde{G}^{\text{CDS}}_j, \tilde{G}^{\text{CDS}}_k \rangle_t$. This is done in two steps.

Step 1: martingale representation. In this step we represent all gains processes as stochastic integrals with respect to the *compensated jump measure* of the Markov chain Γ^Y. Recall that $S^{\Gamma^Y} = \{0,1\}^m \times S^\Psi$, and denote by

$$E^{\Gamma^Y} := \{(\gamma_1, \gamma_2) \in S^{\Gamma^Y} \times S^{\Gamma^Y} : \gamma_1 \neq \gamma_2\}$$

the set of possible transitions of Γ^Y; elements of E^{Γ^Y} are written in the form $\boldsymbol{e} = (\gamma_1, \gamma_2)$. The *counting measure* μ^{Γ^Y} associated with the Markov-chain Γ^Y is a measure on $[0,\infty) \times E^{\Gamma^Y}$; see Section VIII.1 of Brémaud (1981) for the general definition. According to standard measure theory, μ^{Γ^Y} is uniquely defined by its values on sets of the form $[0,t] \times \{(\gamma_1,\gamma_2)\}$, $(\gamma_1, \gamma_2) \in E^{\Gamma^Y}$, $t > 0$; here we have

$$\mu^{\Gamma^Y}\big([0,t] \times \{(\gamma_1,\gamma_2)\}\big) = \sum_{s \leq t} 1_{\{\Gamma^Y_{s-} = \gamma_1, \Gamma^Y_s = \gamma_2\}} \,. \tag{26}$$

The *predictable compensator* of μ^{Γ^Y} is a measure ν^{Γ^Y} on $[0,\infty) \times E^{\Gamma^Y}$ such that for any bounded predictable function $Z \colon \Omega \times [0,\infty) \times E^{\Gamma^Y} \to \mathbb{R}$ the process

$$M^Z_t = \int_0^t \int_{E^{\Gamma^Y}} Z(\omega; s, \boldsymbol{e})(\mu^{\Gamma^Y} - \nu^{\Gamma^Y})(ds \times d\boldsymbol{e})$$

is a martingale. In our case ν^{Γ^Y} is given by

$$\nu^{\Gamma^Y}\big([0,t] \times \{(\gamma_1,\gamma_2)\}\big) = \int_0^t 1_{\{\Gamma^Y_{s-} = \gamma_1\}} q^{\Gamma^Y}_s(\gamma_1,\gamma_2) ds \,, \quad (\gamma_1,\gamma_2) \in E^{\Gamma^Y}, t \geq 0 \,, \tag{27}$$

$q_s^{\Gamma^Y}(\gamma_1, \gamma_2)$ the transition rates of Γ^Y as introduced in Assumption 2.1. It is well-known that every $(\mathcal{F}_t^Y \vee \mathcal{F}_t^\Psi)$-adapted martingale M has a representation of the form

$$M_t = M_0 + \int_0^t \int_{E^{\Gamma^Y}} Z^M(\omega; s, \boldsymbol{e})(\mu^{\Gamma^Y} - \nu^{\Gamma^Y})(ds \times d\boldsymbol{e})$$

for some predictable random function Z^M; see Brémaud (1981), Chapter VIII, Theorem T8 for details. Applying this result to the discounted gains processes $\tilde{G}^{[l,u]}$, $\tilde{G}_k^{\mathrm{CDS}}$, we get the existence of predictable functions $Z^{[l,u]}$, Z_k^{CDS} such that

$$\tilde{G}_t^{[l,u]} = \tilde{G}_0^{[l,u]} + \int_0^t \int_{E^{\Gamma^Y}} Z^{[l,u]}(\omega; s, \boldsymbol{e})(\mu^{\Gamma^Y} - \nu^{\Gamma^Y})(ds \times d\boldsymbol{e}) \tag{28}$$

$$\tilde{G}_{t,k}^{\mathrm{CDS}} = \tilde{G}_{0,k}^{\mathrm{CDS}} + \int_0^t \int_{E^{\Gamma^Y}} Z_k^{\mathrm{CDS}}(\omega; s, \boldsymbol{e})(\mu^{\Gamma^Y} - \nu^{\Gamma^Y})(ds \times d\boldsymbol{e})\,; \tag{29}$$

the computation of $Z^{[l,u]}$ and Z_k^{CDS} is discussed below.

Step 2: computation of the quadratic covariations. We concentrate on computing $\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}} \rangle_t$; the predictable covariation between the discounted gains processes of the CDSs can be computed analogously. As all processes involved have trajectories of finite variation, we get that the pathwise covariation $[\tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}}]$ is given by

$$[\tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}}]_t = \int_0^t \int_{E^{\Gamma^Y}} Z^{[l,u]}(s, \boldsymbol{e}) Z_k^{\mathrm{CDS}}(s, \boldsymbol{e})\, \mu^{\Gamma^Y}(ds \times d\boldsymbol{e})\,.$$

Since $[\tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}}]_t - \langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}} \rangle_t$ is a martingale we thus have

$$\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}} \rangle_t = \int_0^t \int_{E^{\Gamma^Y}} Z^{[l,u]}(s, \boldsymbol{e}) Z_k^{\mathrm{CDS}}(s, \boldsymbol{e})\, \nu^{\Gamma^Y}(ds \times d\boldsymbol{e}) \tag{30}$$

$$= \int_0^t \sum_{\gamma \in S^{\Gamma^Y}, \gamma \neq \Gamma_{s-}^Y} Z^{[l,u]}\big(s, (\Gamma_{s-}^Y, \gamma)\big) Z_k^{\mathrm{CDS}}\big(s, (\Gamma_{s-}^Y, \gamma)\big) q_s^{\Gamma^Y}(\Gamma_{s-}^Y, \gamma)\, ds\,. \tag{31}$$

In differential notation we therefore have $d\langle \tilde{G}^{[l,u]}, \tilde{G}_k^{\mathrm{CDS}} \rangle_t = \xi_t^{[l,u],k} dt$ where the predictable process $\xi^{[l,u],k}$ is given by

$$\xi_t^{[l,u],k} = \sum_{\gamma \in S^{\Gamma^Y}, \gamma \neq \Gamma_{t-}^Y} Z^{[l,u]}\big(t, (\Gamma_{t-}^Y, \gamma)\big) Z_k^{\mathrm{CDS}}\big(t, (\Gamma_{t-}^Y, \gamma)\big) q_t^{\Gamma^Y}(\Gamma_{t-}^Y, \gamma).$$

Using the form of the transition intensities of Γ^Y we get

$$\begin{aligned}\xi_t^{[l,u],k} =& \sum_{i=1}^m (1 - Y_{t-,i}) \lambda_i(t, \Psi_{t-}, Y_{t-}) (Z^{[l,u]} \cdot Z_k^{\mathrm{CDS}})\big(t, \big((Y_{t-}, \Psi_{t-}), (Y_{t-}^i, \Psi_{t-})\big)\big) \\ &+ \sum_{\psi \in S^\Psi, \psi \neq \Psi_{t-}} q^\Psi(\Psi_{t-}, \psi) (Z^{[l,u]} \cdot Z_k^{\mathrm{CDS}})\big(t, \big((Y_{t-}, \Psi_{t-}), (Y_{t-}, \psi)\big)\big)\,. \end{aligned} \tag{32}$$

Similarly we get for the gains processes of two CDSs $d\langle \tilde{G}_j^{\mathrm{CDS}}, \tilde{G}_k^{\mathrm{CDS}} \rangle_t = \xi_t^{j,k} dt$, where $\xi_t^{j,k}$ is given by (32) with $Z^{[l,u]}$ replaced by Z_j^{CDS}.

Computation of $Z^{[l,u]}$ and Z_k^{CDS}. It is immediate from (28) and (29) that $Z^{[l,u]}(\omega;t,\cdot)$ and $Z_k^{\mathrm{CDS}}(\omega;t,\cdot)$ are determined by the jumps of $\tilde{G}_t^{[l,u]}$ and $\tilde{G}_{t,k}^{\mathrm{CDS}}$ induced by transitions of Γ^Y. Recall that the discounted market values are given by

$$\tilde{V}_t^{[l,u]} = \tilde{v}^{[l,u]}(t, L_t, \Psi_t) \text{ and } \tilde{V}_{t,k}^{\mathrm{CDS}} = \tilde{v}_k^{\mathrm{CDS}}(t, Y_t, \Psi_t)\,,$$

for functions $\tilde{v}^{[l,u]}\colon [0,T]\times(0,1]^m \times S^\Psi \to \mathbb{R}$ and $\tilde{v}_k^{\mathrm{CDS}}\colon [0,T]\times\{0,1\}^m\times S^\Psi \to \mathbb{R}$. For notational simplicity we model spread payments by an absolutely continuous payment stream and work with the gains processes (10) and (13). Hence we get at a transition from Y_t to Y_t^i

$$Z^{[l,u]}\big(\omega;t,\cdot\big) = p(0,t)\Delta L_t^{[l,u]}(\omega)|_{\tau_i=t} + \tilde{v}^{[l,u]}\Big(t, \big(L_{t,1},\dots,\delta_i(Y_t),\dots,L_{t,m}\big), \Psi_t\Big) - \tilde{v}^{[l,u]}\big(t, L_t, \Psi_t\big)\,, \tag{33}$$

$$Z_k^{\mathrm{CDS}}\big(\omega;t,\cdot\big) = 1_{\{i=k\}}p(0,t)\delta_i(Y_t) + \tilde{v}_k^{\mathrm{CDS}}(t, Y_t^i, \Psi_t) - \tilde{v}_k^{\mathrm{CDS}}(t, Y_t(\omega), \Psi_t)\,. \tag{34}$$

Similarly we obtain at a transition from Ψ_t to $\tilde{\psi}$ that

$$Z^{[l,u]}\big(\omega;t,\cdot\big) = \tilde{v}^{[l,u]}(t, L_t, \tilde{\psi}) - \tilde{v}^{[l,u]}(t, L_t, \Psi_t)\,, \tag{35}$$

$$Z_k^{\mathrm{CDS}}\big(\omega;t,\cdot\big) = \tilde{v}_k^{\mathrm{CDS}}(t, Y_t, \tilde{\psi}) - \tilde{v}_k^{\mathrm{CDS}}(t, Y_t, \Psi_t)\,. \tag{36}$$

Summarizing we have

Proposition 5.4. *The risk-minimizing hedging strategy $\boldsymbol{\theta} = (\theta_{t,1},\dots,\theta_{t,m})_{0\le t\le T}$ is given as solution of the system*

$$\xi_t^{[l,u],j} = -\sum_{k=1}^m \theta_{t,k}\xi_t^{k,j}\,, \quad j = 1,\dots,m,\ 0 \le t \le T\,, \tag{37}$$

with coefficients $\xi^{[l,u],j}$ and $\xi^{k,j}$ defined in (32) *and* (33) *to* (36).

Proof. It is well-known that a risk-minimizing strategy $\boldsymbol{\theta}$ exists and that it is a predictable process solving the system (19); see for instance Föllmer & Sondermann (1986). Since all quadratic variations involved are absolutely continuous with respect to Lebesgue-measure, the system (19) reduces to (37) and the claim follows. □

Note that we have determined all ingredients necessary to set up the system (37), so that $\boldsymbol{\theta}$ is easily computed. In the homogeneous-portfolio case things simplify further since θ_t solves the one-dimensional equation $\xi_t^{[l,u]} = -\theta_t\big((m-1)\xi_t^{1,2} + \xi_t^{1,1}\big)$. The computation of risk-minimizing hedging strategies for the case where ψ follows a diffusion process is discussed in (Backhaus 2008).

5.4 Numerical experiments.

We conclude this section with a small numerical study. Here we focus on two issues: first we compute risk-minimizing hedging strategies for a random factor process Ψ and study the impact of the spread-volatility on the hedging strategy; second we look at the impact of portfolio-heterogeneity on the form and the performance of hedging strategies.

Risk minimization when Ψ is a Markov chain. In order to illustrate the results of Section 5.3 we compute risk-minimizing hedging strategies for the homogeneous models introduced in Example 2.2. Table 4 gives the value θ of the risk-minimizing hedging strategy and compares with the jump-to-default ration J^{def} and with the spread delta Δ^{spread} introduced in Section 4. As one would expect, with low spread volatility (small range of the factor state space) the risk-minimizing strategy is close to J^{def}; with increasing spread volatility θ comes closer to Δ^{spread}. Hence the risk-minimizing strategy provides a model-based endogenous interpolation between the hedging of spread-risk and the hedging against of jump-to-default risk.

Hedging in heterogenous portfolios. While computationally convenient, the homogeneous-portfolio assumption only an imperfect description of most real-world credit portfolios. It is therefore interesting to study the impact of heterogeneity in the portfolio on the form and the performance of the ensuing hedge ratios. For simplicity, we assume that Ψ is deterministic, so that dynamic hedging strategies coincide with the jump-to-default ration as computed in Section 4. We consider a portfolio with two industry groups of varying credit quality, indexed by $\kappa = 1, 2$. The default intensity of firms from group κ is modelled by a function $h_\kappa(t, n_\kappa, n)$ which depends on time t, on the number of defaults in group κ, denoted n_κ, and on the overall number of defaults n. Denote by m_κ the number of firms in group κ and by $\mu_\kappa(t)$ the expected number of defaults in group κ. In our simulations we take

$$h_\kappa(t, n_\kappa, n) = \lambda_{\kappa,0} + \frac{\lambda_1}{\lambda_2}\Big\{ \exp\Big(\lambda_2\gamma_\kappa \frac{(n_\kappa - \mu_\kappa(t))^+}{m_\kappa} + \lambda_2(1-\gamma_\kappa)\frac{(n-\mu(t))^+}{m}\Big) - 1\Big\}, \qquad (38)$$

with parameters $\lambda_{\kappa,0} > 0$, $\lambda_1 \geq 0$, $\lambda_2 \geq 0$ and $\gamma_\kappa \in [0, 1]$. The first term in the argument of the exponential function in (38) reflects the interaction between firms from the same group; the second term captures the global interaction between defaults in the entire portfolio. The relative strength of these effects is governed by the parameter γ_κ: for γ_κ close to one, firms from group κ are mainly impacted by defaults within group κ; for γ_κ close to zero, the global (portfolio-wide) interaction dominates. In our analysis we compare three different portfolios/parameterizations.

A) Here we consider a homogeneous portfolio of 125 firms with fair CDS spread equal to $s^k = 73$bp for all k.

B) Here we consider a portfolio consisting of 100 'good' names (Group 1, $s^k = 20$bp, $k = 1, \ldots, 100$) and 25 'bad' names (Group 2, $s^k = 317$bp, $k = 101, \ldots, 125$); the spread of a CDS index on the whole portfolio is $s^{\text{Ind}} = 73$bp so that the average credit quality is the same as in Portfolio A. Moreover, we put $\gamma_1 = \gamma_2 = 0$ so that there is only global interaction between defaults. Intuitively, this parameterization corresponds to a portfolio with one-factor structure.

C) Here the portfolio consists of 25 good names (Group 1) with $s^k = 17.5$bp and 100 medium-quality names (Group 2) with $s^k = 88$bp; again $s^{\text{Ind}} = 73$bp. We assume that there is a strong interaction within the group of good firms and put $\gamma_1 = 0.7$; γ_2 is set to 0.2 so that the medium-quality firms are mainly affected by the global portfolio state. Intuitively, Group 1 could be viewed as a set of major financial firms, as the default of a major financial institution typically has a strong negative impact on the remaining firms; Group 2 could be viewed as medium-sized non-financial corporations.

The remaining parameters in (38) have been chosen so that we get (roughly) the same CDO spreads in all three parameterizations.

Recall that for a deterministic factor process the hedging strategy $\boldsymbol{\theta}$ solves the linear system (15) and that the coefficients of this system are given by the change in the gains process of the tranche to be hedged and of the CDSs used as hedging instrument. Hence these changes are largely responsible for the form of the hedge ratios. Numerical values are reported in Table 5 in the appendix: we see that the results for Parameterizations A and B are roughly similar. Note however, that under Parametrization B the default of a bad name (a firm from Group 2) always leads to a smaller absolute change in the gains process than a default of a good name (a firm from Group 1). The reason for this is that in the former case the quality of the remaining portfolio is higher than in the latter case. The results for Parametrization C on the other hand differ widely from the homogeneous-portfolio case. In particular, for the given parameters the default of a good name leads to a substantial deterioration of the credit quality of the overall portfolio, as can be seen from the huge change in the gains process of the index. The hedge ratios for all three parameterizations are given in Table 6 (for $t = 0$). As with the changes in the gains processes, for Portfolio A and B the ensuing hedge ratios are qualitatively similar. However, there are substantial quantitative differences. For Portfolio C on the other hand the strong and asymmetric contagion effects lead to qualitatively different hedge ratios. In particular, in order to hedge the equity- and mezzanine tranches one has to take a protection-seller position in the CDS issued by the good names in Group 1.

An alternative way to assess the model risk due to the simplifying assumption of a homogeneous portfolio is to look at the performance at a default-time T_1 of the homogeneous-portfolio strategy, assuming that the actual change in the gains processes corresponds to an inhomogeneous situation. More precisely, we consider a portfolio consisting of a protection-seller position in a CDO tranche or in the CDS-index and of an offsetting protection-buyer position in the CDSs; the size of this protection-buyer position was computed using Parametrization A (see the Portfolio A row of Table 6). Next we compute the hedging error (the change in the gains process of this portfolio) at the first default time T_1, assuming that $\Delta G_{T_1}^{[l,u]}$ and $\Delta G_{k,T_1}^{\mathrm{CDS}}$ are generated by Parameterizations B or C (see the Portfolio B and the Portfolio C row of Table 5). The results of this exercise are contained in Table 7. We report the *relative hedging error* for each tranche (the hedging error normalized by the overall notional); in this way results for different tranches can be compared. We note the following: in case where the actual gains processes are generated by Parametrization B, the homogeneous-portfolio strategy performs quite well. On the other hand, if the actual gains processes are generated by Parametrization C the performance of the homogeneous-portfolio strategy is poor, at least at a default of a firm in Group 1. In fact, the hedging error for the equity- and the junior mezzanine tranche is in the order of 50% of total notional.

These findings suggest that hedging strategies based on the assumption of a homogeneous portfolio - and in particular all hedges computed within the top-down approach - might perform well if the real portfolio is heterogeneous with respect to credit quality but relatively homogeneous with respect to the interaction between firms; on the other hand, if the real portfolio exhibits strongly asymmetric contagion effects the homogeneous-portfolio assumption might lead to poorly performing strategies. Limitations of the top-down approach for the purpose of hedging portfolio credit derivatives are also discussed in the recent paper Bielecki, Crepey & Jeanblanc (2009).

6 Conclusion

This paper has studied the (dynamic) hedging of CDO tranches in a portfolio credit risk model with default contagion and random fluctuations in credit spreads. This model was constructed and analyzed with Markov-chain techniques. From our analysis the following findings emerged. First, we studied the impact of default contagion on the market-standard sensitivity-based hedging strategies. It turned out that even a small amount of default contagion has a substantial impact on the form of the ensuing hedge ratios, essentially because of the impact of the default event on the quality of the remaining portfolio. Second, we showed how to compute theoretically consistent dynamic hedging strategies using incomplete-market theory, more specifically the concept of risk-minimization. The main tool in the derivation of these strategies is stochastic calculus for marked point processes. Third, we carried out numerical experiments to study the properties of these strategies. It turned out that risk-minimizing hedging strategies interpolate between the hedging of spread- and default risk in an endogenous fashion. Moreover, we showed that deviations from the popular assumption of a homogeneous portfolio can have a substantial impact on the form and on the performance of hedging strategies.

These are important results on dynamic hedging in credit markets. In particular, the sizeable differences between the market-standard sensitivity-based hedging strategies computed in the copula framework and the dynamic hedging strategies derived in our setup with spread risk and default contagion show that the current hedging practice is subject to a substantial amount of model risk. A systematic study of the model risk associated with the hedging of credit derivatives is therefore a logical next step. Due to its versatility the Markov-chain model proposed in the first part of the present paper could be a useful tool in this analysis. However, such a study is a major undertaking and is therefore deferred to further research.

A Tables

A1. Risk-minimization when Ψ is a Markov chain

	[0,3]-tranche			[3,6]-tranche			[6,9]-tranche		
	θ	J^{def}	Δ^{spread}	θ	J^{def}	Δ^{spread}	θ	J^{def}	Δ^{spread}
S_0^Ψ	0.344	0.344	-	0.138	0.138	-	0.058	0.058	-
S_1^Ψ	0.348	0.345	0.476	0.138	0.138	0.143	0.057	0.058	0.049
S_2^Ψ	0.414	0.366	0.491	0.136	0.134	0.138	0.050	0.053	0.045
S_3^Ψ	0.469	0.414	0.526	0.127	0.126	0.128	0.041	0.045	0.038

	[9,12]-tranche			[12,22]-tranche		
	θ	J^{def}	Δ^{spread}	θ	J^{def}	Δ^{spread}
S_0^Ψ	0.039	0.039	-	0.107	0.107	-
S_1^Ψ	0.039	0.039	0.031	0.106	0.107	0.082
S_2^Ψ	0.033	0.037	0.029	0.093	0.103	0.079
S_3^Ψ	0.028	0.032	0.025	0.084	0.096	0.074

Table 4: Comparison of the risk-minimizing hedging strategy θ with J^{def} and Δ^{spread} for different choices of the state spaces S^Ψ; Δ^{spread} has been computed using (14)

A2. Hedging in heterogenous portfolios

	ΔG^{CDS_1}	ΔG^{CDS_2}	ΔG^{Ind}	$\Delta G^{[0,3]}$	$\Delta G^{[3,6]}$	$\Delta G^{[6,9]}$	$\Delta G^{[9,12]}$	$\Delta G^{[12,22]}$
Portfolio A	0.0161	0.0161	-2.600	-0.471	-0.582	-0.236	-0.134	-0.315
Portfolio B								
default in Group 1	0.0177	0.0234	-2.752	-0.525	-0.772	-0.387	-0.233	-0.474
default in Group 2	0.0149	0.0197	-2.201	-0.467	-0.645	-0.322	-0.193	-0.392
Portfolio C								
default in Group 1	0.1386	0.1019	-13.451	-0.660	-1.295	-0.929	-0.807	-2.355
default in Group 2	0.0083	0.0146	-1.617	-0.466	-0.558	-0.211	-0.113	-0.258

Table 5: Changes due to a default in the gains process of the non-defaulted CDSs, of the index and of various CDO tranches for Portfolios A, B and C. The numbers were computed for $t = 0$.

Product		Index	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Portfolio A	J^{def}	1.000	0.181	0.224	0.091	0.051	0.121
Portfolio B	J_1^{def} (good firms)	1.029	0.155	0.237	0.119	0.072	0.146
	J_2^{def} (bad firms)	0.802	0.168	0.185	0.090	0.053	0.108
Portfolio C	J_1^{def} (good firms)	1.039	-0.575	-0.514	-0.043	0.084	0.370
	J_2^{def} (bad firms)	0.979	0.286	0.325	0.108	0.049	0.088

Table 6: Hedge-ratio (or equivalently J^{def}) for the CDS index and for various CDO tranches at $t = 0$, assuming that $T_1 > 0$.

Product	Index	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
$\Delta G^{[l,u]}$, ΔG^{CDS} as in Parametrization B						
Rel. hedging error at default in Group 1	0.1%	0.2%	-3.0%	-3.2%	-2.2%	-0.9%
Rel. hedging error at default in Group 2	0.3%	-0.1%	-1.9%	-2.4%	-1.7%	-0.7%
$\Delta G^{[l,u]}$, ΔG^{CDS} as in Parametrization C						
Rel. hedging error at default in Group 1	0.5%	50.5%	49.8%	9.5%	-2.3%	-5.2%
Rel. hedging error at default in Group 2	0.5%	-1.6%	-1.4%	-0.2%	0.1%	0.1%

Table 7: Performance at $t = T_1$ of the hedging strategy computed for Parametrization A, assuming that the actual change in the gains processes is generated by Parameterizations B or C. The hedging error is represented as percentage of the total notional of the tranche. The latter equals 125 for the index, 3.75 for the $[0,3]$ up to the $[9,12]$ tranche and 12.5 for the $[12,22]$ tranche

Product	Index	$[0,3]$	$[3,6]$	$[6,9]$	$[9,12]$	$[12,22]$
Observed spread	36 bp	26%	84bp	25bp	12bp	6bp

Table 8: iTraxx-spreads from January 2006 used in the numerical examples.

A3. ITraxx spreads

References

Andersen, L. & Sidenius, J. (2004), 'Extensions to the Gaussian copula: Random recovery and random factor loadings', *Journal of Credit Risk* **1**, 29–70.

Arnsdorf, M. & Halperin, I. (2007), 'BSLP: Markovian bivariate spread-loss model for portfolio credit derivatives', working paper, available from `http://arXiv.org/abs/0901.3398`.

Backhaus, J. (2008), Pricing and Hedging of Credit Derivatives in a Model with Interacting Default Intensities: a Markovian Approach, PhD thesis, Department of mathematics, Universität Leipzig.

Bielecki, T., Crepey, S. & Jeanblanc, M. (2009), 'Up and down credit risk', preprint, Evry University, available on www.defaultrisk.com.

Bielecki, T., Jeanblanc, M. & Rutkowski, M. (2004), Hedging of defaultable claims, *in* 'Paris-Princeton Lectures on Mathematical Finance', Vol. 1847 of *Springer Lecture Notes in Mathematics*, Springer.

Bielecki, T., Jeanblanc, M. & Rutkowski, M. (2007), 'Hedging of basket credit derivatives in the Credit Default Swap market', *Journal of Credit Risk* **3**, 91–132.

Bielecki, T. Vidozzi, A. & Vidozzi, L. (2008), 'A Markov copulae approach to pricing and hedging of credit index derivatives and rating-triggered step-up bonds', to appear in Journal of Credit Risk.

Brémaud, P. (1981), *Point Processes and Queues: Martingale Dynamics*, Springer, New York.

Colwell, D., El-Hassan, N. & Kwon, O. K. (2007), 'Hedging diffusion processes by local risk minimization with applications to index tracking', *J. Econom. Dynam. Control* **31**, 2135–2151.

Cont, R. & Minca, A. (2008), 'Recovering portfolio default intensities implied by CDO quotes', working paper, Center for Financial Engineering,Columbia University, available from `http://ssrn.com/abstract=1104855`.

Crépey, S. & Carmona, R. (2008), 'Importance sampling and interacting particles for the estimation of Markovian credit portfolio loss distributions', preprint, Evry University, forthcoming in Int. Jour. Theor. Appl. Fin.

Davis, M. & Lo, V. (2001), 'Infectious defaults', *Quant. Finance* **1**, 382–387.

El Karoui, N., Jeanblanc-Picqué, M. & Shreve, S. (1998), 'Robustness of the Black and Scholes formula', *Math. Finance* **8**, 93–126.

Elouerkhaoui, Y. (2006), Etude des Problèmes de Corrélation et d'Incomplétude dans les Marchés de Crédit, PhD thesis, Université Paris IX Dauphine. in English.

Föllmer, H. & Schweizer, M. (1991), Hedging of contingent-claims under incomplete information, *in* 'Applied Stochastic Analysis', Gordon & Breach, London, pp. 205–223.

Föllmer, H. & Sondermann, D. (1986), Hedging of non-redundant contingent-claims, *in* W. Hildenbrand & A. Mas-Colell, eds, 'Contributions to Mathematical Economics', North Holland, pp. 147–160.

Frey, R. & Backhaus, J. (2008), 'Pricing and hedging of portfolio credit derivatives with interacting default intensities', *International Journal of Theoretical and Applied Finance* **11**(6), 611–634.

Giesecke, K. & Goldberg, L. (2007), 'A top-down approach to multi-name credit', working paper, Department of Management Science and Engineering, Stanford University.

Giesecke, K. & Weber, S. (2006), 'Credit contagion and aggregate losses', *J. Econom. Dynam. Control* **30**, 741–767.

Herbertsson, A. (2008), 'Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach', *The Journal of Credit Risk* **4**, 3–35.

Jarrow, R. & Yu, F. (2001), 'Counterparty risk and the pricing of defaultable securities', *J. Finance* **56**, 1765–1799.

Laurent, J., Cousin, A. & Fermanian, J. (2008), 'Hedging default risk of CDOs in Markovian contagion models', workinp paper, ISFA Actuarial School, Université de Lyon.

Lopatin, A. & Misirpashaev, T. (2007), 'Two-dimensional Markovian model for dynamics of adequate credit loss', working paper, available from `www.defaultrisk.com`.

Neugebauer, M. (2006), 'Understanding and hedging risks in synthetic CDO tranches', Fitch Special Report.

Rosen, D. & Saunders, D. (2009), 'Analytical methods for hedging systematic credit risk with linear factor portfolios', *J. Econom. Dynam. Control* **33**, 37–52.

Schönbucher, P. (2006), 'Portfolio loss and the term-structure of loss transition rates: a new methodology for the pricing of portfolio credit derivatives', working paper, ETH Zürich.

Schweizer, M. (2001), A guided tour through quadratic hedging approaches, *in* E. Jouini, J. Cvitanic & M. Musiela, eds, 'Option Pricing, Interest Rates and Risk Management', Cambridge University Press, pp. 538–574.